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Pre-RMO 2020

Pre-Regional Mathematical Olympiad

Prashant Jain

Pearson

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PREFACE

In 2017, PreRMO (Pre-Regional Mathematics Olympiad) was introduced as a region-wise examination to select 300+ students in 30+ regions across India to reduce subjective copy checking work at the stage of RMO. PRMO was held in states such as Maharashtra and West Bengal earlier but in 2017 it became a compulsory exercise across the nation.

In this scenario, a need arose to construct good study material for this examination as some of the books were too tough while others were too easy for this examination. To cater to this need, I decided to produce this book with a very simple yet intuitive architecture—the architecture of non-routine mathematics.

In this book, I have tried to build the problem level from NSEJS to KVPY to PRMO level so that students can sequentially develop their thinking for PRMO. Please note that this book can be used as a prerequisite for studying for RMO, which is considered as one of the toughest examinations for mathematics at the school level.

The theory has been included with solved examples to ensure that students are able to understand the topics properly. For problems, I have included two exercise sets, divided into three sections each, as per the difficulty level:

1. NSEJS Level
2. KVPY Level
3. PRMO Level

In exercise 2, previous years' problems have been included to ensure that the students are always aware of their progress.

The success and final outcome of this project required a lot of assistance from many people. I would like to express special gratitude to Ashish Chawla for his insights on the chapters on Geometry and to Himanshu Nagar for his insights on the chapters on Algebra. Their constructive suggestions have contributed immensely in the pursuit of this project. I must thank Srijon Sarkar, one of my students, for his tireless work checking the relevance of questions. I sincerely convey my thanks to Aditi Vyas for completing the final proof-reading and Iqbal Ansari and Hariom Gautam for their extraordinary efforts in typesetting.

Every care has been taken to ensure that the book is error free, but since it is a one-of-a-kind book, there may be some errors, and we are very happy to listen to them so that we may review and remove them in subsequent editions. To better understand this book, you are welcome to attend my lectures on YouTube. All the best for PRMO!

Prashant Jain

ABOUT THE AUTHOR

Prashant Jain is a B.Tech graduate in Computer Science from IIT Powai. He secured All India Rank 42 in JEE 2008.

Prashant Jain is an author, trader, analyst, teacher, educationalist, dancer and also a motivational speaker. He was a former Intern at BMC Software, former Trader at Optiver and former academic coordinator, PCCP (Pre-Foundation Career Care Program) Division at Resonance Eduventures. From 2018 to 2019, he was the Program Head of PCCP Division at Nucleus Education. In 2019, he started his own venture Talking Textbooks for Grades VI to X.

Prashant Jain is a passionate mathematics teacher of PRMO, and his course based on PRMO on YouTube, with more than 500K views, is very popular among students.

Chapter

1 Basics of Algebra

Learning Outcomes

After reading this chapter, you will be able to understand

- Comparison of Ratios and Types of Ratios
- Practical Mathematics %, Time and Work, etc.
- Variation Direct, Indirect, Joint
- Polynomials Value of a Polynomial, Remainder Theorem, Factor Theorem
- Factorization of Polynomials
- Algebraic Identities and their Value Forms
- Understating Symmetric and Homogeneous Expressions
- Solving Linear Equation in Two Variable and Condition of Solvability
- Integer Solutions to Linear Equation

Ratio and Proportion

Ratio

A ratio is a comparison of two numbers (quantities) by division. The ratio of a to b is written as $a : b$ or $\frac{a}{b}$. In the ratio $a : b$, a and b are called terms of the ratio. 'a' is the antecedent and 'b' is the consequent.

Properties of Ratio

(a) In a ratio, two quantities are compared. So, the quantities must be of the same kind i.e., they must be expressed in the same units.

(b) The order of the terms in a ratio $a : b$ is very important, since $2 : 3$ is not same as $3 : 2$.

(c) The value of a ratio remains unaltered· if the given ratio is multiplied or divided by the same non-zero quantity. If a, b and m are non-zero real numbers, then $a : b = ma : mb$ and $a : b = a/m : b/m$.

(d) If the ratio of any two quantities can be expressed exactly by the ratio of two integers, the quantities are said to be commensurable otherwise incommensurable.

To Divide a Number into Parts in a Given Ratio

Let 'X' be the given number, the given ratio is $x_1 : x_2$.

X is to be divided into two parts such that the

Value of first part: Value of second part $= x_1 : x_2$

Therefore,

$$\text{First part} = \left(\frac{x_1}{x_1 + x_2}\right) \times X$$

$$\text{Second part} = \left(\frac{x_2}{x_1 + x_2}\right) \times X$$

Compare the ratio $3 : 4$ and $1 : 2$

$$\frac{3}{4} = \frac{3}{4} \times \frac{1}{1} = \frac{3}{4}$$

$$\frac{1}{2} = \frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$$

$$3 > 2, \frac{3}{4} > \frac{2}{4} \text{ i.e. } 3 : 4 > 1 : 2$$

Comparison of Ratio

Let $a : b$ and $c : d$ are two given ratios, then

(a) $a : b > c : d$ if $ad > bc$

(b) $a : b < c : d$ if $ad < bc$

(c) $a : b = c : d$ if $ad = bc$

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(d) Ratios can also be compared by expressing the ratios as fractions and then converting them to decimal numbers.

(e) They can also be compared by converting them to their equivalent fraction of common denominator.

Ratio of Greater Inequality or Lesser Inequality

(a) A ratio $a : b$ is called a ratio of greater inequality If $a > b$ e.g. 5 : 2 is a ratio of greater inequality. A ratio of greater inequality is decreased if a same positive number is added to both the terms of the ratio and increased if a same positive number is subtracted from both of its terms.

(b) A ratio $a : b$ is called ratio of less inequality if $a < b$. A ratio of less inequality is decreased if a same positive number is subtracted from both the terms of the ratio and increased if a same positive quantity is added to both of its terms.

More Terms Related to Ratios

(a) Compounded Ratio: The compounded ratio of $a : b$ and $c : d$ is $ac : bd$.

(b) Duplicate Ratio: The duplicate ratio of $a : b$ is $a^2 : b^2$

(c) Triplicate Ratio: The triplicate ratio of $a : b$ is $a^3 : b^3$

(d) Sub-duplicate Ratio: The sub-duplicate ratio of $a : b$ is $\sqrt{a} : \sqrt{b}$

(e) Sub-triplicate Ratio: The sub-triplicate ratio of $a : b$ is $\sqrt[3]{a} : \sqrt[3]{b}$

(f) Inverse Ratio: The inverse ratio or reciprocal ratio of $a : b$ is $\frac{1}{a} : \frac{1}{b}$ i.e.,

$b : a$ ratio × its Inverse ratio = 1.

Proportion

Equality of two ratios is called proportion.

If $a : b = c : d$, then $a, b, c,$ and d are in proportion, $a : b = c : d$ is also represented as $a : b :: c : d$.

1. Each quantity $a, b, c,$ and d of a proportion $a : b :: c : d$ is called a term or proportional. a, b, c and d are, respectively, called first, second, third, and fourth term or proportional of the proportion $a: b :: c: d$.

2. The first and the fourth (last) terms are called extremes and the second and third terms are called middle terms or means.

3. In a proportional $a : b :: c : d$ i.e., **product** of extremes = product of means.

4. In a proportion, fourth term is called fourth proportional of the other three terms.

5. If $a : b :: c : d$ then $b : a :: d : c$ or $\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{b}{a} = \frac{d}{c}$

This is known as **Invertendo**.

6. If $a : b :: c : d$ then $a : c :: b : d$ or $\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a}{c} = \frac{b}{d}$

This result is known as **Alternendo**.

7. If $a : b :: c : d \Rightarrow \frac{a}{b} + 1 = \frac{c}{d} + 1 \Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$ or

$a : b :: c : d \Leftrightarrow (a + b) : b :: (c + d) : d$ ..... (1)

This result is known as **Componendo**.

8. If $a : b :: c : d \Rightarrow \frac{a}{b} - 1 = \frac{c}{d} - 1 \Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$ or

$a : b :: c : d \Leftrightarrow (a - b) : b :: (c - d) : d$ ..... (2)

This result is known as **Dividendo**.

9. Dividing equation (1) and (2), we get

$$\frac{\frac{a+b}{b}}{\frac{a-b}{b}} = \frac{\frac{c+d}{d}}{\frac{c-d}{d}} \Rightarrow (a+b) : (a-b) :: (c+d) : (c-d) \text{ or}$$

$a : b :: c : d \Leftrightarrow$ This result is known as **Componendo** and **Dividendo**.

10. a, b, c, d are said to be in continued proportion if

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

11. If a, b, c are in continued proportion then $\frac{a}{b} = \frac{b}{c} \Rightarrow b^2 = ac \Rightarrow b = \pm\sqrt{ac}, b$ is called mean proportional of 'a' and 'c' and its value is $\sqrt{ac}$. $(b > 0)$ The first is to the third is the duplicate ratio of first to the second.

$$\frac{a}{b} = \frac{b}{c}$$

$$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$$

12. If $a : b = c : d$ and $e : f = g : h$, then $ae : bf = cg : dh$

If $a : b = c : d$, $b : x = d : y$,

then $a : x = c : y$,

13. $a : b = c : d = e : f = (ka + lc + me) / (kb + ld + mf)$. It follows that if all the ratios are finite and denominator $kb + ld + mf = 0$, then it implies $ka + lc + me = 0$

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Example 1

If $a : b = 5 : 9$ and $b : c = 4 : 7$, find $a : b : c$. (Combining Ratios)

Solution

$$a : b = 5 : 9 \text{ and } b : c = 4 : 7 = \left(4 \times \frac{9}{4}\right) : \left(7 \times \frac{9}{4}\right) = 9 : \frac{63}{4}$$

$$\Rightarrow a : b : c = 5 : 9 : \frac{63}{4} = 20 : 36 : 63.$$

Example 2

Find out:

(i) the fourth proportional to 4, 9, 12.
(ii) the third proportional to 16 and 36.
(iii) the mean proportional between 0.08 and 0.18.

Solution

(i) Let the fourth proportional to 4, 9, 12 be x.
Then, $4 : 9 :: 12 : x \Rightarrow 4 \times x = 9 \times 12$

$$\Rightarrow x = \frac{9 \times 12}{4} = 27$$

$\therefore$ Fourth proportional to 4, 9, 12 is 27.

(ii) Let the third proportional to 16 and 36 is x.
Then, $16 : 36 :: 36 : x \Rightarrow 16 \times x = 36 \times 36$

$$\Rightarrow x = \frac{36 \times 36}{16} = 81$$

$\therefore$ Third proportional to 16 and 36 is 81.

(iii) Mean proportional between 0.08 and 0.18

$$= \sqrt{0.08 \times 0.18} = \sqrt{\frac{8}{100} \times \frac{18}{100}} = \sqrt{\frac{144}{100 \times 100}}$$

$$= \frac{12}{100} = 0.12$$

Example 3

The monthly incomes of two persons are in the ratio 4 : 5. The monthly expenditure of them are in the ratio 7 : 9. If each saves ₹1000 per month find their income.

Solution

The incomes are in the ratio 4 : 5.
$\therefore$ Their incomes are $4K$ and $5K$ (say)
Their expenditures are $4K - 1000$ and $5K - 1000$.

Given $\frac{4K - 1000}{5K - 1000} = \frac{7}{9}$

$\Rightarrow 36K - 9000 = 35K - 7000 \Rightarrow K = 2000.$

Their incomes are ₹8,000 and ₹10,000.

Example 4

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that $\frac{a^3b + 2c^2e - 3ae^2f}{b^4 + 2d^2f - 3bf^3} = \frac{ace}{bdf}$

Solution

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$;

then $a = bk$, $c = dk$, $e = fk$;

$$\therefore \frac{a^3b + 2c^2e - 3ae^2f}{b^4 + 2d^2f - 3bf^3} = \frac{b^4k^3 + 2d^2fk^3 - 3bf^3k^3}{b^4 + 2d^2f - 3bf^3}$$

$$k^3 = \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$$

Example 5

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that $\frac{x^2 + a^2}{x + a} + \frac{y^2 + b^2}{y + b} + \frac{z^2 + c^2}{z + c} = \frac{(x + y + z)^2 + (a + b + c)^2}{x + y + z + a + b + c}$

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Solution

Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k$, so that $x = ak$, $y = bk$, $z = ck$;

$$\text{then } \frac{x^2 + a^2}{x + a} = \frac{a^2k^2 + a^2}{ak + a} = \frac{(k^2 + 1)a}{k + 1}$$

$$\therefore \frac{x^2 + a^2}{x + a} + \frac{y^2 + b^2}{y + b} + \frac{z^2 + c^2}{z + c} = \frac{(k^2 + 1)a}{k + 1} + \frac{(k^2 + 1)b}{k + 1} +$$

$$\frac{(k^2 + 1)c}{k + 1} = \frac{(k^2 + 1)(a + b + c)}{k + 1}$$

$$\left[\text{Multiply and Divide by } \frac{(a + b + c)}{(a + b + c)}\right]$$

$$= \frac{k^2(a + b + c)^2 + (a + b + c)^2}{k(a + b + c) + a + b + c}$$

$$= \frac{(ka + kb + kc)^2 + (a + b + c)^2}{(ka + kb + kc) + a + b + c}$$

$$= \frac{(x + y + z)^2 + (a + b + c)^2}{x + y + z + a + b + c}$$

Example 6

If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, prove that

$$\frac{x+y+z}{a+b+c}=\frac{x(y+z)+y(z+x)+z(x+y)}{2(ax+by+cz)}$$

Solution

Each of the given fractions $=\frac{\text{sum of numerators}}{\text{sum of denominators}}$

$$=\frac{x+y+z}{a+b+c} \quad \ldots\ldots \text{(i)}$$

Again, if we multiply both numerator and denominator of the three given fractions by $y+z$, $z+x$, $x+y$, respectively, each fraction

$$\frac{x(y+z)}{(y+z)(b+c-a)}=\frac{y(z+x)}{(z+x)(c+a-b)}=\frac{z(x+y)}{(x+y)(a+b-c)}$$

$$=\frac{\text{sum of numerators}}{\text{sum of denominators}}$$

$$=\frac{x(y+z)+y(z+x)+z(x+y)}{2ax+2by+2cz} \quad \ldots\ldots \text{(ii)}$$

$\therefore$ from (i) and (ii).

$$\frac{x+y+z}{a+b+c}=\frac{x(y+z)+y(z+x)+z(x+y)}{2ax+2by+2cz}$$

Example 7

If $x=\frac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$. Solve that $x^3-3mx^2+3x-m=0$.

Solution

We have $\frac{x}{1}=\frac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$

By componedo and dividendo, we get

$$\frac{x+1}{x-1}=\frac{2\sqrt[3]{m+1}}{2\sqrt[3]{m-1}} \Rightarrow \frac{(x+1)^3}{(x-1)^3}=\frac{m+1}{m-1}$$

Again by componendo and dividendo, we get

$$\frac{(x+1)^3+(x-1)^3}{(x+1)^3-(x-1)^3}=\frac{m+1+m-1}{m+1-(m-1)}$$

$$\Rightarrow \frac{x^3+3x^2+3x+1+x^3-3x^2+3x-1}{x^3+3x^2+3x+1-(x^3-3x^2+3x-1)}=\frac{m}{1}$$

$$\Rightarrow \frac{2x^3+6x}{6x^2+2}=m$$

$$\Rightarrow \frac{x(x^2+3)}{(3x^2+1)}=m$$

$$x^3+3x=3mx^2+m \Rightarrow x^3-3mx^2+3x-m=0$$

Example 8

If $\frac{x^2+y^2}{p^2+q^2}=\frac{xy}{pq}$. Prove that $\frac{x}{y}=\frac{p}{q}$.

Solution

From the given, we get,

$$\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$$

$$\Rightarrow \frac{x^2+y^2}{2xy}=\frac{p^2+q^2}{2pq}$$

By componendo and dividendo

$$\frac{x^2+y^2+2xy}{x^2+y^2-2xy}=\frac{p^2+q^2+2pq}{p^2+q^2-2pq}$$

$$\Rightarrow \frac{(x+y)^2}{(x-y)^2}=\frac{(p+q)^2}{(p-q)^2}$$

$$\Rightarrow \frac{x+y}{x-y}=\frac{p+q}{p-q}$$ (One of the possibilities out of positive and negative)

Again by componendo and dividendo

$$\frac{x+y+x-y}{x+y-(x-y)}=\frac{p+q+p-q}{p+q-(p-q)}$$

$$\Rightarrow \frac{2x}{2y}=\frac{2p}{2q}$$

$$\Rightarrow \frac{x}{y}=\frac{p}{q}.$$

Example 9

4 men and 3 women finish a job in 6 days and 5 men and 7 women can do the same job in 4 days. How long will 1 man and 1 women take to do the work?

(A) $22\left(\frac{2}{7}\right)$ days **(B)** $25\left(\frac{1}{2}\right)$ days

(C) $5\left(\frac{1}{7}\right)$ days **(D)** $12\left(\frac{7}{22}\right)$ days

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Solution

Let men complete m part in a day and women complete w part in a day. Then,

$$4m + 3w = \frac{1}{6} \quad \ldots(i)$$

$$5m + 7w = \frac{1}{4} \quad \ldots(ii)$$

After simplifying, we get

$$20m + 15w = \frac{5}{6}$$

$$20m + 28w = 1$$

So $13w = \frac{1}{6}$

$$w = \frac{1}{78} \quad \ldots(iii)$$

From (i) and (iii) we get, $m = \frac{5}{156}$

$$m + w = \frac{5}{156} + \frac{1}{78} = \frac{7}{156}$$

So, it requires $\frac{156}{7}$ days $= 22\frac{2}{7}$ **Ans. (A)**

Example 10

A person distributes his pens among four friends A, B, C, and D in the ratio $\frac{1}{3} : \frac{1}{4} : \frac{1}{5} : \frac{1}{6}$. What is the minimum number of pens that the person should have?

Solution

Ratio of pens = A: B: C: $D = \frac{1}{3} : \frac{1}{4} : \frac{1}{5} : \frac{1}{6}$

$= 20 : 15 : 12 : 10$

Let number of pens given to A, B, C, D are $20x$, $15x$, $12x$, $10x$

So, minimum number of pens (for $x = 1$)

$= 20 + 15 + 12 + 10 = 57$

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Know Your Progress - 1

1. Two numbers are, respectively, 20% and 50% more than a third number. Find the ratio of the two numbers.
2. The ratio of the number of boys and girls in a college is 7 : 8. If the percentage increase in the number of boys and girls be 20 % and 10 %, respectively, then find the new ratio.
3. The prices of a scooter and a TV are in the ratio 7 : 5. If the scooter costs ₹8000 more than a TV set, then find the price of a TV set.
4. Find the ratio whose terms differ by 40 and the measure of which is $\frac{2}{7}$.
5. The ages of a man and his son is in the ratio of 7 : 2. After 15 years, they would be in the ratio of 2 : 1. Find the father's age when son was born.
6. What quantity must be added to each term of the duplicate ratio of $a : b$ to make it equal to $a : b$?
7. If $\frac{ay - bx}{c} = \frac{cx - az}{b} = \frac{bz - cy}{a}$, show that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.
8. If $a : b = c : d = e : f$, show that $27(a + b)(c + d)(e + f) = bdf\left(\frac{a+b}{b} + \frac{c+d}{d} + \frac{e+f}{f}\right)^3$
9. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, show that $(ax + by + cz)$ is the mean proportional between $(a^2 + b^2 + c^2)$ and $(x^2 + y^2 + z^2)$.
10. If y varies as the sum of two quantities, of which one varies directly as x and the other inversely as x; and if $y = 6$ when $x = 4$, and $y = 3\frac{1}{3}$ when $x = 3$, then find the equation between x and y.
11. A diamond falls and breaks into pieces whose weights are in the ratio 2 : 3 : 5. The value of the diamond is directly proportional to the square of its weights. Find the loss incurred, if the actual cost of the diamond is ₹96000 (in ₹).

Theorems Related To Polynomials

General expression for the polynomial in one variable can be as follows:

$$f(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$$

where b_i are real numbers (or complex numbers as the case given in problems) and b_n is not equal to 0. b_0 is constant term and n is whole number.

Some Definitions

Degree of a Polynomial

In a polynomial, degree is the largest power of x having non-zero coefficient. In above expression, it is 'n'. Degree of a polynomial is unique and can only be greater or equal to 0. Thus, $P(x) = 6$ is also a polynomial with degree 0. It is called a constant polynomial. Degree of a polynomial is a whole number.

Number of Terms in a Polynomial

As the name suggests, it is the number of distinct terms in a polynomial in irreducible form. Irreducible form means that it cannot be reduced further e.g., $P(x) = x^3 + 4x + 6 + 9x$ has 3 terms and not 4 terms in the reduced form i.e., $P(x) = x^3 + 13x + 6$

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Zero Polynomial and Constant Polynomial

$P(x) = a$ for some constant value a is called constant polynomial. When the value of constant is 0. $P(x)$ is called zero polynomial. **Degree of zero polynomial is not defined** while degree of other constant polynomials is 0.

Coefficients

They are real numbers which come in front of a power of x or constant term b_0, e.g., coefficient of x^4 is b_4 in general representation of polynomials.

Condition of Whole Number Power

All the powers of the variable in different terms of any polynomial should be whole number, otherwise it will not be called as a polynomial.

E.g., $x^3 + x^2 + 3$, it will be a polynomial as the power of variable in the 3 terms are 3, 2, and 0, respectively, but in expression $x^{\frac{3}{2}} + x - 4$ will not be called a polynomial as the power of x are $\frac{3}{2}$, 1, and 0 out of which $\frac{3}{2}$ is not a whole number.

Different Types of Polynomials

Generally, we divide the polynomials in the following categories.

Based on Degrees

There are four types of polynomials based on degrees. These are listed below:

1. **Zero degree polynomial:** Any non-zero number (constant) is regarded as a polynomial of degree zero or zero degree polynomial i.e., $f(x) = a$, where $a \neq 0$ is a zero degree polynomial, since we can write $f(x) = a$ as $f(x) = a\,x^0$
2. **Linear Polynomial**: A polynomial of degree one is called a linear polynomial. The general form of linear polynomial is $ax + b$, where a and b are any real constant and $a \neq 0$.
3. **Quadratic Polynomial**: A polynomial of degree two is called a quadratic polynomial. The general form of a quadratic polynomial is $ax^2 + bx + c$, where $a \neq 0$.
4. **Cubic Polynomial**: A polynomial of degree three is called a cubic polynomial. The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where $a \neq 0$.
5. **Biquadratic (or quartic) Polynomial**: A polynomial of degree four is called a biquadratic (quartic) polynomial. The general form of a biquadratic polynomial is $ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$.

Note

1. A polynomial of degree five or more than five does not have any particular name. Such a polynomial is usually called polynomial of degree five or six, etc.
2. If $a = 1$ in above classification, the polynomials are said to be monic polynomials. "a" is also called the leading coefficient i.e., coefficient of x^n

Based on Number of Terms

There are three types of polynomials based on number of terms. These are as follows:

1. **Monomial**: A polynomial is said to be a monomial if it has only one term. E.g., $x, 9x^2, 5x^3$ all are monomials.
2. **Binomial**: A polynomial is said to be a binomial if it contains two terms. E.g., $2x^2 + 3x, x + 5x^3, 8x^3 + 3$ all are binomials.

3. **Trinomial**: A polynomial is said to be a trinomial if it contains three terms. E.g., x^2+x+1, x^3+4x+1, $x^{54}+x+3$ are all trinomials.

Note

A polynomial having four or more than four terms does not have particular name. These are simply called polynomials.

Zeroes of Polynomials

Let's consider a polynomial $P(x) = x + 4$ to see what the solution of polynomial means. If we put $x = -4$ in this polynomial, we get $P(-4) = 0$. The values of x for which $P(x) = 0$ are called zeros of the polynomial or solutions of the polynomial.

Remainder Theorem

When 30 is divided by 4, you have 2 as remainder. 30 can be written as follows,

$$30 = 4 \times 7 + 2$$

Note that the remainder is always less than the divisor (obvious). Can such an expression be written for polynomials? Yes, it can be written for polynomials too.

Before doing that, let's discuss multiplication of polynomial first. Multiplication of polynomial is done term by term as shown in the example below.

Example 11

Let's multiply $P(x)=x+2$ and $Q(x)=x^2+4$

Solution

We will write multiplication as follows:

$$R(x)=P(x)Q(x)=(x+2)(x^2+4)$$

$$=x(x^2+4)+2(x^2+4)$$

$$=x^3+4x+2x^2+8$$

This is like algebraic multiplication that you have done in your earlier classes $(a + b)(c + d) = a(c + d) + b(c + d)$.

Find $R(x)$ and also write degree of $R(x)$ in the following cases:

$P(x)$	$Q(x)$	$R(x)=P(x)Q(x)$	Degree of $R(x)$
x^3+x^2+x+1	x		4
$x^{54}-x+1$	x^2+1		
6	x^2+1	$6x^2+6$	
$x^4+3x^2+2x+2x^2$	0		

Can you observe any relation between the degrees of $R(x), P(x)$, and $Q(x)$, when $R(x) = P(x)Q(x)$ the relation will be, degree of $R(x)$ = degree of $P(x)$ + degree of $Q(x)$.

Having done multiplication, what about if we do some division with polynomials. Let's see the following example:

Example 12

Divide x^2+4x+4 by x.

Solution

The dividend polynomial can be written in the form

Dividend = (Divisor × Quotient) + Remainder

(The famous division algorithm for polynomials)

$$B(x)=Q(x)A(x)+R(x)$$

As, $x^2+4x+4=x(x+4)+4$

If you compare this expression with the expression used when 30 was divided by 4, you can see a very interesting comparison. In above expression, x^2+4x+4 is $B(x)$(dividend), x is $A(x)$ (divisor), $x+4$ is $(Q(x)$ (quotient,) and 4 is $(R(x))$ (remainder). In number case, remainder cannot be greater than divisor. In polynomial division, there is a small twist. Here, **degree of remainder cannot be greater than that of divisor.** The interesting part is that we can indeed write polynomial division in our very own long division method as shown below with $A(x)=x^3-2x^2-4, B(x)=x-3$. Here $Q(x)=x^2+x+3$, $R(x)=5$

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$$
\begin{array}{r|l}
x-3 & x^3-2x^2+0x-4 \quad (x^2+x+3 \text{ as quotient}) \\
 & -x^3+3x^2 \\
\hline
 & +x^2+0x \\
 & -x^2+3x \\
\hline
 & +3x-4 \\
 & +3x-9 \\
\hline
 & +5
\end{array}
$$

At this point, parallels between numeric division and polynomial division ends. In numeric division, you cannot find remainder directly without actually dividing the dividend. In polynomial division, however, there is a method to find remainders when the divisor is linear i.e., polynomial of degree 1. We will illustrate by using above polynomials.

Here, $A(x)=x^3-2x^2-4, B(x)=x-3.3$ is a zero of B(x). Put this value in A(x). You will get A (3) to be 5, which is the same remainder as found by long division method. This method can be formalized using the very famous Remainder Theorem.

Statement of Remainder Theorem

Let P(x) to be any polynomial of degree greater than or equal to 1 and let α be any real number. If $P(x)$ is divided by linear polynomial $x-a$, then the remainder is $P(a)$.

Proof

Let Q(x) be the quotient and $R(x)$ be the remainder obtained when the polynomial $P(x)$ is divided by $x-a$. Then,

$$P(x)=(x-a)Q(x)+R(x),$$

where $R(x)=0$ or some constant.

Let $R(x)=c$, where c is some constant. Then.

$$P(x)=(x-a)Q(x)+c$$

Putting $x=a$, in $P(x)=(x-a)\,q(x)+c$, we get

$$P(a)=(a-a)Q(a)+c \quad P(a)=0\times Q(a)+c \quad P(a)=c$$

This shows that the remainder is $P(a)$ when $P(x)$ is divided by $x-a$.

Statement of Factor Theorem

If $P(x)$ is a polynomial of degree $n\geq 1$ and a is any real number, then $x-a$ is a factor of $P(a)=0$.

Proof

Before starting the proof, let's understand what "iff" means. Its full form is "if and only if". This here means that

(a) If $P(a)=0$, $x-a$ is a factor of $P(x)$

(b) If $x-a$ is a factor of $P(x)$, $P(a)=0$

Thus, the statement is true in both directions. This occurs more often in higher classes. We will prove both directions now. By remainder theorem, $P(x)$ can be written as $P(x)=(x-a)Q(x)+P(a)$

(a) If $P(a)=0, P(x)=(x-a)Q(x)$, which means $x-a$ is a factor of $P(x)$.

(b) If $x-a$ is a factor of $P(x), P(x)=(x-a)Q(x)$ for some $Q(x)$. Putting $x=a$ in this equation we get $P(a)=0$.

Example 13

If $x=2$ and $x=0$ are zeros of the polynomials $f(x)=2x^3-5x^2+ax+b$, then find the values of a and b.

Solution

$$f(2)=2(2)^3-5(2)^2+a(2)+b=0$$

$$\Rightarrow 16-20+2a+b=0 \Rightarrow 2a+b=4$$

$$\Rightarrow f(0)=2(0)^3-5(0)^2+a(0)+b=0$$

$$\Rightarrow b=0$$

$$\Rightarrow 2a=4 \Rightarrow a=2,\ b=0.$$

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Example 14

Find the sum of all the coefficients in $f(x)=(x-2)(x+3)(x-4)(x-5)(x+6)$.

Solution

The sum of all the coefficients is given by $f(1)$

$$f(1)=(1-2)(1+3)(1-4)(1-5)(1+6)=-336$$

Example 15

If $f\left(2x+\frac{1}{x}\right)=x^2+\frac{1}{4x^2}+1\ (x\neq 0)$, then find the value of $f(x)$.

Solution

$$f\left(2x+\frac{1}{x}\right)=x^2+\frac{1}{4x^2}+1=\frac{1}{4}\left(4x^2+\frac{1}{x^2}+4\right)=\frac{1}{4}\left(2x+\frac{1}{x}\right)^2$$

$$\Rightarrow f(x)=\frac{1}{4}x^2$$

Example 16

Find the remainder, when $f(x)=x^3-6x^2+2x-4$ is divided by $g(x)=1-2x$.

Solution

$$f(x) = x^3 - 6x^2 + 2x - 4$$

Let, $1 - 2x = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

$$\text{Remainder} = f\left(\frac{1}{2}\right)$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 4 = \frac{1}{8} - \frac{3}{2} + 1 - 4$$

$$= \frac{1 - 12 + 8 - 32}{8} = -\frac{35}{8}.$$

Example 17

Find the remainder, when $f(x) = x^3 + 3x^2 - 2x + 6$ is divided by $g(x) = x^2 - 1$

Solution

$$f(x) = g(x)q(x) + r(x)$$

$$\Rightarrow x^3 + 3x^2 - 2x + 6 = (x^2 - 1)\, q(x) + (ax + b)$$

(since divisor is quadratic, remainder should be linear or constant)

$$\Rightarrow x^3 + 3x^2 - 2x + 6 = (x - 1)(x + 1)\, q(x) + (ax + b) \quad \ldots (i)$$

Putting $x = 1$ in (i), we get $a + b = 8$
Putting $x = -1$ in (i), we get $-a + b = 10$
Upon solving, $a = -1$, $b = 9$
Hence remainder is $(-x + 9)$

Example 18

Show that $x + 1$ and $2x - 3$ are factors of $2x^3 - 9x^2 + x + 12$.

Solution

To prove that $(x + 1)$ and $(2x - 3)$ are factors of $2x^3 - 9x^2 + x + 12$ it is sufficient to show that $p(-1)$ and

$$p\left(\frac{3}{2}\right) \text{ both are equal to zero.}$$

$$p(-1) = 2(-1)^3 - 9(-1)^2 + (-1) + 12$$
$$= -2 - 9 - 1 + 12 = -12 + 12 = 0.$$

$$\text{and } p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 12$$

$$= \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12 = \frac{27 - 81 + 6 + 48}{4} = \frac{-81 + 81}{4} = 0.$$

Hence, $(x + 1)$ and $(2x - 3)$ are the factors $2x^3 - 9x^2 + x + 12$.

Example 19

Find the values of a and b so that the polynomials $x^3 - ax^2 - 13x + b$ has $(x - 1)$ and $(x + 3)$ as factors.

Solution

Let $f(x) = x^3 - ax^2 - 13x + b$

Because $(x - 1)$ and $(x + 3)$ are the factors of $f(x)$,

$$\therefore f(1) = 0 \text{ and } f(-3) = 0$$

$$f(1) = 0 \Rightarrow (1)^3 - a(1)^2 - 13(1) + b = 0 \Rightarrow 1 - a - 13 + b = 0$$

$$\Rightarrow -a + b = 12 \quad \ldots. (i)$$

$$f(-3) = 0$$

$$\Rightarrow (-3)^3 - a(-3)^2 - 13(-3) + b = 0$$

$$\Rightarrow 27 - 9a + 39 + b = 0$$

$$\Rightarrow -9a + b = -12 \quad \ldots (ii)$$

Subtracting equation (ii) from equation (i)

$$(-a + b) - (-9a + b) = 12 + 12$$

$$\Rightarrow -a + 9a = 24 \Rightarrow 8a = 24 \Rightarrow a = 3.$$

Put $a = 3$ in equation (i)

$$-9 + b = 12 \Rightarrow b = 15.$$

Hence, $a = 3$ and $b = 15$.

Example 20

What must be subtracted from $x^3 - 6x^2 - 15x + 80$ so that the result is exactly divisible by $x^2 + x - 12$.

Solution

Let $ax + b$ be subtracted from $p(x) = x^3 - 6x^2 - 15x + 80$ so that it is exactly divisible by $x^2 + x - 12$.

$$\therefore s(x) = x^3 - 6x^2 - 15x + 80 - (ax + b) = x^3 - 6x^2$$

$$-(15 + a)x + (80 - b)$$

Dividend = Divisor × quotient + remainder

But remainder will be zero.

$$\therefore \text{Dividend} = \text{Divisor} \times \text{quotient}$$

$$\Rightarrow s(x) = (x^2 + x - 12) \times \text{quotient}$$

$$\Rightarrow s(x) = x^3 - 6x^2 - (15 + a)x + (80 - b)$$

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$$x^2+x-12\overline{)x^3-6x^2-x(15+a)+80-b}\ \Big|\ x-7$$

$$-x^3 \pm x^2 \mp 12x$$

$$-7x^2+12x-(15+a)x+80-b$$

$$-7x^2+x(-3-a)+80-b$$

$$\mp 7x^2 \mp 7x \pm 84$$

$$x(4-a)+(-4-b)=0$$

Hence, $x(4-a)+(-4-b)=0.x+0$

$\Rightarrow 4-a=\&(-4-b)=0 \quad \Rightarrow a=4$ and $b=-4$

Hence, if in $p(x)$ we subtract $4x-4$ then it is exactly divisible by x^2+x-12.

Example 21

If $x^2-x-1=0$, then find the value of x^3-2x+1

Solution

$$x^3-2x+1=x(x^2-x-1)+(x^2-x-1)+2$$

Putting $x^2-x-1=0$, we get $x^3-2x+1=2$

Example 22

Find all the zeros of the polynomial $f(x)=2x^4-2x^3-7x^2+3x+6$, if two of its zeros are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$

Solution

Since, $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$ are zeros of $f(x)$.

Therefore, $\left(x+\sqrt{\frac{3}{2}}\right)\left(x-\sqrt{\frac{3}{2}}\right)=\left(x^2-\frac{3}{2}\right)=\frac{2x^2-3}{2}$

or $2x^2-3$ is a factor of $f(x)$.

$$2x^2-3\overline{)2x^4-2x^3-7x^2+3x+6}\ \Big|\ x^2-x-2$$

$$2x^4 \quad \mp 3x^2$$

$$-2x^3-4x^2+3x+6$$

$$\mp 2x^3 \quad \pm 3x$$

$$-4x^2+6$$

$$\mp 4x^2 \pm 6$$

$$0$$

$\therefore\ 2x^4-2x^3-7x^2+3x+6=(2x^2-3)(x^2-x-2)$

$=(2x^2-3)(x-2)(x+1)$

$=\left(x+\sqrt{\frac{3}{2}}\right)\left(x-\sqrt{\frac{3}{2}}\right)(x-2)(x+1)$

So, the zeros are $-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 2, -1$.

Example 23

If x^2+x-1 is a factor of $x^4+px^3+qx^2-1$, then find the values of p and q.

Solution

$$x^4+px^3+qx^2-1=(x^2+x-1)(x^2+bx+c)$$

$$\Rightarrow x^4+px^3+qx^2-1=x^4+(b+1)x^3+(x+c-1)x^2+(c-b)x-c$$

Upon comparing coefficients, we get

$$b+1=p\ ;\ b+c-1=q\ ;\ c-b=0\ ;\ -c=-1$$

Upon solving, $c=1=b,\ p=2,\ q=1$

Example 24

Try to find the zeros of $p(x)=x^3+6x^2+11x+6$ by hit and trial method (choose the value and then put that value of x in the polynomial to check).

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Solution

Choose $x=-1,-2$ or -3, will get $P(x)=0$. This mean that $\{-1,-2,-3\}$ is the set of all values of x, which are zeros of the given polynomials. Hit and trial method is often used to find the solutions of polynomials when these solutions are simple.

Try to find the solution of $P(x)=x^2+20x+91$ using hit and trial method. You will see that this polynomial cannot be solved easily by hit and trial method.

Example 25

Find the value of $a^3+b^3+c^3-3abc$

If $a=500$

$b=502$

$c=504$

Solution

$$\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\left(\frac{a+b+c}{2}\right)$$

$$=[4+4+16]\left(\frac{1506}{2}\right)$$

$$=12\times 1506$$

$$=18072$$

Example 26

Find remainder when $P(x) = x^3 + 3x^2 + 3x + 1$ is divided by $x + 1$

Solution

Here,

$$P(x) = x^3 + 3x^2 + 3x + 1$$

Putting $x = -1$ for remainder, we get

$$P(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1$$

$$P(-1) = 0$$

Example 27

Factorize $x^3 + 19x^2 + 71x - 91$ into linear factors.

Solution

By the look of this polynomial, it is not very easy to solve directly. However, we can see that 1 is a solution of this polynomial and hence $x - 1$ is a factor of this polynomial. By long division method, the given polynomial can be written as

$$x^3 + 19x^2 + 71x - 91 = (x - 1)(x^2 + 20x + 91)$$

In earlier example, we solved $x^2 + 20x + 91$. Thus, writing in linear factors,

$$x^3 + 19x^2 + 71x - 91 = (x - 1)(x + 7)(x + 13)$$

Given a cubic polynomial, it's not very easy to find its factors. Below mentioned table summarizes some cases to find simple factors of a cubic polynomial $ax^3 + bx^2 + cx + d$.

Factor	Condition
$x - 1$	$a + b + c + d = 0$
$x + 1$	$a - b + c - d = 0$
$x - 2$	$8a + 4b + 2c + d = 0$
$x + 2$	$8a - 4b + 2c - d = 0$

Example 28

Factorize $6x^4 - 19x^3 - 23x^2 + 10x + 8$.

Solution

Let $f(x) = 6x^4 - 19x^3 - 23x^2 + 10x + 8$ on hit and trial basis, we find that $f(-1) = 0$ and $f(4) = 0$

$\therefore (x + 1)$ as well as $(x - 4)$ is a factor $f(x)$. on dividing $f(x) = 6x^4 - 19x^3 - 23x^2 + 10x + 8$ by $(x^2 - 3x - 4)$, we get $(6x^2 - x - 2)$.

$$\therefore \quad 6x^4 - 19x^3 - 23x^2 + 10x + 8$$

$$= (x^2 - 3x - 4)\ (6x^2 - x - 2)$$

$$= (x + 1)\ (x - 4)\ (6x^2 - 4x + 3x - 2)$$

$$= (x + 1)(x - 4)[2x(3x - 2) + 1(3x - 2)]$$

$$= (x + 1)(x - 4)(3x - 2)(2x + 1)$$

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Know Your Progress - 2

1. Using remainder theorem, find the remainder when $f(x)$ is divided by $g(x)$ and verify the result by actual division. Given $f(x) = x^3 - 6x^2 + 2x - 4$ and $g(x) = 1 - 2x$.
2. (1) If the polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$, when divided by $(x - 2)$ leave the same remainder, find the value of a.
3. Determine whether $x - 3$ is a factor of polynomial $p(x) = x^3 - 3x^2 + 4x - 12$.
4. Using factor theorem, prove that $p(x)$ is divisible by $g(x)$ if $P(x) = 4x^4 + 5x^3 - 12x^2 - 11x + 5$, $g(x) = 4x + 5$
5. Find the remainder, when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by $g(x) = 1 + 2x$.
6. The polynomials $ax^3 + 3x^2 - 3$ and $2x^3 - 5x + a$ when divided by $(x - 4)$ leaves remainder R_1 and R_2, respectively, then find the value of 'a' if $2R_1 - R_2 = 0$.
7. If $ax^3 + bx^2 + x - 6$ has $x + 2$ as a factor and leaves a remainder 4 when divided by $(x - 2)$, find the values of a and b?
8. The remainder when $x^5 + kx^2$ is divided by $(x - 1)$ $(x - 2)$ $(x - 3)$ contains no terms in x^2. Find 'k' without performing division.

Algebraic Identities and Expressions

In previous sections, we discussed about polynomials in one variable (i.e., x). Polynomials of 2 variables are just extension of earlier learnt concepts and we will use these concepts to discuss them and their algebraic properties.

Examples of polynomials in two variables are:

1. $P(x,y) = x^2 + 2xy + y^2$
2. $P(x,y) = x^3 + 3x^2y + 2xy^2$
3. $P(x,y) = x$ (coefficient of powers of y is 0)
4. $P(x,y) = 6$ (coefficients of powers of x and y is 0)

Factorization of Polynomials in Two Variables

In our discussions here, we will discuss polynomials in which sum of powers of x and y in a particular term is constant for all the terms. For example, each term in $x^2 + 2xy + y^2$ has this sum 2, likewise, $x^3 + 3x^2y + 2xy^2$ has this sum to be 3 in each of its term. These polynomials are called **homogeneous polynomials or uniform polynomials** as this sum is same for each term.

Such a choice is necessary to apply concepts learnt in previous sections here. For example,

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(a) $x^2 + 2xy + y^2 = (x+y)(x+y)$ (By splitting middle term)

(b) $x^3 + 3x^2y + 2xy^2 = x(x+y)(x+2y)$ (By splitting middle term)

Algebraic Identities

Following identities are handy to solve common questions in polynomial algebra. Proofs of these identities can be done by simple mathematics.

(a) $(a+b)^2 = a^2 + 2ab + b^2$

(b) $(a-b)^2 = a^2 - 2ab + b^2$

(c) $a^2 - b^2 = (a+b)(a-b)$

(d) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(e) $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

(f) $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

(i) $(a+b)^3 = a^3 + b^3 + 3ab\,(a+b)$

(j) $(a-b)^3 = a^3 - b^3 - 3ab\,(a-b)$

(k) $\left(a^2 + b^2 + c^2 - ab - bc - ac\right) = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

Special case: if $\left(a^2 + b^2 + c^2 - ab - bc - ac\right) = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] = 0$ then $a = b = c$

(l) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$

(m) $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

Special case I: if $a+b+c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

Special case II: if $a^3 + b^3 + c^3 - 3abc = 0$, then either $a+b+c = 0$ or $a = b = c$

Value Form

(a) $a^2 + b^2 = (a+b)^2 - 2ab$, if $a + b$ and ab are given.

(b) $a^2 + b^2 = (a-b)^2 + 2ab$, if $a - b$ and ab are given.

(c) $a + b = \sqrt{(a-b)^2 + 4ab}$, if $a - b$ and ab are given.

(d) $a - b = \sqrt{(a+b)^2 - 4ab}$, if $a + b$ and ab are given.

(e) $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$, if $a + \frac{1}{a}$ is given.

(f) $a^2 + \frac{1}{a^2} = \left(a - \frac{1}{a}\right)^2 + 2$, if $a - \frac{1}{a}$ is given.

(g) $a^3 + b^3 = (a+b)^3 - 3ab\,(a+b)$, if $(a+b)$ and ab are given.

(h) $a^3 - b^3 = (a-b)^3 + 3ab\,(a-b)$, if $(a-b)$ and ab are given.

(i) $a^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right)$, if $a + \frac{1}{a}$ is given

(j) $a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)^3 + 3\left(a - \frac{1}{a}\right)$, if $a - \frac{1}{a}$ is given

(k) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = [(a+b)^2 - 2ab](a+b)(a-b)$.

(l) $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$ (Sophie Germain identity)

(m) $(x^n - y^n)$ is always divisible by $(x-y)$

(n) $(x^n + y^n)$ is divisible by $(x+y)$ when 'n' is odd.

(o) $(x^n - y^n)$ is divisible by $(x+y)$ when 'n' is even.

(p) $a^4 + b^4 + a^2b^2 \equiv (a^2 + ab + b^2)(a^2 - ab + b^2)$

(q) $x^2 + y^2 + xy \equiv \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2$

Algebraic Expressions

Homogeneous Expressions

An expression is said to be homogeneous when all its terms are of the same degree. For example, $ax + by$ is homogeneous expression of the first degree in x and y, $ax^2 + bxy + cy^2$ is a homogeneous expression of the second degree in x and y, and $ax^3 + by^3 + cz^3 - xyz$ is homogeneous expression of the third degree.

If a homogeneous expression of the first degree, say $x + y + z$, is multiplied by a homogeneous expression of the second degree, say $ax^2 + by^2 + cz^2$, then the product $ax^3 + by^3 + cz^3 + ax^2y + ax^2z + by^2x + by^2z + cz^2x + cz^2y$ is a homogeneous expression of the third degree ($\therefore$, $1 + 2 = 3$). Hence, we have the following law:

The product of two homogeneous expressions of degrees m and n, respectively, is a homogeneous expression of degrees $m + n$.

Symmetric Expressions

An integral function is said to be symmetric with respect to a pair of variables if the expression remains unaltered when the variables are interchanged, i.e., an expression $f(x,y)$ is said to be symmetric if $f(x,y) = f(y,x)$.

Thus, $x^2 + 2xy + y^2$ is symmetric with respect to x and y, since when x and y are interchanged, the resulting function, $y^2 + 2yx + x^2$, is the same as the original function. Further, $xy + yz + zx$ is a symmetric expression, since by interchanging x and y, it becomes $yx + xz + zy$, which is the same as $xy + yz + zx$. Similarly, $xy + yz + zx$ is symmetric with respect to the pair (y, z) and (z, x), and hence $xy + yz + zx$ is symmetric in x, y, and z.

Similarly, $x + y + z$, $x^2 + y^2 + z^2 - xy - yz - zx$, $x^3 + y^3 + z^3 - 3xyz$, $x + y + 1$, and $x^3 + y^3 - x - y$ are symmetric expressions.

Note that $x - y + 1$ is not symmetric in x and y and $ab + bc - ca$ is not symmetric in a, b, and c.

A symmetric expression is not necessarily a homogenous, nor a homogeneous necessarily symmetrical.

For example, $x + y - 1$ is a symmetric function but not a homogenous function. Again, $x^2 + xy$ is a homogeneous function but not a symmetric function.

Properties of Symmetrical Expressions

1. In a symmetric function, all terms of the same type must occur and their coefficients must be equal.
   Terms which can be obtained from one another by successively interchanging the variables are known as same type terms.
   Thus, if x, y, and z are the variables, then y^2z and z^2y are terms of the same type, since the former is changed to the latter by interchanging the pair y, z. Similarly, z^2x, x^2z, y^2x, and x^2y are all terms belonging to the same type as y^2z and z^2y, but x^3, y^3, and z^3 are not terms of the same y^2z type, since the interchanging of the letters x, y, and z will not transform y^2z into x^3, y^3, or z^3.
2. The sum, difference, product, or quotient of two symmetric expressions is a symmetric expression.
   If we consider two letters x and y, then, in general, the symmetric homogeneous expressions of the first and second degrees, respectively, are the following:
   (a) $k(x + y)$
   (b) $k(x^2 + y^2) + mxy$, where k and m are constants
   If we consider three letters x, y, and z, then, in general, the symmetric homogenous expressions of the first and second degrees, respectively; are the following:
   (a) $k(x + y + z)$
   (b) $k(x^2 + y^2 + z^2) + m(xy + yz + zx)$, where k and m are any constants.
   The values of these constants can be found out by comparing the coefficients.

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Cyclic Symmetric Expressions

An expression $f(x, y, z)$ is said to be a cyclic expression if $f(x, y, z) = f(y, z, x) = f(z, x, y)$.

For the sake of brevity, we use the symbols $\sum$ and $\prod$ to write the cyclic expressions.

$\sum$ is used for sum of terms and $\prod$ is used for product of terms.

The addition, subtraction, multiplication, and/or division of two cyclic polynomials always results in a cyclic polynomial.

Here are some common factors and their cyclic representatives for a cyclic polynomial with 3 variables:

Expression	Expansion
$\sum a$	$a + b + c$
$\sum a^2$	$a^2 + b^2 + c^2$
$\sum ab$	$ab + bc + ca$
$\sum a^2(b + c)$	$a^2(b + c) + b^2(c + a) + c^2(a + b)$
$\sum ka$	$k\sum a$

(Continued)

Expression	Expansion
$\prod a$	abc
$\prod (a+b)$	$(a+b)(b+c)(c+a)$

HCF and LCM of Polynomials

A polynomial $D(x)$ is a divisor of the polynomial $P(x)$ if it is a factor of $P(x)$. Where $Q(x)$ is another polynomial such that $P(x) = D(x) \times Q(x)$

1. **HCF/GCD (Greatest Common Divisor):** A polynomial $h(x)$ is called the HCF or GCD of two or more given polynomials, if $h(x)$ is a polynomial of highest degree dividing each of one of the given polynomials.
2. **L.C.M. (Least Common Multiple):** A polynomial $P(x)$ is called the LCM of two or more given polynomials, if it is a polynomial of smallest degree which is divided by each one of the given polynomials. For any two polynomials $P(x)$ and $Q(x)$, we have:
$P(x) \times Q(x)$ = [HCF of $P(x)$ and $Q(x)$] × [LCM of $P(x)$ and $Q(x)$]

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Example 29

Expand:

(i) $\left(2x - \frac{1}{3x}\right)^2$

(ii) $(3x^2 + y)^2$

(iii) $(\sqrt{2}\,x - 3y)\,(\sqrt{2}\,x + 3y)$

(iv) $\left(\frac{1}{4}a - \frac{1}{2}b + 1\right)^2$

Solution

(i) $\left(2x - \frac{1}{3x}\right)^2 = (2x)^2 - 2(2x)\left(\frac{1}{3x}\right) + \frac{1}{(3x)^2}$

$= 4x^2 - \frac{4}{3} + \frac{1}{9x^2}$.

(ii) $(3x^2 + 5y)^2 = (3x^2)^2 + 2(3x^2)(5y) + (5y)^2$
$= 9x^4 + 30x^2y + 25y^2$

(iii) $\left(\sqrt{2}x - 3y\right)\left(\sqrt{2}x + 3y\right) = \left(\sqrt{2}x\right)^2 - (3y)^2 = 2x^2 - 9y^2$

(iv) $\left(\frac{1}{4}a - \frac{1}{2}b + 1\right)^2 = \left(\frac{1}{4}a\right)^2 + \left(-\frac{1}{2}b\right)^2 + (1)^2$

$+ 2\left(\frac{1}{4}a\right)\left(-\frac{1}{2}b\right) + 2\left(-\frac{1}{2}b\right)(1) + 2(1)\left(\frac{1}{4}a\right)$

$= \frac{1}{16}a^2 + \frac{1}{4}b^2 + 1 - \frac{ab}{4} - b + \frac{a}{2}$.

Example 30

Simplify:

(i) $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

(ii) $(2x + y)(2x - y)\,(4x^2 + y^2)$

Solution

(i) $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

$= \left(x^2 - \frac{1}{x^2}\right) = \left(x^2 + \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

$= \left[(x^2)^2 - \left(\frac{1}{x^2}\right)^2\right]\left(x^4 + \frac{1}{x^4}\right) = \left(x^4 - \frac{1}{x^4}\right)\left(x^4 + \frac{1}{x^4}\right)$

$= (x^4)^2 - \left(\frac{1}{x^4}\right)^2 = x^8 - \frac{1}{x^8}$

(ii) $(2x + y)(2x - y)(4x^2 + y^2) = [(2x)^2 - (y)^2](4x^2 + y^2)$
$= (4x^2 - y^2)(4x^2 + y^2) = (4x^2)^2 - (y^2)^2 = 16x^4 - y^4$.

Example 31

Find the value of $x - y$ when $x + y = 9$ and $xy = 14$:

Solution

$$x + y = 9$$

On squaring both sides $x^2 + y^2 + 2xy = 81$

Putting value of $xy = 14$

$$x^2 + y^2 + 28 = 81$$

$$x^2 + y^2 = 81 - 28 = 53 \qquad \text{.....(i)}$$

$$(x - y)^2 = x^2 + y^2 - 2xy$$

Putting $xy = 14$ and (i)

$$(x - y)^2 = 53 - 2\,(14) = 53 - 28$$

$$\Rightarrow (x - y)^2 = 25 \quad \Rightarrow x - y = \pm\sqrt{25} = \pm 5$$

Example 32

If $x^2 + \frac{1}{x^2} = 23$, find the values of $\left(x + \frac{1}{x}\right)$, $\left(x - \frac{1}{x}\right)$, and $\left(x^4 + \frac{1}{x^4}\right)$.

Solution

$$x^2 + \frac{1}{x^2} = 23 \qquad \text{.....(i)}$$

$\Rightarrow x^2 + + 2 = 25$ [Adding 2 on both sides of (i)]

$$\Rightarrow (x)^2 + \left(\frac{1}{x}\right)^2 + 2 \cdot x \cdot \frac{1}{x} = 25 \qquad \Rightarrow \left(x - \frac{1}{x}\right)^2 = (5)^2$$

$$\Rightarrow x + \frac{1}{x} = \pm 5$$

$$\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 1 \qquad \Rightarrow \left(x - \frac{1}{x}\right)^2 = 23 - 2 = 21$$

$$\Rightarrow \left(x - \frac{1}{x}\right) = \pm\sqrt{21}\,.$$

$$\left(x^2 + \frac{1}{x^2}\right)^2 = \left(x^4 + \frac{1}{x^4}\right) + 2$$

$$\Rightarrow \left(x^4 + \frac{1}{x^4}\right) = \left(x^2 + \frac{1}{x^2}\right)^2 - 2$$

$$\Rightarrow \left(x^4 + \frac{1}{x^4}\right) = (23)^2 - 2 = 529 - 2 \Rightarrow \left(x^4 + \frac{1}{x^4}\right) = 527$$

Example 33

Find the value of $\left[\frac{a^2 - 5ab}{a^2 - 6ab + 5b^2} \times \frac{a^2 - b^2}{a^2 + ab}\right]$.

Solution

$$\frac{a^2 - 5ab}{a^2 - 6ab + 5b^2} \times \frac{a^2 - b^2}{a^2 + ab} = \frac{a(a-5b)}{(a-b)(a-5b)} \times \frac{(a-b)(a+b)}{a(a+b)} = 1$$

Example 34

Find the value of $\frac{7.98 \times 7.98 - 2.02 \times 2.02}{5.96}$.

Solution

$$\frac{7.98 \times 7.98 - 2.02 \times 2.02}{5.96} = \frac{(7.98 + 2.02)(7.98 - 2.02)}{5.96}$$

$$= \frac{10 \times 5.96}{5.96} = 10.$$

Example 35

Simplify:

(i) $(3x + 4)^3 - (3x - 4)^3$

(ii) $\left(x + \frac{2}{x}\right)^3 + \left(x - \frac{2}{x}\right)^3$

Solution

(i) $(3x + 4)^3 - (3x - 4)^3 = [(3x)^3 + (4)^3 + 3\,(3x)\,(4)\,(3x + 4)] - [(3x)^3 - (4)^3 - 3\,(3x)\,(4)\,(3x - 4)]$
$= [27x^3 + 64 + 36x\,(3x + 4)] - [27x^3 - 64 - 36x\,(3x - 4)]$
$= [27x^3 + 64 + 108x^2 + 144x] - [27x^3 - 64 - 108x^2 + 144x]$
$= 27x^3 + 64 + 108x^2 + 144x - 27x^3 + 64 + 108x^2 - 144x = 128 + 216x^2.$

(ii) $\left(x + \frac{2}{x}\right)^3 + \left(x - \frac{2}{x}\right)^3 = x^3 + \left(\frac{2}{x}\right)^3 + 3(x)\left(\frac{2}{x}\right)\left(x + \frac{2}{x}\right) + x^3 - \left(\frac{2}{x}\right)^3 - 3(x)\left(\frac{2}{x}\right)\left(x - \frac{2}{x}\right)$

$$= x^3 + \frac{8}{x^3} + 6x + \frac{12}{x} + x^3 - \frac{8}{x^3} - 6x + \frac{12}{x} = 2x^3 + \frac{24}{x}.$$

Example 36

Evaluate:

(i) $(1005)^3$

(ii) $(997)^3$

Solution

(i) $(1005)^3 = (1000 + 5)^3 = (1000)^3 + (5)^3 + 3(1000)(5)\,(1000 + 5)$
$= 1000000000 + 125 + 15000\,(1000 + 5)$
$= 1000000000 + 125 + 15000000 + 75000$
$= 1015075125.$

(ii) $(997)^3 = (1000 - 3)^3 = (1000)^3 - (3)^3 - 3 \times 1000 \times 3 \times (1000 - 3)$
$= 1000000000 - 27 - 9000 \times (1000 - 3)$
$= 1000000000 - 27 - 9000000 + 27000$
$= 991026973.$

Example 37

If $x - \frac{1}{x} = 5$, find the value of $x^3 - \frac{1}{x^3}$.

Solution

We have, $x - \frac{1}{x} = 5$ ....(i)

$$\Rightarrow \left(x - \frac{1}{x}\right)^3 = (5)^3 \qquad \text{[Cubing both sides of (i)]}$$

$$\Rightarrow x^3 - \frac{1}{x} - 3x \cdot \frac{1}{x} \cdot \left(x - \frac{1}{x}\right) = 125$$

$$\Rightarrow x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 125$$

$$\Rightarrow x^3 - \frac{1}{x^3} - 3 \times 5 = 125 \text{ [Substituting } \left(x - \frac{1}{x}\right) = 5]$$

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$\Rightarrow x^3 - \frac{1}{x^3} - 15 = 125$

$\Rightarrow x^3 - \frac{1}{x^3} = (125 + 15) = 140.$

Example 38

Find the products of the following expression:

(i) $(4x + 3y)\ (16x^2 - 12xy + 9y^2)$

(ii) $(5x - 2y)\ (25x^2 + 10xy + 4y^2)$

Solution

(i) $(4x + 3y)\ (16x^2 - 12\ xy + 9y^2) = (4x + 3y)\ [(4x)^2 - (4x) \times (3y) + (3y)^2]$
$= (a + b)\ (a^2 - ab + b^2)$ [Where $a = 4x$, $b = 3y$]
$= a^3 + b^3 = (4x)^3 + (3y)^3 = 64x^3 + 27y^3.$

(ii) $(5x - 2y)\ (25x^2 + 10xy + 4y^2) = (5x - 2y)\ [(5x)^2 + (5x) \times (2y) + (2y)^2]$
$= (a - b)\ (a^2 + ab + b^2)$ [Where $a = 5x$, $b = 2y$]
$= a^3 - b^3$
$= (5x)^3 - (2y)^3 = 125x^3 - 8y^3.$

Example 39

If $a + b + c = 9$ and $ab + bc + ac = 26$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Solution

We have $a + b + c = 9$ ...(i)

$\Rightarrow (a + b + c)^2 = 81$ [On squaring both sides of (i)]

$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ac) = 81$

$\Rightarrow a^2 + b^2 + c^2 + 2 \times 26 = 81$ [$\because ab + bc + ac = 26$]

$\Rightarrow a^2 + b^2 + c^2 = (81 - 52)$ $\Rightarrow a^2 + b^2 + c^2 = 29.$

Now, we have

$a^3 + b^3 + c^3 - 3abc = (a + b + c)\ (a^2 + b^2 + c^2 - ab - bc - ac)$

$= (a + b + c)\ [(a^2 + b^2 + c^2) - (ab + bc + ac)] = 9 \times [(29 - 26)] = (9 \times 3) = 27.$

Example 40

If $(x + a)^2 + (y + b)^2 = 4\ (ax + by)$, where x, a, y, b are real, then find the value of $xy - ab$.

Solution

$$(x + a)^2 + (y + b)^2 = 4\ (ax + by)$$

$$\Rightarrow (x + a)^2 - 4ax + (y + b)^2 - 4by = 0$$

$$\Rightarrow (x - a)^2 + (y - b)^2 = 0$$

Since, sum of perfect squares can be zero only when individual perfect squares are zero.

Hence, $(x - a) = 0$, $(y - b) = 0$

$$\Rightarrow x = a,\ y = b \Rightarrow xy = ab \Rightarrow xy - ab = 0$$

Example 41

Simplify: $\dfrac{\left(a^2 - b^2\right)^3 + \left(b^2 - c^2\right)^3 + \left(c^2 - a^2\right)^3}{(a - b)^3 + (b - c)^3 + (c - a)^3}$

Solution

Here, $(a^2 - b^2) + (b^2 - c^2) + (c^2 - a^2) = 0$

$$\therefore (a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3(a^2 - b^2)\ (b^2 - c^2)\ (c^2 - a^2)$$

Also, $(a - b(+ (b - c) + (c - a) = 0$

$$\therefore (a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)\ (b - c)\ (c - a)$$

$\therefore$ Given expression,

$$= \frac{3\left(a^2 - b^2\right)\left(b^2 - c^2\right)\left(c^2 - a^2\right)}{3(a - b)(b - c)(c - a)}$$

$$= \frac{3(a - b)(a - b) + (b - c)(b - c) + (c - a)(c - a)}{3(a - b)(b - c)(c - a)}$$

$$= (a + b)\ (b + c)\ (c + a)$$

Example 42

Find the value of $(28)^3 - (78)^3 + (50)^3$.

Solution

Let $a = 28$, $b = -78$, $c = 50$

Then, $a + b + c = 28 - 78 + 50 = 0$

$$\therefore a^3 + b^3 + c^3 = 3abc.$$

So, $(28)^3 + (-78)^3 + (50)^3 = 3 \times 28 \times (-78) \times 50 = -327600.$

Example 43 Factorization by Taking out the Common Factor

Factorize: $6x^3 + 8x^2 - 10x$

Solution

$$6x^3 + 8x^2 - 10x = 2x\ (3x^2 + 4x - 5)$$

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Example 44 Factorization by Grouping

Factorize: $ax + by + ay + bx$

Solution

$$ax + by + ay + bx = ax + ay + bx + by = a(x + y) + b(x + y) = (x + y)(a + b)$$

Example 45 Factorization by Making a Perfect Square

Factorize: $4x^2 + 12x + 9$

Solution

$$4x^2 + 12x + 9 = (2x)^2 + 2(2x)(3) + 3^2 = (2x + 3)^2$$

Example 46 Factorization the Difference of Two Squares

Factorize: $4x^2 - 25$.

Solution

$$4x^2 - 25 = (2x)^2 - (5)^2 = (2x - 5)(2x + 5)$$

Example 47 Factorization of a Quadratic Polynomial by Splitting the Middle Term

Factorize: $x^2 + 4\sqrt{2}\,x + 6$

Solution

$$x^2 + 4\sqrt{2}\,x + 6 = x^2 + 3\sqrt{2}\,x + \sqrt{2}\,x + 6$$

$$= x(x + 3\sqrt{2}) + \sqrt{2}\,(x + 3\sqrt{2}) = (x + 3\sqrt{2})(x + \sqrt{2})$$

Example 48 Factorization of a Algebraic Expression as the Sum or Difference of Two Cubes

Factorize: $16a^3b - 250b^4$.

Solution

$$16a^3b - 250b^4 = 2b(8a^3 - 125b^3) = 2b\{(2a)^3 - (5b)^3\}$$
$$= 2b\{(2a - 5b)(4a^2 + 25b^2 + 10ab)\}$$

Example 49 Factorization of a Algebraic Expression of the Form $a^3 + b^3 + c^3 - 3abc$

Factorize:

(i) $2\sqrt{2}\,a^3 + 8b^3 - 27c^3 + 18\sqrt{2}\,abc$

(ii) $(x - y)^3 + (y - z)^3 + (z - x)^3$

Solution

(i) $2\sqrt{2}\,a^3 + 8b^3 - 27c^3 + 18\sqrt{2}\,abc = \left(\sqrt{2}a\right)^3 + (2b)^3 + (-3c)^3\ 3(\sqrt{2}\,a)(2b)(-3c) = (\sqrt{2}\,a + 2b - 3c)$

$[(\sqrt{2}\,a + (2b)^2 + (-3c)^2 - (\sqrt{2}\,a + (2b) - (-3c)(\sqrt{2}\,a)]$

$= (\sqrt{2}\,a + 2b - 3c)\,[2a^2 + 4b^2 + 9c^2 - 2\sqrt{2}\,ab + 6bc + 3\sqrt{2}\,ac]$

(ii) $(x - y)^3 + (y - z)^3 + (z - x)^3$

Let $a = x - y$, $b = y - z$ and $c = z - x$

So, $(x - y)^3 + (y - z)^3 + (z - x)^3 = a^3 + b^3 + c^3$

Now, $a + b + c = x - y + y - z + z - x = 0$

So, $a^3 + b^3 + c^3 = 3abc$

$\therefore (x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$

Example 50

If a,b,c,d are positive integers such that $a = bcd$, $b = cda$, $c = dab$ and $d = abc$, then find the value of $\dfrac{(a+b+c+d)^4}{(ab+bc+cd+da)^2}$.

Solution

$a = bcd$, $b = cda$, $c = dab$ and $d = abc$

$\Rightarrow abcd = (abcd)^3$ (on multiplication)

$\Rightarrow abcd = 1$ (since a,b,c,d are positive integers)

$\Rightarrow a = b = c = d = 1$

$$\Rightarrow \frac{(a+b+c+d)^4}{(ab+bc+cd+da)^2} = \frac{256}{16} = 16$$

Example 51

If $p(x) = (x + 2)(x^2 - 4x - 21)$, $Q(x) = (x - 7)(2x^2 + x - 6)$ find the HCF and LCM of $P(x)$ and $Q(x)$.

Solution

$$P(x) = (x + 2)(x^2 - 4x - 21) = (x + 2)(x^2 - 7x + 3x - 21)$$
$$= (x + 2)(x - 7)(x + 3)$$

$$Q(x) = (x - 7)(2x^2 + x - 6) = (x - 7)(2x^2 + 4x - 3x - 6)$$
$$= (x - 7)\,[2x(x + 2) - 3(x + 2)]$$
$$= (x - 7)(2x - 3)(x + 2)$$

$$\text{HCF} = (x + 2)(x - 7)$$

$$\text{LCM} = (x + 3)(x - 7)(2x - 3)(x + 2).$$

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Example 52

If HCF and LCM of $P(x)$ and $Q(x)$ are $(x + 2)$ and $(x + 3)$ $(x^2 + 9x + 14)$, respectively, if

$$P(x) = x^2 + 5x + 6, \text{ find } Q(x).$$

Solution

$$P(x) = (x^2 + 5x + 6) = (x + 2)\ (x + 3)$$

$$\text{LCM} = (x + 3)\ (x^2 + 9x + 14) = (x + 3)(x + 7)(x + 2)$$

We know that HCF × LCM = $P(x) \times Q(x)$

$$Q(x) = \frac{(x+2)(x+3)(x+7)(x+2)}{(x+2)(x+3)} = (x + 7)\ (x + 2)$$
$$= x^2 + 9x + 14.$$

Example 53

Factorize $8p^3 - 1 - m^3 - 6m$

Solution

$$8p^3 - 1 - m^3 - 6m$$

Here, a is $2p$, b is -1, and c is $-m$

$$= (2p)^3 + (-1)^3 + (-m)^3 - 3(2p)(-1)(-m)$$

$$= (2p - 1 - m)(4p^2 + 1 + m^2 - 2p - m + 2pm)$$

Example 54

Factorize $\sum x^2y^2(x - y)$

Solution

$$\sum x^2y^2(x - y) = x^2y^2(x - y) + y^2z^2(y - z) + z^2x^2(z - x)$$

$$= x^3y^2 - x^2y^3 + y^3z^2 - y^2z^3 + z^2x^2(z - x)$$

$$= y^2(x^3 - z^3) - y^3(x^2 - z^2) + z^2x^2(z - x)$$

$$= y^2(x - z)(x^2 + xz + y^2) - y^3(x - z)(x + z) - z^2x^2(x - z)$$

$$= (x - z)[y^2(x^2 + xz + z^2) - y^3(x + z) - z^2x^2]$$

$$= (x - z)[y^2x^2 - xy^3 + xy^2z - y^3z + y^2z^2 - z^2x^2]$$

$$= (x - z)[xy^2(x - y) + z^2(y - x)(y + x) + y^2z(x - y)]$$

$$= (x - z)(x - y)[xy^2 - z^2x + y^2z - yz^2]$$

$$= (x - z)(x - y)[x(y - z)(y + z) + yz(y - z)]$$

$$= (x - z)(x - y)(y - z)(xy + yz + zx)$$

$$= -(x - y)(y - z)(z - x)(xy + yz + zx)$$

Example 55

Factorize $P + 3abc$ if $P = \Sigma bc(b + c)$

Solution

$$P + 3abc$$

$$= bc(b + c) + ca(c + a) + ab(a + b) + 3abc$$

$$= bc(b + c) + abc + ca(c + a) + abc + ab(a + b) + abc$$

$$= bc(a + b + c) + ca(a + b + c) + ab(a + b + c)$$

$$= (a + b + c)(bc + ca + ab)$$

Example 56

Factorize $(a + b + c)^3 - a^3 - b^3 - c^3$

Solution

$$(a + b + c)^3 - a^3 - (b^3 + c^3)$$

$$= (a+b+c-a)[(a+b+c)^2 + (a+b+c)a + a^2]$$
$$- [(b+c)(b^2 - bc + c^2)]$$

$$= (b + c)(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca + a^2 + ab + ac + a^2$$
$$- b^2 + bc - c^2) = (b + c)(3a^2 + 3ab + 3bc + 3ca)$$

$$= 3(b + c)[a(a + b) + c(a + b)]$$

$$= 3(a + b)(b + c)(c + a)$$

Example 57

Factorize $a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2)$

Solution

When we put $b = c$ in the expression, the expression becomes zero. Therefore, $(b - c)$ is a factor. Similarly, $(a - b)$ and $(c - a)$ are also factors. As the expression is of fifth degree symmetric and homogeneous and the expression formed by $(a-b)(b-c)(c-a)$ is of third degree, there will be one more factor of second degree in a, b, and c,

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which will be symmetric, homogeneous, and of the form $[m(a^2+b^2+c^2)+n(ab+bc+ca)]$ .Therefore,

$$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)=(a-b)(b-c)(c-a)[m(a^2+b^2+c^2)+n(ab+bc+ca)]$$

Comparing the terms, say a^3b^2 on both the sides, we get $a^3b^2 = -a^2b \times nab = -na^3b^2$

$$\therefore n = -1$$

Again comparing the terms a^4b on both the sides, we get $0 = -a^2b \times ma^2$

$$\therefore m = 0$$

Hence, $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2) = -(a-b)(b-c)(c-a)(ab+bc+ca)$

Know Your Progress - 3

1. Factorize: $(x+1)(x+2)(x+3)(x+4)-3$
2. (i) Factorize: x^4+5x^2+9
   (ii) If $p-q=9$, prove that $p^3-q^3-27pq=729$.
   (iii) Factorize: $30p^2+85pq+35q^2$
   (iv) Find the quotient when $x^{54}-y^{54}$ is divided by $x-y$.
3. Evaluate:
   (i) $(5x+4y)^2$ (ii) $(4x-5y)^2$
   (iii) $\left(2x-\frac{1}{x}\right)^2$
4. If $a^4+\frac{1}{a^4}=119$, then find the value of $a^3-\frac{1}{a^3}$.
5. If $x+y=3$ and $xy=-18$, find the value of x^3+y^3.
6. Prove that : $a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)[(a-b)^2(b-c)^2(c-a)^2]$
7. If $a+b+c=15$, $a^2+b^2+c^2=83$, then find the value of $a^3+b^3+c^3-3abc$.
8. If $a+b+c=0$ where a,b,c are non-zero real numbers, that the value of $(a^2-bc^2)-(b^2-ca)(c^2-ab)$ is.
9. If $xy=6$ and $x^2y+xy^2+x+y=63$ then $x^2+y^2=$?
   (A) 69 (B) 45 (C) 63 (D) 66
10. Expand the following:
    (i) $\sum_{a,b,c} a^2b$ (ii) $\sum_{a,b,c} bc(b-c)$
    (iii) $\sum_{a,b,c} a(b-c)^2$
11. Simplify the following:
    (i) $\sum_{x,y,z}(x+y)$ (ii) $\sum_{a,b,c} b(c-a)$
    (iii) $\sum_{a,b,c}(a^n-b^n)$
12. Express the following using Σ notation:
    (i) $a^2bc+b^2ca+c^2ab$
    (ii) $(x-y)^2+(y-z)^2+(z-x)^2$
13. Factorize the following:
    (i) $\sum z(x^2-y^2)$
    (ii) $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$
    (iii) $\sum a(b-c)^3$
    (iv) $(a+b+c)^5-a^5-b^5-c^5$

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Linear Equations

A statement of equality of two algebraic expressions, which involve one or more unknown quantities is known as an equation. If there are two unknown quantities then equation is called linear equation in two variables.

A linear equation is an equation that involves linear polynomials.

A value of the variable that makes the two sides of the equation equal is called the solution of the equation.

Same quantity can be added/subtracted to/from both the sides of an equation without changing the equality.

Both the sides of an equation can be multiplied/divided by the same non-zero number without changing the equality.

Note

To find value of variables in any equation, we require number of equation equal to number of variables in equation.

General Form of Pair of Linear Equations

$$\left.\begin{aligned} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{aligned}\right\}$$

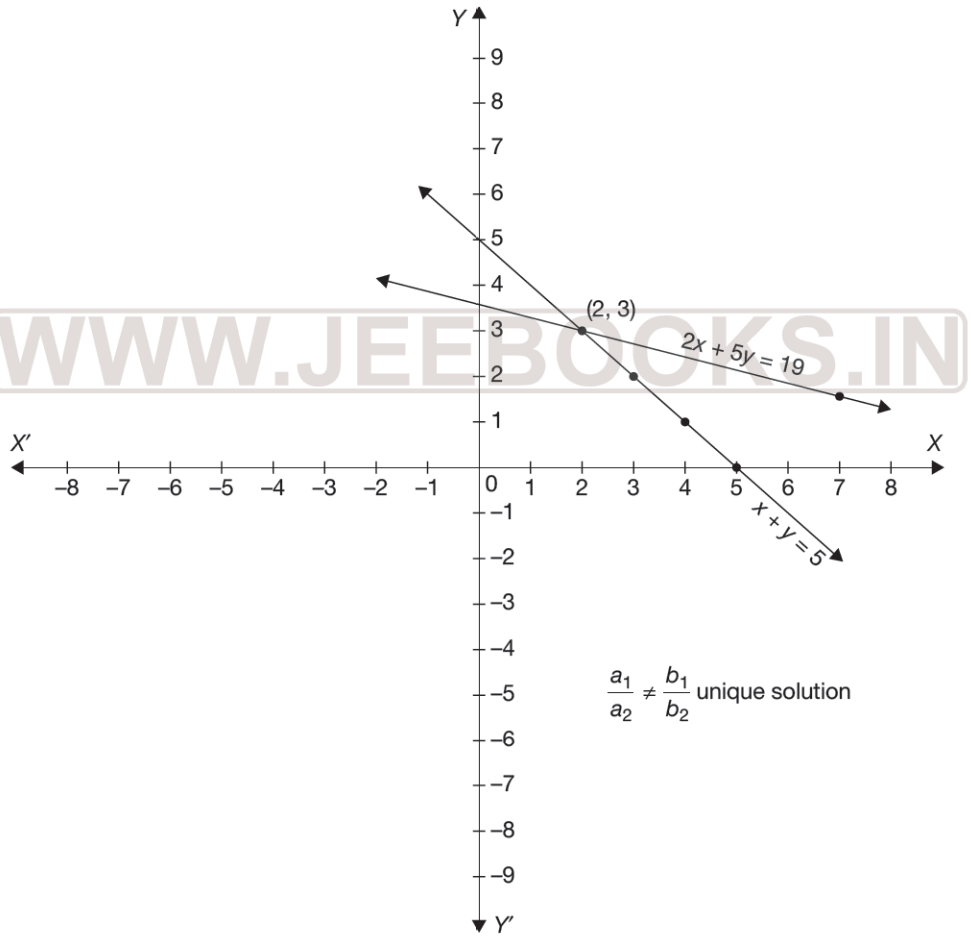
where a_1, b_1, c_1 and a_2, b_2, c_2 are constants.

Methods of solving linear equations and condition of solvability in two variables:

Graphical Method

Linear equations in two variables represent a line on a graph. Two equations represent two lines. We claim that the intersection point of these two lines give the solution that we want. For illustration, we take these two equations,

(a) $x + y = 5$ (b) $2x + 5y = 19$



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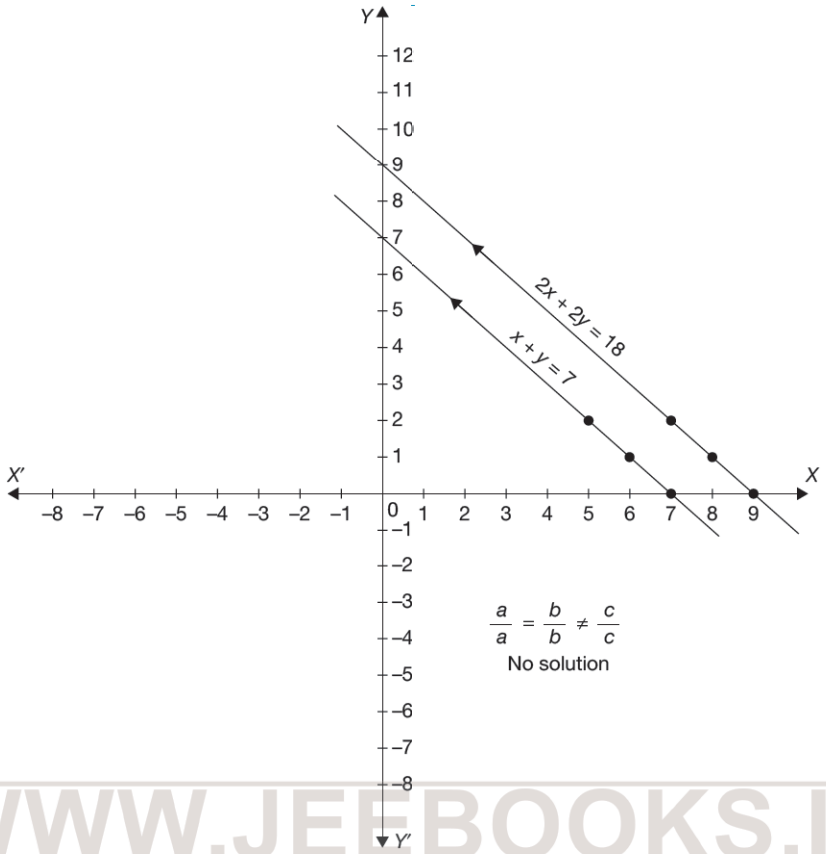
In the above plot, it can be seen clearly that line intersects at $(x, y) = (2, 3)$, which gives our required solution. This method does not include any computation. A simple plotting gives you the solution. The graph technique also gives physical significance to the solution of pair of linear equations. If you think carefully, it is not surprising at all. The solution satisfies both the equations and hence lies on both the lines. As two non-parallel and non-coincident lines meet at a single point in a coordinate plane, this single point should be the solution of this simultaneous set of equations.

Now we should consider other possibilities through graph as shown below.



$\frac{a}{a} = \frac{b}{b} \neq \frac{c}{c}$

No solution

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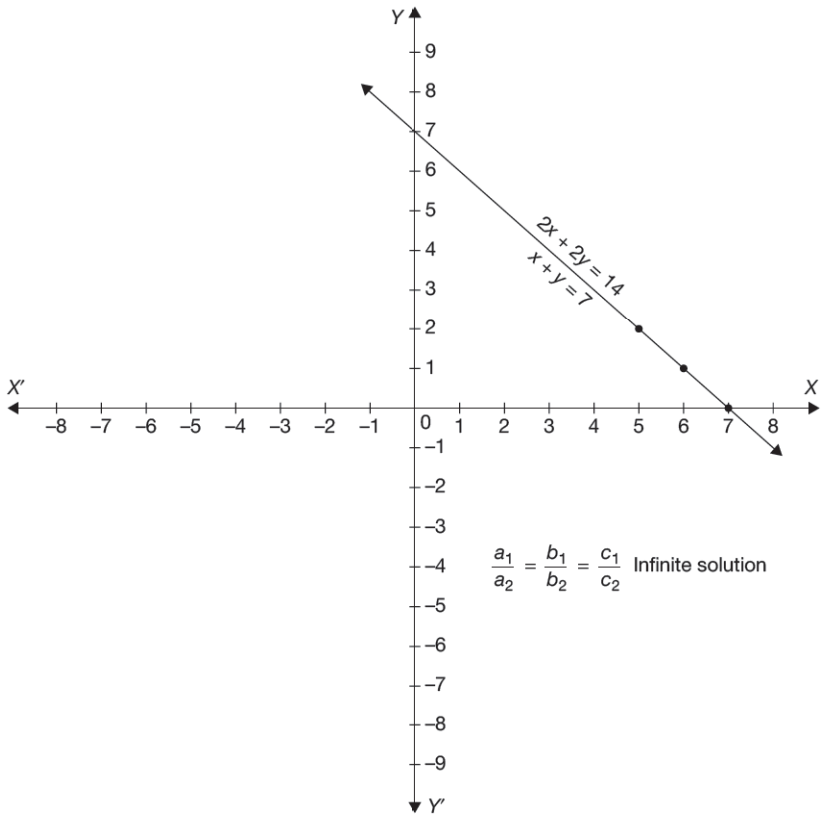

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ Infinite solution

After seeing the 3 possibilities of the graphical representation for the set of equations, we can write the working rule to find the solution using graphical method.

Following steps summarises this method as follows:

Step 1: Graph the first equation.
Step 2: Graph the second equation on the same coordinate system as the first.
Step 3: Find the solution.

- **(a)** If the two lines intersect at one place, then the point of intersection is the solution to the system.
- **(b)** If the two lines are parallel, then they never intersect, so there is no solution.
- **(c)** If the **two lines lie on top of each other (coincident lines)**, then they are the same line and you have an infinite number of solutions. In this case, you can write down either equation as the solution to indicate they are the same line.

Step 4: Check the proposed ordered pair solution in BOTH equations.

The observations about graphical representation for the set of equations can be summarised in the table below:

Graph No.	Pair of Lines	Possibility	Interpretation
1	$x+y=5; 2x+5y19$	Intersecting Lines	Exactly one solution
2	$x+y=7; 2x+2y=14$	Coincident Lines	Infinitely many solutions
3	$x+y=7; x+y=9$	Parallel Lines	No Solutions

You are encouraged to plot these lines to see for yourself. We will formalize this result in algebraic form in the next method of solving the simultaneous equations.

Elimination Method

We changed this word problem into two mathematical equations as follows,

$$x + y = 7 \quad (1)$$

$$2x + 5y = 23 \quad (2)$$

Multiplying equation (1) by 2 we get,

$$2x + y = 14 \quad (3)$$

Subtracting equation (3) from equation (2), we get

$$(2x - 2x) + (5y - 2y) = 23 - 14$$

Or, $$y = 3$$

Substituting $y = 3$ in equation (1), we get $x = 4$.

Thus, we eliminated x from equations by making its co-efficient equal in both equations and doing relevant mathematical operation (here we subtracted equation (3) from equation (1)). Now we formalize this method using general equations in two variables. These are,

$$a_1x + b_1y = c_1 \quad (1)$$

$$a_2x + b_2y = c_2 \quad (2)$$

Multiplying equation (1) with a_2 and equation (2) with a_1 we get,

$$a_2a_1x + a_2 b_1y = a_2c_1 \quad (3)$$

$$a_1a_2x + a_1 b_2y = a_1c_2 \quad (4)$$

Subtracting equation (4) from equation (3) we get,

$$(a_2a_1x - a_2a_1x) + (a_2b_1y - a_1b_2y) = a_2c_1 - a_1c_2$$

This gives,

$$y = (a_2c_1 - a_1c_2) / (a_2b_1 - a_1b_2)$$

Substituting value of y in equation (1) we get,

$$x = (b_2c_1 - b_1c_2) / (a_1b_2 - a_2b_1)$$

Observations based on these values of x and y :

- **(a)** Here if the denominator i.e., $a_1b_2 - a_2b_1 = 0$ and numerators are not equal to 0, then solution will not exist. This is what we saw in graphical method also as $a_1b_2 - a_2b_1 = 0$ is a necessary condition for equations (1) and (2) to represent pair of parallel lines that does not have any intersecting point and thus does not have any solution.
- **(b)** If $a_1b_2 - a_2b_1 \neq 0$, then there is no problem and we will have a unique solution.
- **(c)** However if $a_1b_2 - a_2b_1 = 0$ and numerators are also equal to 0, then the fractions can assume infinitely many values and a particular value cannot be ascertained. To understand this point more clearly, graphical approach is beneficial. This case represents pair of co-incident lines. This means when plotted on graph paper, these two equations will represent same lines. Thus, they will have infinitely many solutions.

  Table for above observations for system of two linear equations: For this purpose, our equations are as under,

$$a_1x + b_1y = c_1 \quad (1)$$

$$a_2x + b_2y = c_2 \quad (2)$$

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Condition	Graphical Interpretation	Outcomes	Example
$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$	Pair of parallel lines	No solution	$x+y=7;x+y=9$
$\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$	Intersecting lines	Unique solution	$x+y=5;2x+5y19$
$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$	Co-incident lines	Infinite solution	$x+y=7;2x+2y=14$

Cross Multiplication Method

Let us solve the following system of equations

$$a_1x+b_1y+c_1=0 \quad ....(1)$$

$$a_2x+b_2y+c_2=0 \quad ....(2)$$

Multiplying equation (1) by b_2 and equation (2) by b_1, we get

$$a_1b_2x+b_1b_2y+b_2c_1=0 \quad ....(3)$$

$$a_2b_1x+b_1b_2y+b_1c_2=0 \quad ....(4)$$

Subtracting equation (4) from equation (3), we get

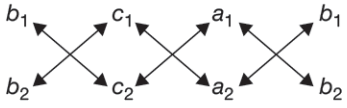
$$(a_1b_2-a_2b_1)\,x+(b_2c_1-b_1c_2)=0$$

$$\Rightarrow x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$$

$$\left[\begin{array}{c} a_1b_2-a_2b_1\neq 0 \\ \frac{a_1}{a_2}\neq\frac{b_1}{b_2} \end{array}\right]$$

Similarly, $y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

These values of x and y can also be written as



$$\frac{x}{b_1c_2-b_2c_1}=\frac{-y}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

Substitution Method

This is very similar method to elimination method and can be illustrated with the following example. Consider two equations as

$$x+y=7 \quad (1)$$

$$2x+5y=23 \quad (2)$$

By (1),

$$x=7-y \quad (3)$$

Substituting this value of x from equation (3) in equation (2)

$$2(7-y)+5y=23$$

Or, $3y=9$

Or, $y=3$

Substituting value of y in equation (1), we get $x=4$.

Thus, we saw that, in this method, we found value of one variable in terms of another variable using one of the equations and substitute this value in another equation. When such a thing is convenient, this method should be used otherwise substitution can be a tedious task as in the following example,

$$23x+41y=64 \quad (1)$$

$$37x+53y=90 \quad (2)$$

Conditions for Solvability and Finding Integral Solutions

Let the two equations be:

$$a_1x+b_1y+c_1=0 \quad ...(i)$$

$$a_2x+b_2y+c_2=0 \quad ...(ii)$$

Unique Solution

If the Denominator $a_1b_2-a_2\,b_1\neq 0$, then the given system of equations have unique solution (i.e., only one solution)

$$\therefore\ a_1b_2-a_2b_1\neq 0\Rightarrow\frac{a_1}{a_2}\neq\frac{b_1}{b_2}.$$

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For two lines: lines are said to be consistent (i.e., they meet at one point) when the given system of equation has unique solution.

No Solution

If the denominator $a_1b_2 - a_2b_1 = 0$, then the given system of equations have no solution.

i.e. $$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

For two lines: lines are said to be inconsistent (i.e., they does not meet) when the given the system of equation has no solution.

Many Solutions (Infinite Solutions)

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then system of equations has many solutions and lines are said to be consistent.

For two lines: two lines are coincident when they have many solutions.

Finding Integral Solution for Indeterminate Equations

If the number of unknown quantities is greater than the number of independent equations, there will be an unlimited number of solutions, and the equations are said to be indeterminate. If we confine our attention to positive integral values of the unknown quantities, we can express the solutions in a very simple form.

Example 58

Solve using substitution method: $x + 4y = 14$ and $7x - 3y = 5$.

Solution

$$x + 4y = 14$$

$$x = 14 - 4y \quad \text{....(i)}$$

$$7x - 3y = 5 \quad \text{....(ii)}$$

Substitute the value of x in equation (ii)

$$\Rightarrow 7(14 - 4y) - 3y = 5 \quad \Rightarrow 98 - 28y - 3y = 5$$

$$\Rightarrow 98 - 31y = 5 \quad \Rightarrow 93 = 31y \quad \Rightarrow y = \frac{93}{31} \Rightarrow y = 3.$$

Now substitute value of y in equation (ii)

$$\Rightarrow 7x - 3(3) = 5 \quad \Rightarrow 7x - 3(3) = 5$$

$$\Rightarrow 7x = 14 \quad \Rightarrow x = \frac{14}{7} = 2$$

So, solution is $x = 2$ and $y = 3$.

Example 59

Solve using elimination by equating the coefficients method: $9x - 4y = 8$ and $13x + 7y = 101$.

Solution

$$9x - 4y = 8 \quad \text{..... (i)}$$

$$13x + 7y = 101 \quad \text{.... (ii)}$$

Multiply equation (i) by 7 and equation (ii) by 4, we get

$$63x - 28y = 56$$

$$\text{Add } \underline{52x + 28y = 404}$$

$$115x \qquad = 460$$

$$\Rightarrow x = \frac{460}{115} \Rightarrow x = 4.$$

Substitute $x = 4$ in equation (i)

$$9(4) - 4y = 8 \quad \Rightarrow 36 - 8 = 4y \quad \Rightarrow 28 = 4y$$

$$\Rightarrow y = \frac{28}{4} = 7.$$

So, solution is $x = 4$ and $y = 7$.

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Example 60

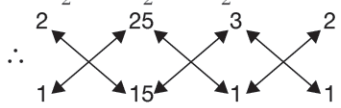
Solve using cross multiplication method: $3x + 2y + 25 = 0$ and $x + y + 15 = 0$.

Solution

Here, $a_1 = 3$, $b_1 = 2$, $c_1 = 25$

$a_2 = 1$, $b_2 = 1$, $c_2 = 15$

$\therefore$



$$\frac{x}{2\times15 - 25\times1} = \frac{y}{25\times1 - 15\times3} = \frac{1}{3\times1 - 2\times1}$$

$$\Rightarrow \frac{x}{30 - 25} = \frac{y}{25 - 45} = \frac{1}{3 - 2}$$

$$\Rightarrow \frac{x}{5} = \frac{y}{-20} = \frac{1}{1} \quad \text{.....(i)}$$

$$\Rightarrow \frac{x}{5} = 1, \frac{y}{-20} = \frac{1}{1} \Rightarrow x = 5, y = -20$$

So, solution is $x = 5$ and $y = -20$.

Example 61

Solve the following by reducing in system of linear equations in two variables:

$$\frac{6}{x+y}=\frac{7}{x-y}+3, \frac{1}{2(x+y)}=\frac{1}{3(x-y)}$$

where $x+y\neq 0$ and $x-y\neq 0$.

Solution

$$\frac{6}{x+y}=\frac{7}{x-y}+3 \qquad ...(i)$$

Put $x+y=a, \quad x-y=b$

Now, the given equation reduces to

$$\frac{6}{a}=\frac{7}{b}+3 \qquad ...(ii)$$

Another equation is $\frac{1}{2(x+y)}=\frac{1}{3(x-y)}$ ...(iii)

$$\frac{1}{2a}=\frac{1}{3b}$$

or, $$a=\frac{3b}{2} \qquad ...(iv)$$

Put (iv) in (ii)

$$\frac{6}{3b}\times 2=\frac{7+3b}{b}$$

$$4=7+3b$$

$$b=-1 \qquad ...(v)$$

Put (v) in (iv)

$$a=-\frac{3}{2}$$

From our assumptions,

$$a=x+y=-\frac{3}{2}$$

$$b=x-y=-1$$

Adding the two equations given above,

$$2x=-\frac{5}{2} \qquad \Rightarrow x=-\frac{5}{4}$$

$$\therefore\ y=-\frac{5}{4}+1 \qquad \Rightarrow y=-\frac{1}{4}.$$

Example 62

If $\frac{1}{2(x+2y)}+\frac{5}{3(3x-2y)}=-\frac{3}{2}, \frac{5}{4(x+2y)}-\frac{3}{5(3x-2y)}=\frac{61}{60}$,

where $x+2y\neq 0$ and $3x-2y\neq 0$. Then what will be the values of x and y?

Solution

Let $\frac{1}{x+2y}=U, \frac{1}{3x-2y}=V \quad \Rightarrow \quad \frac{U}{2}+\frac{5V}{3}=-\frac{3}{2}$

$$\Rightarrow 3U+10V=-9 \qquad ... (i)$$

$$\Rightarrow \frac{5U}{4}-\frac{3V}{5}=\frac{61}{60}$$

$$\Rightarrow 75U-36V=61 \qquad ... (ii)$$

Equation (i) is multiplied by 25

$$75U+250\ V=-225$$

$$75U-36\ V=61$$

$$\underline{\quad - \quad + \quad\quad - \quad}$$

Subtracting, $286V=286$

$$V=-1,\ U=\frac{1}{3}$$

$$\therefore\ \frac{1}{x+2y}=\frac{1}{3}, \frac{1}{3x-2y}=-1$$

$$x+2y=3, -3x+2y=1$$

$$x+2y=3$$

$$3x+2y=1$$

$$\underline{\quad + - - - \quad}$$

Subtracting, $4x=2$

So, $x=\frac{1}{2}, y=\frac{5}{4}$

Example 63

Solve: $3\sqrt{2}\ x-5\sqrt{3}\ y+\sqrt{5}=0$

$2\sqrt{3}\ x+7\sqrt{2}\ y-2\sqrt{5}=0.$

Solution

$3\sqrt{2}\ x-5\sqrt{3}\ y+\sqrt{5}=0$ ....(i)

$2\sqrt{3}\ x+7\sqrt{2}\ y-2\sqrt{5}=0.$ ....(ii)

Multiplying (i) by $2\sqrt{3}$ and (iii) by $3\sqrt{2}$

$$6\sqrt{6}\ x-10\times 3y=-2\sqrt{15}$$

$$6\sqrt{6}\ x+21\times 2y=6\sqrt{10}$$

$$\underline{\quad - \quad - \quad}$$

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Subtracting, $-72y = -(6\sqrt{10} + 2\sqrt{15})$

or, $72y = 6\sqrt{10} + 2\sqrt{15}$

$$y = \frac{6\sqrt{10} + 2\sqrt{15}}{72}$$

$$\therefore x = \frac{10\sqrt{15} - 7\sqrt{10}}{72}$$

Example 64

A two digit number is such that product of its digits is 18. When 63 is subtracted from the number, the digits interchange their place. Find the number.

Solution

$$xy = 18 \quad \ldots(i)$$

Let the given number be $10x + y$

As per the question,

$$(10x + y) - 63 = 10y + x \Rightarrow 10x - 10y - x + y = 63$$

$$\Rightarrow 9x - 9y = 63$$

$$\Rightarrow x - y = 7 \quad \ldots(ii)$$

Put $x = \frac{18}{y}$ in (ii)

$$\Rightarrow \frac{18}{y} - y = 7 \quad \Rightarrow 18 - y^2 = 7y \quad \Rightarrow y^2 + 7y - 18 = 0$$

$$\Rightarrow y^2 + 9y - 2y - 18 = 0 \quad \Rightarrow y(y + 9) - 2(y + 9) = 0$$

$$\Rightarrow (y + 9)(y - 2) = 0$$

$$\Rightarrow y = 2, -9$$

$y = -9$ is not valid

$$\therefore y = 2, x = 9.$$

So, the number $= 10x + y = 10(9) + 2 = 92$.

Example 65

The sum of two numbers is 2490. If 6.5 % of one number is equal to 8.5 % of the other, find the numbers.

Solution

Let, the numbers be x and y.

Then, $x + y = 2490 \quad \ldots(i)$

$$\Rightarrow \frac{x \times 6.5}{100} = \frac{8.5}{100} \times y \Rightarrow x = \frac{8.5}{100} y \Rightarrow x = \frac{17}{13} y \quad \ldots(ii)$$

Put (ii) in (i)

$$\frac{17}{13} y + y = 2490 \quad \Rightarrow 30y = 13 \times 2490$$

$$\Rightarrow y = \frac{13 \times 2490}{30} = 1079$$

$$\therefore x = 1411$$

So, the numbers are $x = 1411$ and $y = 1079$.

Example 66

A and B each has a certain number of mangoes. A says to B, 'if you give 30 of your mangoes I will have twice as many as left with you.' B replies 'if you give me 10, I will have thrice as left with you.'

Find how many mangoes does each have.

Solution

Say, A has x mangoes and B has y mangoes initially.

As per the statement of A to B,

$$x + 30 = 2(y - 30) \quad \text{or,} \quad x - 2y = -90 \quad \ldots(i)$$

and as per statement of B to A,

$$3(x - 10) = y + 10 \quad \text{or,} \quad 3x - y = 40 \quad \ldots(ii)$$

Now, we have

$$x - 2y = -90$$

$$3x - y = 40$$

$$3x - 6y = -270$$

$$3y - y = 40$$

$$- \quad + \quad -$$

$$-5y = -310$$

$$\Rightarrow y = 62$$

$\therefore x = 34$. So, A have 34 mangoes and B have 62 mangoes.

Example 67

Find the value of 'M' for which the given system of equation has only one solution.

(i.e., unique solution). $Mx - 2y = 9$ and $4x - y = 7$.

Solution

$$a_1 = M, b_1 = -2, c_1 = 9$$

$$a_2 = 4, b_2 = -1, c_2 = 7$$

Condition for unique solution is $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

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$$\Rightarrow \frac{M}{4} \neq \frac{+2}{+1} \Rightarrow M \neq \frac{8}{1} \Rightarrow M \neq 8$$

$\therefore$ M can have all real values except 8.

Example 68

What is the value of a, for which the system of linear equations $ax + 3y = a - 3$; $12x + ay = a$ has no solution.

Solution

Condition for no solution is $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

$$\frac{a}{12} = \frac{3}{a} \neq \frac{a-3}{a}$$

$$a^2 = 36 \text{ or, } a = \pm 6$$

$a = +6$ is not possible because it gives

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

or, $a = -6$ is the answer.

Example 69

Find the values of α and β for which the following system of linear equations has infinite number of solution: $2x + 3y = 7$ and $2ax + (\alpha + \beta)\, y = 28$.

Solution

For infinite solution:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{2}{2\alpha} = \frac{3}{\alpha+\beta} = \frac{7}{28}$$

$$\Rightarrow \frac{1}{\alpha} = \frac{3}{\alpha+\beta} = \frac{1}{4}$$

$\Rightarrow \alpha = 4$ [From the first and third term]

$$\Rightarrow \frac{3}{4+\beta} = \frac{1}{4} \Rightarrow 4 + \beta = 12$$

$$\Rightarrow \beta = 8.$$

Example 70

Solve $7x + 12y = 220$ in positive integers.

Solution

Divide throughout by the smaller coefficient 7, we get

$$x + y + \frac{5y}{7} = 31 + \frac{3}{7}$$

$$x + y + \frac{5y-3}{7} = 31$$

Since x and y are to be integers, we must have

$$\frac{5y-3}{7} = \text{integer}$$

Multiplying by 3 in order to make the coefficient of y differ by unity from a multiple of 7, we get

$$\Rightarrow \frac{15y-9}{7} = \text{integer}$$

$$\Rightarrow 2y - 1 + \frac{y-2}{7} = \text{integer}$$

$$\Rightarrow \frac{y-2}{7} = \text{integer} = p$$

$$\Rightarrow \left.\begin{matrix} y = 7p+2 \\ x = 28-12p \end{matrix}\right\} \text{This is called general solution}$$

Now, for $p > 2$, x is negative which gives y as negative. Thus,

$$p = 0, 1, 2$$

$$x = 28, 16, 4$$

$$y = 2, 9, 16$$

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Example 71

Find the number of positive integral solution of $14x - 11y = 29$.

Solution

Divide throughout by the smaller coefficient 11, we get

$$x + \frac{3x}{11} - y = 2 + \frac{7}{11}$$

$$\Rightarrow \frac{3x-7}{11} = 2 - x + y = \text{integer}$$

Multiplying by 4 in order to make the coefficient of y differ by unity from a multiple of 11,

$$\Rightarrow \frac{12x-28}{11} = \text{integer}$$

$$\Rightarrow x - 2 + \frac{x-6}{11} = \text{integer}$$

$$\Rightarrow \frac{x-6}{11} = \text{integer} = p$$

$$\Rightarrow \left.\begin{matrix} x = 11p+6 \\ y = 14p+5 \end{matrix}\right\} \text{This is called general solution}$$

Thus, we have

$$p = 0, 1, 2, 3, \ldots$$

$$x = 6, 17, 28, 39, \ldots$$

$$y = 5, 19, 33, 47, \ldots$$

Hence, the number of positive integral solutions is infinite.

Example 72

In how many ways can ₹5 be paid in denomination of 25 paise and 10 paise?

Solution

Let x be the number of 25 paise coins and y be the number of 10 paise coins.

$$\therefore 25x + 10y = 500$$

$$\Rightarrow x + \frac{2}{5}y = 20 \Rightarrow x = 20 - \frac{2}{5}y = \text{integer}$$

$$\therefore y = 0, 5, 10, 15, 20, \ldots, 45, 50$$

Thus, number of ways = 11

Example 73

The expenses of a party having 43 guests were ₹229; if each man paid ₹10, each woman paid ₹5, and each child ₹2, how many were there of each?

Solution

Let x, y, z denote the number of men, women, and children, respectively, thus

$$x + y + z = 43 \quad \ldots(i)$$

and $$10x + 5y + 2z = 229 \quad \ldots(ii)$$

Multiplying (i) by 2 and then subtracting it from (ii), we get

$$8x + 3y = 143$$

The general solution of this equation is $x = 3p + 1$, $y = 45 - 8p \Rightarrow z = 5p - 3$

$$\therefore p = 1, 2, 3, 4, 5$$

$$\Rightarrow x = 4, 7, 10, 13, 16$$

$$y = 37, 29, 21, 13, 5$$

$$z = 2, 7, 12, 17, 22$$

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Know Your Progress - 4

1. A person covers a distance of 500 km in his car, train, and bus. He goes 50 km in his car followed by 300 km in train and remaining by bus in total of 7 hours. For the same journey, he observed that if he covers 150 km by car, 200 km by train and remaining by bus, it takes 8 hours to complete the journey. Lastly, he observed that time taken to complete a particular distance in car and bus are equal. Find the speed of car, bus, and train?

2. Two pipes can fill a large tank in 10 hours. If one pipe is used for 5 hours and other for 20 hours then also they can fill the tank. How long would each pipe take to fill the tank alone?

3. Solve for x and y :
   (a) $px + qy = r$ and $qx - py = 1 + r$.
   (b) $(a - b)\, x + (a + b)\, y = (a^2 - 2ab + b^2)\ (a + b)$
   $(x + y) = a^2 + b^2$

4. Solve:
   (a) $37x + 43y = 123$
   $43x + 37y = 117$
   (b) $119x - 381y = 643$
   $381x - 119y = -143$

5. Solve for x and y : $\frac{x}{3} + y = 1.7$ and $\frac{11}{x + \frac{y}{3}} = 10 \ \forall \left[x + \frac{y}{3} \neq 0\right]$

6. If a certain number is divided by the sum of its two digits , the quotient is 6 and remainder is 3. If the digits are interchanged and the resulting number is divided by the sum of the digits, then the quotient is 4 and the remainder is 9. Find the number.

7. 2 men and 3 boys together can do a piece of work in 8 days. The same work is done in 6 days by 3 men and 2 boys together. How long would 1 boy alone or 1 man alone take to complete the work.

8. A man sold a chair and a table together for ₹1520 thereby making a profit of 25% on chair and 10% on table. By selling them together for ₹1535, he would

have made a profit of 10% on the chair and 25% on the table. Find cost price of each.

9. Abdul travelled 300 km by train and 200 km by taxi taking 5 hours 30 minutes. But, if he travels 260 km by train and 240 km by taxi, he takes 6 minutes longer. Find the speed of the train and that of the taxi.

10. A train covered a certain distance at a uniform speed. If the train would have been 6 km/h faster, it would have taken 4 hours less than the scheduled time. And, if the train were slower by 6 km/h, it would have taken 6 hours more than the scheduled time. Find the length of the journey.

11. Find the number of solutions of the following pair of linear equations :
$x + 2y - 8 = 0$
$2x + 4y = 16$.

12. Find the value of k for which the given system of equations has a unique solution.
$3x + 5y = 12$; $4x - 7y = k$.

13. Find the value of k for which the following system of linear equation becomes infinitely many solution or represent the coincident lines.
$6x + 3y = k - 3$; $2k\,x + 6y = k$.

14. Find the value of k or which the following systems of equations be Inconsistent.
$2x + ky + k + 2 = 0$; $kx + 8y + 3k = 0$.

15. For what value of a and b, the following system of equations have an infinite number of solutions.
$2x + 3y = 7$; $(a - b)\,x + (a + b)\,y = 3a + b - 2$.

16. A person spends ₹461 in buying horses and cows, If each horse costs ₹23 and each cow costs ₹16, how many of each does he buy?

17. Find the positive integral solutions of $3x + 8y = 103$.

18. Find the number of positive integral solutions of $13x + 11y = 414$.

19. Divide 81 into two parts so that one may be a multiple of 8 and the other of 5.

20. Divide 136 into two parts so that one of which when divided by 5 leaves remainder 2, and the other divided by 8 leaves remainder 3.

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Practical Mathematics

Profit and Loss

In our daily routine, we have to buy some articles from various shops. The shopkeepers purchase these articles either from wholesalers or directly from the manufacturers by paying a certain price. Generally, the shopkeeper sells his articles at a different price. These prices and difference in these prices are given special names such as cost price, selling price, profit, loss etc.

Cost Price

The price for which an article is purchased is called the cost price and abbreviated as C.P.

Selling Price

The price for which an article is sold is called the selling price and abbreviated as S.P.

Profit

If selling price is more than cost price, then the difference between selling price and the cost price is called the profit.

$\therefore$ Profit = Selling Price − Cost Price

Loss

If selling price is less than cost price, then the difference between the selling price and cost price is called loss.

$\therefore$ Loss = Cost Price − Selling Price

Overheads

Usually, a merchant has to spend some money on freight or transport, labour, or maintenance of the purchased articles. These extra expenditures are called overheads. The overheads are an essential part of cost price.

$\therefore$ Cost Price = (Payment made while purchasing the articles) + overhead charges

Some useful formulae regarding Discount, Marked Price, Selling Price, etc.

1. Net Selling Price = Marked Price − Discount
2. Discount = Marked Price − Net Selling Price
3. Marked Price = Net Selling Price + Discount
4. $\text{Discount } \% = \left(\frac{\text{Discount}}{\text{Marked Price}}\right) \times 100\%$
5. $\text{S.P.} = \text{M.P.}\left(\frac{100 - \text{Discount}\,\%}{100}\right)$
6. $\text{M.P.} = \frac{100 \times S.P.}{(100 - \text{Discount}\,\%)}$

Successive Discounts

When a series of discounts are given, we call them successive discounts .

Note

(i) When an article is sold after two successive discounts of $p\%$ and $q\%$, then the final selling price is given by,

$$\text{Selling price} = \frac{\text{M.P.}(100-p)(100-q)}{(100)(100)}$$

(ii) When an article is sold after successive discount of $p\%$, $q\%$, and $r\%$ then its final selling price is given by,

$$\text{Selling price} = \frac{\text{M.P.}(100-p)(100-q)(100-r)}{(100)(100)(100)}$$

Simple Interest

(i) Simple Interest

$$= \frac{\text{Principal} \times \text{Rate of interest} \times \text{Time}}{100}$$

(ii) Amount = Principal + Simple Interest.

Computation of Compound Interest by Using Formulae

Formula 1

Let P be the principal and the rate of interest be $R\%$ per annum. If the interest is compounded annually, then the amount A and the compound interest C.I. at the end of n years are given by

$$A = P\left(1 + \frac{R}{100}\right)^n$$

$$\text{C.I.} = A - P = P\left\{\left(1 + \frac{R}{100}\right)^n - 1\right\}$$

Formula 2

Let P be the principal and the rate of interest be $R\%$ per annum. If the interest is compounded annually, then the amount A and the compound interest C.I. at the end of n years are given by

$$A = P\left(1 + \frac{R}{100k}\right)^{nk}$$

and, $\text{C.I.} = A - P = P\left\{\left(1 + \frac{R}{100k}\right)^{nk} - 1\right\}$, respectively.

Here, interest is payable k times in a year.

Particular Cases

Case 1: When the interest is compound half-yearly or semi-annually.

In this case, $k = 2$

$$\therefore A = P\left(1 + \frac{R}{2 \times 100}\right)^{2n}$$

and $$\text{C.I.} = P\left[\left(1 + \frac{R}{2 \times 100}\right)^{2n} - 1\right]$$

Case 2: When interest is compounded quarterly.

In this case, $k = 4$

$$\therefore A = P\left(1 + \frac{R}{4 \times 100}\right)^{4n}$$

and $$\text{C.I.} = P\left[\left(1 + \frac{R}{4 \times 100}\right)^{4} - 1\right]$$

Case 3: When the rate of interest for successive years are different.

If the rate of interest of different for every years say r_1, r_2, r_3, for the first , second , third year, then the amount after 3 years is given by

$$A = P\left(1 + \frac{r_1}{100}\right)\left(1 + \frac{r_2}{100}\right)\left(1 + \frac{r_3}{100}\right)$$

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Work and Time

Work is defined as the amount of job assigned or the amount of job actually done.

Work is always considered as a whole or 1.

Units of work

Work is measured by many units i.e., men-days, men-hours, men-minutes, machine-hours, or in general person-time, machine-time.

If A and B can do a piece of work in x and y days, respectively, while working alone, then they will take $\left(\frac{xy}{x+y}\right)$ days to complete the work if both are working together.

Proof: A's one day work $= \frac{1}{x}$ and B's one day work $= \frac{1}{y}$

and $(A + B)$'s one day work $= \frac{1}{x}+\frac{1}{y}$

$$(A + B)\text{'s one day work} = \frac{x+y}{xy}$$

$\therefore$ Time taken by both A and B (working together) to complete the work $= \frac{xy}{x+y}$.

If A, B, C can do a piece of work in x, y, z days respectively while working alone, then they will together take $\frac{1}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$ days to complete the work.

Example 74

If 40% of the numbers exceeds the 25% of it by 54, find the number.

Solution

$$\frac{\text{Any value}}{\text{Its rate \% of number}} = \text{number (i.e., base number)}$$

Here, 54 stands for the difference of (40% and 25% of number)

$$\Rightarrow \frac{54}{(40-25)\%} = \text{number}$$

$$\Rightarrow \frac{54}{40-25}\times 100 = 300 \quad \text{...Ans.}$$

Example 75

$P_1\%$ of number N_1 is equal to $P_2\%$ of number N_2. Find what percent of N_1 is N_2?

Solution

$$\therefore \text{Required percentage} = \frac{N_2}{N_1}\times 100\%$$

It is given that $\frac{P_1}{100}\times N_1 = \frac{P_2}{100}\times N_2 \left(\therefore \frac{N_2}{N_1}\times\frac{P_1}{P_2}\right)$

Putting the value of $\frac{N_2}{N_1}$, we find the required percentage $= \left(\frac{P_1}{P_2}\times 100\right)\%$

$\therefore N_2$ is equal to $\left(\frac{P_1}{P_2}\times 100\right)\%$ of N_1 ...**Ans.**

Example 76

The ratio of salary of a worker in July to that in June was $2\frac{1}{2} : 2\frac{1}{4}$. By what % was the salary of July more than Salary of June? Also find by what % salary of June was less than of July.

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Solution

Let Salary of July $= \frac{5}{2}x$

and Salary of June $= \frac{9}{4}x$

Here the basis of comparison is either of salary of June or the Salary of July.

Salary of July more than that of June by percent. $= \frac{\text{Difference}}{\text{Salary of June}}\times 100$

$$= \frac{\left(\frac{5}{2}-\frac{9}{4}x\right)}{\frac{9}{4}x}\times 100 = 11\frac{1}{9}\%$$

Salary of June less than that of July by percent:

$$= \frac{\text{Difference}}{\text{Salary of June}}\times 100 = \frac{\left(\frac{5}{2}-\frac{9}{4}x\right)}{\frac{5}{2}x}\times 100 = 10\ \% \quad \text{...Ans.}$$

Example 77

The side of a square increases by $p\%$, then find by what % does its area increase?

Solution

Suppose, side of a square = b

Original area of the square = b^2, i.e., result = $A \times B$

Here, both sides are increased by $p\%$

Net % change in area = $x + y + \frac{xy}{100}$, where, $x = y = +p$

$\Rightarrow$ Net % change in area = $p + p + \frac{p^2}{100} = 2p + \frac{p^2}{100}$

Hence, area increase by $\left(2p + \frac{p^2}{100}\right)\%$

Note

This formula is also applicable when the radius of circle is increased by p% then its area increased by $\left(2p + \left(\frac{p^2}{100}\right)\right)\%$ ...**Ans.**

Example 78

The daily wage is increased by 15%, and a person now gets ₹23 per day. What was his daily wage before the increase?

Solution

$$\text{Original daily wage} = \frac{\text{Increased daily wage}}{100 + \%\text{ increase}} \times 100$$

$$= \frac{23}{115} \times 100 = 20$$

Note

In case of decrease, use (–)ve sign before % value.

Example 79

A student X passes his examination with 515 marks, having scored 3% above the minimum. If Y had obtained 710 marks, what % would he have been above the minimum?

Solution

$$\frac{\text{Marks of } Y}{\text{Marks of } X} = \frac{100 + \%\text{ above minimum of } Y}{100 + \%\text{ above minimum of } X}$$

$$\frac{710}{515} = \frac{100 + Y}{100 + 3} \Rightarrow Y = +42\%$$

Hence Y gets 42% above minimum.

Note

Similarly, if the % marks is below minimum, formula would have been.

$$\frac{\text{Marks of } Y}{\text{Marks of } X} = \frac{100 - \%\text{ above minimum of } Y}{100 - \%\text{ above minimum of } X}$$

Remember,

$$\frac{\text{Marks of } Y}{\text{Marks of } X} = \frac{100 \pm \%\text{ above/below minimum of } Y}{100 \pm \%\text{ above/below minimum of } X}$$

Example 80

The ratio of number of boys and girls in a school is 3 : 2, if 20% of the girls are holding scholarship, find the % of school students who

(a) hold scholarship
(b) do not hold scholarship

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Solution

Percentage of scholarship holders

= (Boys × % boys who are scholarship holders) + (Girls × % Girls who are scholarship holders)

$$\left(\frac{3}{2+3} \times 20\right) + \left(\frac{3}{3+2} \times 25\right) = 22\%$$

Similarly, percentage of non-scholarship holders

$$= \left(\frac{3}{2+3} \times 80\right) + \left(\frac{3}{2+3} \times 75\right) = 78$$

(Since 100 – 20 = 80, 100 – 25 = 75).

Example 81

A reduction of ₹2 per kg enables a man to purchase 4 kg more sugar for ₹16, Find the original price of Sugar.

Solution

Here expenditure is fixed (₹16), but as rate reduces (by ₹2/kg), so the quantity of sugar available increases (by 4 kg.)

Let original price be ₹x/kg.

$$\frac{\text{Expenditure}}{x} + \text{change in quantity available}$$

$$= \frac{\text{Expenditure}}{\text{New rate}}$$

$$\Rightarrow \frac{16}{4} + 4 = \frac{16}{x-2} \Rightarrow x^2 - 2x - 8 = 0$$

$$\Rightarrow (x-4)(x-2) = 0 \Rightarrow x = 4 \text{ or } -2$$

Considering the +ve value original price = ₹4 per kg.

Example 82

If 10% of an electricity bill is deducted. ₹45 is still to be paid. How much was the bill?

Solution

Here ₹45 refers to (100 − 10)% of the bill since 90% of bill = 45 $\Rightarrow$ 100% of bill = $\frac{45}{90} \times 100 = 50$

Hence the bill was 50.

Example 83

If weight of a sand bag is 40 kg. In a hurry, it was weighed as 40.8 kg. Find the error percentage.

Solution

$$\% \text{ Error} = \frac{\text{False weight} - \text{Actual weight}}{\text{Actual weight}} \times 100$$

$$= \frac{40.8 - 40}{40} \times 100 = 2\%$$

$\therefore$ The error is 2%

Example 84

The ratio of number of boys and girls in a school is 4 : 3. If there are 480 boys in the school, find the number of girls in the school.

Solution

$$4 : 3 = 480 : x$$

$$\Rightarrow x = \frac{3 \times 480}{4} = 360 \text{ girls.}$$

Example 85

The ratio between two numbers is 3 : 5. If each number is increased by 4, the ratio becomes 2 : 3. Find the number.

Solution

Let the numbers be $3x$ and $5x$,

$$\Rightarrow \frac{3x+4}{5x+4} = \frac{2}{3}$$

$$\Rightarrow x = 1$$

$\therefore$ Numbers are 12 and 20.

Example 86

The ratio of number of ladies to gents at a party was 1 : 2, but when 2 ladies and 2 gents left, the ratio becomes 1 : 3. How many people were originally present at the party?

Solution

Let the numbers of ladies = x, then

the number of gents = $2x$

As per question, $\frac{x-2}{2x-2} = \frac{1}{3} \Rightarrow x = 4$

$\therefore$ Total number of people originally present = $x + 2x = 3 \times 4 = 12$.

Example 87

A bag contains an equal number of one rupee, 50 paise, and 25 paise coins, respectively. If the total value is ₹35, how many coins of each type are there?

Solution

Here number of each type of coin is same. Hence, we may write,

$$\text{Number of each type of coin} = \frac{\text{Total amount}}{\text{Sum of value of each coin}}$$

$$\therefore \text{ Number of each type of coin} = \frac{35}{1 + 0.5 + 0.25}$$

= 20 coins of each type.

Example 88

The sum of the present age of A, B, and C is 90 years. Six years ago, their ages were in the ratio 1 : 2 : 3. What is the present age of C?

Solution

$(A + B + C)$'s age 6 years ago = $90 - 6 \times 3 = 72$

$\therefore$ C's age 6 years ago = $\frac{3}{1+2+3} \times 72 = 36$

C's present age = 36 + 6 = 42 years.

Example 89

The sum of the squares of three numbers is 532 and the ratio of the first to the second as also of the second to the third is 3: 2. What is the second number?

Solution

$$\frac{\text{First number}}{\text{Second number}} = \frac{3}{2} \times \frac{3}{3} = \frac{9}{6}$$

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[Make the second number same in both the ratio, i.e., 6]

and $\frac{\text{Second number}}{\text{Third number}} = \frac{3}{2}\times\frac{2}{2} = \frac{6}{4}$

$\therefore$ First : Second : Third = 9 : 6 : 4

$\therefore (9x)^2 + (6x)^2 + (4x)^2 = 532$ (given)

$133x^2 = 532 \Rightarrow x = 2$

Second number is $6x = 12$.

Example 90

A bag contains 1 rupee, fifty paise, twenty five paise, and ten paise coins in the proportion 1 : 3 : 5 : 7. If the total amount is ₹22.25, find the number of coins of each kind.

Solution

Let the number of coins be x, $3x$, $5x$, $7x$, respectively, as 1 rupee, fifty paise, twenty five paise, and ten paise. Since

Number of coin × Value of coin in rupee = Amount in rupees

Now, value of 50 paise coin in rupee = $\frac{1}{2}$ value of 25 paise coin in rupee = $\frac{1}{4}$ value of 10 paise coin in rupee = $\frac{1}{10}$

$$\therefore \left[(x\times 1)+\left(3x\times\frac{1}{2}\right)+\left(5x\times\frac{1}{4}\right)+\left(7x\times\frac{1}{10}\right)\right]$$

$$= 22.25 \Rightarrow \frac{89x}{20} = 22.25 \Rightarrow x = 5$$

$\therefore$ Number of rupee = 5 × 1 = 5

Number of 50 paise coin = $3x = 15$

Number of 25 paise coin = $5x = 25$

Number of 10 paise coin = $7x = 35$.

Example 91

A person distributes his pens among four friends A, B, C, and D in the ratio $\frac{1}{3}:\frac{1}{4}:\frac{1}{5}:\frac{1}{6}$. What is the minimum number of pens that the person should have?

Solution

Ratio of pens = $A : B : C : D = \frac{1}{3}:\frac{1}{4}:\frac{1}{5}:\frac{1}{6}$

$= 20 : 15 : 12 : 10$

Let number of pens given to A, B, C, D are $20x$, $15x$, $12x$, $10x$

So, minimum number of pens (for $x = 1$)

$= 20 + 15 + 12 + 10 = 57$

Example 92

A is twice as efficient as B, and finish the task 20 days earlier than B. Find number of days required to finish the task by A.

(A) 20 days **(B)** 40 days

(C) 10 days **(D)** 30 days

Solution

Let A can complete the work in x days, then B requires $2x$ days to finish the same work. According to given condition A finish the task 20 days earlier than B i.e., $2x - x = x = 20$ days.

So, A can finish the task in 20 days and B can finish the same task in 40 days. **Ans. (B)**

Example 93

A is three times as efficient as B and finish the task 32 days ahead of B. Find the number of days required to finish the task if both are working simultaneously.

(A) 11 days **(B)** 12 days

(C) 13 days **(D)** 14 days

Solution

Let A requires x days, then

B requires $3x$ days

$3x - x = 32$

$\Rightarrow 2x = 32$

$\Rightarrow x = 16$ days

So, $3x = 48$ days

$\therefore$ Together, they can finish

$\Rightarrow \frac{1}{16}+\frac{1}{48} = \frac{4}{48}$ part $= \frac{1}{12}$ part

Thus, 12 days are required by them to finish the task.

Ans. (B)

Example 94

Vinod can do 25% of a piece of work in 5 days. How many days will he take to complete the work ten times?

(A) 150 days **(B)** 250 days

(C) 200 days **(D)** 180 days

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Solution

25% part or $\frac{1}{4}$ part in 5 days

then 1 part in $\frac{5}{1/4} = 20$ days

$\therefore$ days required to complete ten time work

$= 20 \times 10 = 200$ days **Ans. (C)**

Example 95

6 men can do piece of work in 12 days. How many men are needed to do the work in 18 days?

(A) 3 men **(B)** 6 men

(C) 4 men **(D)** 2 men

Solution

Here Man × Day = $6 \times 12 = 72$

$m \times 18 = 72$, $m = \frac{72}{18} = 4$ **Ans. (C)**

Example 96

4 men and 3 women finish a job in 6 days, and 5 men and 7 women can do the same job in 4 days. How long will 1 man and 1 women take to do the work?

(A) $22\left(\frac{2}{7}\right)$ days **(B)** $25\left(\frac{1}{2}\right)$ days

(C) $5\left(\frac{1}{7}\right)$ **(D)** $12\left(\frac{7}{22}\right)$ days

Solution

Let man completes m part in a day and women completes w part in a day, then

$$4m + 3w = \frac{1}{6} \quad \ldots(i)$$

$$5m + 7w = \frac{1}{4} \quad \ldots(ii)$$

After simplifying, we get

$$20m + 15w = \frac{5}{6}$$

$$20m + 28w = 1$$

So $13w = \frac{1}{6}$

$$w = \frac{1}{78} \quad \ldots(iii)$$

From (i) and (iii) we get, $m = \frac{5}{156}$

$$m + w = \frac{5}{156} + \frac{1}{78} = \frac{7}{156}$$

So, it requires $\frac{156}{7}$ days $= 22\frac{2}{7}$ days **Ans. (A)**

Example 97

If 8 boys and 12 women can do a piece of work in 25 days, in how many days can the work be done by 6 boys and 11 women working together?

(A) 15 days **(B)** 10 days

(C) 12 days **(D)** cannot be determined

Solution

$$8B + 12w = \frac{1}{25}$$

now to calculate

$$6B + 11w = ?$$

Because we have 1 equation in two variables. **Ans. (D)**

Example 98

A can do a piece of work in 10 days and B can do the same work in 20 days. With the help of C, they finish the work in 5 days. How long will it take for C alone to finish the work?

(A) 20 days **(B)** 10 days

(C) 35 days **(D)** 15 days

Solution

Work done by C in 1 day $= \frac{1}{5} - \frac{1}{10} - \frac{1}{20} = \frac{1}{20}$

This is C's one day work so 20 days are required for C alone to finish task **Ans. (A)**

Example 99

A can do a piece of work in 20 days. He works at it for 5 days and then B finishes it in 10 more days. In how many days will A and B together finish the work?

(A) 8 days **(B)** 10 days

(C) 12 days **(D)** 6 days

Solution

A's one day work $= \frac{1}{20}$

In 5 days, A can complete $\frac{1}{20} \times 5 = \frac{1}{4}$ part

remaining work $= \frac{3}{4}$ part

$\therefore$ B finishes $\frac{3}{4}$ part in 10 days

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$\therefore$ B finish 1 part in $\frac{10}{3/4} = \frac{40}{3}$ days

Together they can finish it in

$$\frac{1}{\frac{1}{20}+\frac{3}{40}} = \frac{1}{\frac{1}{8}} = 8 \text{ days}$$ **Ans. (A)**

Example 100

Pravi can do a piece of work in 25 days and Rishi can do it in 20 days. They work for 5 days and then Pravi goes away. In how many more days will Rishi finish the work?

(A) 10 days **(B)** 12 days
(C) 14 days **(D)** None of these

Solution

Pravi's day work = $\frac{1}{25}$

Rishi day work = $\frac{1}{20}$

together they can finish $\frac{1}{25}+\frac{1}{20} = \frac{9}{100}$ part in a day

$\therefore$ Work in 5 days $\frac{9}{100}\times 5 = \frac{45}{100}$ part

$\frac{55}{100}$ part is remaining. Rishi can finish it in $\frac{\left(\frac{55}{100}\right)}{\left(\frac{1}{20}\right)} = 11$ days **Ans. (D)**

Example 101

Saroj takes twice as much time as Ajay and thrice as much as Vijay to finish a place of work. Together, they finish the work in 1 day. What is the time taken by Saroj to finish the work?

(A) 6 days **(B)** 3 days
(C) 2 days **(D)** None of these

Solution

$$M : A : V$$
$$= 2 : 1 : 2/3 \text{ or } 6 : 3 : 2$$

Let Saroj takes $6x$ days, Ajay takes $3x$ days and Vijay takes $2x$ days, then $\frac{1}{6x}+\frac{1}{3x}+\frac{1}{2x} = 1$

$$\frac{1}{x}\left[\frac{1}{6}+\frac{1}{3}+\frac{1}{2}\right] = 1$$

$$\frac{1}{x} = 1,$$

So $x = 1$

Saroj will take 6 days **Ans. (A)**

Example 102

In a fort, there was sufficient food for 200 soldiers for 31 day. After 27 days, 120 soldiers left the fort. For how many extra days will the rest of the food last for the remaining soldiers?

(A) 12 days **(B)** 10 days
(C) 8 days **(D)** 6 days

Solution

$$\text{Man} \times \text{Day} = \text{Man} \times \text{Day}$$

$$200 \times 31 = 6200$$

After 27 days

$$200 \times 27 = 5400 \text{ Man.}$$

Day is finished

$$\text{remaining} = 800$$

$$\text{Man} \times \text{Day} = 800$$

$$80 \times \text{Day} = 800$$

$$\text{Day} = 10$$ **Ans. (B)**

Example 103

Suppose there are a total of 2011 participants in a mathematics competition, and at least 1000 of them are female. Moreover, given any 1011 participants, at least 11 of them are male. If the number of male participants in this competition is x, then find the value of $\frac{x}{100}$ to the nearest integer?

Solution

It is given that there are at least 1000 female participants. Suppose there are more than 1000 female participants. Then we take a group of 1001 female participants, and add any 10 participants to this group of female participants. This will result in a group of 1011 participants with at most 10 male participants, which contradicts the assumption. Therefore, there are exactly 1000 female participants. Hence, the number of male participants is $2011 - 1000 = 1011$.

Therefore the final answer is 10.

Example 104

Let A be a 4-digit integer. When both the first digit (leftmost) and the third digit are increased by n, and the second

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digit and the fourth digit are decreased by n, the new number is n times A. Find the sum of digits of A.

Solution

Let the 4-digit number be $A = \overline{abcd}$. Then

$$1000(a + n) + 100(b - n) + 10(c + n) + (d - n) = nA.$$

It gives $A + 909n = nA$; or equivalently, $(n - 1)A = 909n$.

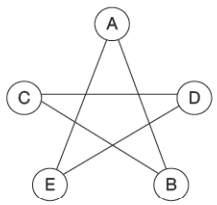
Note that $(n - 1)$ and n are relatively prime and 101 is a prime number. We must have $(n - 1)$ is a factor of 9. So $n = 2$ or $n = 4$.

If $n = 4$, then $A = 1212$, which is impossible since $b < m$. So $n = 2$ and $A = 909 \times 2 = 1818$.

Therefore, the final answer is 18.

Know Your Progress - 5

1. Varun and Mayank run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Varun has run 100 meters. They next meet after Mayank has run 150 meters past their first meeting point. Each boy runs at a constant speed. What is the length of the track in meters after dividing it by 10?
2. The two digits in Sunil's age are the same as the digits in Suresh's age, but in reverse order. In five years Sunil will be twice as old as Suresh will be then. What is the difference in their current ages?
3. Driving at a constant speed, Sunita usually takes 180 minutes to drive from her house to her mother's house. One day Sunita begins the drive at her usual speed, but after driving $\frac{1}{3}$ of the way, she hits a bad snowstorm and reduces her speed by 20 km per hour. This time the trip takes her a total of 276 minutes. Sunita is now H kms far from her mother's house. What is $\frac{H}{5}$?
4. At a Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non–swimmers play soccer?
5. Ram and Rahul are 20 kilometers a part. They bike toward one another with Ram travelling three times as fast as Rahul, and the distance between them decreasing at a rate of 1 kilometers per minute. After 5 minutes, Ram stops biking because of a flat tire and waits for Rahul. After how many minutes from the time they started to bike does Rahul reach Ram?
6. Yesterday Ram drove 1 hour longer than Mohan at an average speed 5 miles per hour faster than Mohan. Akash drove 2 hours longer than Mohan at an average speed 10 miles per hour faster than Mohan. Ram drove 70 miles more than Mohan. If Akash drove x km more than Mohan, then find $\frac{x}{10}$
7. The saxena family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?
8. A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each of the juniors receive on the test?
9. In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9 although not necessarily in this order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$ form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?



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10. A man bought two paintings and then sold them for 300 rupees each. He made a profit of 20% for the first painting, but a loss of 20% for the second painting. State the amount of profit or loss in rupees.

11. The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years, Jack will be twice as old as Bill will be then. What is the difference in their current ages?

12. Sunita and Babita start their new jobs on the same day. Sunita's schedule is 3 work–days followed by 1 rest–day. Babita's schedule is 7 work–days followed by 3 rest–days. On how many of their first 500 days do both have rest days on the same day?

13. Each valve A, B, and C, when open, releases water into a tank as it own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hour, and with only valves B and C open it takes 2 hours. The number of minutes required with only valves A and B open is

14. Ram, Mohan, and Hari were on a bike and were wondering how far away the nearest town was. When Ram said, "We are at least 6 km away," Mohan replied, "We are at most 5 km away." Hari then remarked, "Actually the nearest town is at most 4 km away." It turned out that none of the three statements were true. Let d be the distance in km to the nearest town. Find the sum of all possible values of d?

15. Mohit has a collection of 23 coins, consisting of 5–cent coins, 10–cent coins, and 25–cent coins. He has 3 more 10–cent coins than 5–cent coins, and the total value of his collection is 320 cents. How many more 25–cent coins does Mohit have than 5–cent coins?

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. A quadratic polynomial is exactly divisible by $(x + 1)$ and $(x + 2)$ and leaves the remainder 4 after division by $(x + 3)$ then that polynomial is:
(A) $x^2 + 6x + 4$ **(B)** $2x^2 + 6x + 4$
(C) $2x^2 + 6x - 4$ **(D)** $x^2 + 6x - 4$

2. What is the remainder when the polynomial $p(x) = x^{200} - 2x^{199} + x^{50} - 2x^{49} + x^2 + x + 1$ is divided by $(x - 1)(x - 2)$?
(A) 1 **(B)** 7
(C) $2x + 1$ **(D)** $6x - 5$

3. A cubic polynomial $p(x)$ is such that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, and $p(4) = 5$, then the value of $p(6)$ is:
(A) 16 **(B)** 13
(C) 10 **(D)** 7

4. If $(x + y + z) = 1$, $xy + yz + zx = -1$, $xyz = -1$, then value of $x^3 + y^3 + z^3$ is:
(A) -1 **(B)** 1
(C) 2 **(D)** -2

5. If $4x - 5z = 16$ and $xz = 12$, then $64x^3 - 125z^3 =$
(A) 14512 **(B)** 15676
(C) 25833 **(D)** 15616

6. $\dfrac{x^{-3} - y^{-3}}{x^{-3}y^{-1} + (xy)^{-2} + y^{-3}x^{-1}}$ is equal to:
(A) $x + y$ **(B)** $y - x$
(C) $\dfrac{1}{x} - \dfrac{1}{y}$ **(D)** $\dfrac{1}{x} + \dfrac{1}{y}$

7. If $\dfrac{\left(\sqrt{a} - \sqrt{b}\right)^2 + 4\sqrt{ab}}{a - b} = \dfrac{5}{3}$, then the value of $a : b$ is:
(A) 1 : 16 **(B)** 1 : 4
(C) 4 : 1 **(D)** 16 : 1

8. If $(a^2 + b^2)^3 = (a^3 + b^3)^2$ then $\dfrac{a}{b} + \dfrac{b}{a} =$
(A) $\dfrac{2}{3}$ **(B)** $\dfrac{3}{2}$
(C) $\dfrac{5}{6}$ **(D)** $\dfrac{6}{5}$

9. If $x = \sqrt{2 + \sqrt{2}}$, then $x^4 + \dfrac{4}{x^4}$ is:
(A) $2(3 - \sqrt{2})$ **(B)** $6\sqrt{2} - 2$
(C) $6 - \sqrt{2}$ **(D)** 12

10. If $f(x) = x^4 - 2x^3 + 3x^2 - ax + b$ is a polynomial such that when it is divided by $(x - 1)$ and $(x + 1)$, the remainders are 5 and 19 respectively. If $f(x)$ is divided by $(x - 2)$, then remainder is:
(A) 0 **(B)** 5
(C) 10 **(D)** 2

11. The cubic polynomial $P(x)$ satisfies the condition that $(x - 1)^2$ is a factor of $P(x) + 2$, and $(x + 1)^2$ is a factor of $P(x) - 2$. Then $P(3)$ equals.
(A) 27 **(B)** 18
(C) 12 **(D)** 6

12. If $(x - 1)^3 + (y - 2)^3 + (z - 3)^3 = 3(x - 1)(y - 2)(z - 3)$ and $x - 1 \neq y - 2 \neq z - 3$, then $x + y + z$ is equal to
(A) 2 **(B)** -6
(C) 6 **(D)** 4

13. If $2x^3 - 5x^2 + x + 2 = (x - 2)(ax^2 - bx - 1)$, then a and b are, respectively:
(A) 2, 1 **(B)** 2, -1
(C) 1, 2 **(D)** -1, 1/2

14. The number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$ is
(A) 0 **(B)** 1
(C) 2 **(D)** 3

15. If a, b, c are real, then $a(a - b) + b(b - c) + c(c - a) = 0$, only if
(A) $a + b + c = 0$
(B) $a = b = c$
(C) $a = b$ or $b = c$ or $c = a$
(D) $a - b - c = 0$

16. If $x^3 - 6x^2 + 11x - 6 = (x - a)(x - b)(x - c)$, then $2a + b + c$ is equal to, where $a < b < c$
(A) 6 **(B)** 8
(C) 7 **(D)** -7

17. If $\left(x + \dfrac{1}{x}\right) = 2$, then the value of $\left(x^6 + \dfrac{1}{x^6}\right)$ is:
(A) 2 **(B)** 4
(C) 8 **(D)** None of these

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18. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ where $(a + b + c) \neq 0$ and $abc \neq 0$. What is the value of $(a + b)\ (b + c)\ (c + a)$?
(A) -1 **(B)** 1
(C) 0 **(D)** 2

19. If $(x + y + z) = 6$ and $(xy + yz + zx) = 11$, then the value of $(x^3 + y^3 + z^3 - 3xyz)$ is:
(A) 81 **(B)** 54
(C) 18 **(D)** None of these

20. If $(a^{3/2} - ab^{1/2} + a^{1/2}\, b - b^{3/2})$ is divided by $(a^{1/2} - b^{1/2})$, then the quotient is:
(A) $a + b$ **(B)** $a - b$
(C) $a^{1/2} + b^{1/2}$ **(D)** $a^2 - b^2$

21. If $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{1}{2}$, find the value of $\frac{a^2+ab+b^2}{a^2-ab+b^2}$.
(A) $\frac{91}{73}$ **(B)** $\frac{73}{91}$
(C) $\frac{91}{55}$ **(D)** None of these

22. The ratio of the prices of two motorcycles was 4 : 5. Three years later, the price of the first had risen by 10% and that of the second by ₹6000, and the ratio of the prices now is 11 : 15. Find the original prices of the two vehicles.
(A) ₹24000, ₹30000
(B) ₹28000, ₹35000
(C) ₹20000, ₹25000
(D) None of these

23. If $\frac{2x+3y}{a-2b}=\frac{4y+7z}{3b-c}=\frac{6z+5x}{2c-3a}$, find the value of $11x + 17y + 20z$.
(A) 0
(B) 1
(C) 2
(D) None of these

24. If a, b, c, d are in continued proportion, then $(b - c)^2 + (c - a)^2 + (d - b)^2 =$?
(A) 0 **(B)** $(d - a)^2$
(C) $(a - b)^2$ **(D)** $a^2 + d^2$

25. If y varies as the sum of three quantities of which the first is constant, the second varies as x, and the third as x^2, and if $y = 0$ when $x = 1$, $y = 1$ when $x = 2$, and $y = 4$ when $x = 3$; find y when $x = 7$.
(A) 6 **(B)** 32
(C) 40 **(D)** 36

26. $\frac{x^2-5x+6}{x^2-9x+20}\div\frac{x^2-3x+2}{x^2-5x+4}$:
(A) $\left(\frac{x-4}{x-3}\right)$ **(B)** $\left(\frac{x-3}{x-5}\right)$
(C) $\left(\frac{x-2}{x-5}\right)$ **(D)** $\left(\frac{x-5}{x-3}\right)$

27. If $ax = (x + y + z)y$, $ay = (x + y + z)z$, $az = (x + y + z)x$, then:
(A) $3(x + y + z) = a$ **(B)** $2a = x + y + z$
(C) $x + y + z = 0$ **(D)** $x = y = z = \frac{a}{3}$

28. If a, b, c are such that $a + b + c = 2$, $a^2 + b^2 + c^2 = 6$, $a^3 + b^3 + c^3 = 8$, then $a^4 + b^4 + c^4$ is equal to:
(A) 10 **(B)** 12
(C) 18 **(D)** None of these

29. If $xy + yz + zx = 1$, then the expression $\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}$ is equal to
(A) $\frac{1}{x+y+z}$ **(B)** xyz
(C) $x + y + z$ **(D)** $\frac{1}{xyz}$

30. Factorize $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$ and find its value, when $b + c - a = 7$, $c + a - b = 10$, and $a + b - c = 3$.
(A) 3500
(B) 2100
(C) 4200
(D) Cannot be determined

31. How many four digit numbers leave a remainder 16 when divided by 39, and remainder 27 when divided by 56?
(A) 0 **(B)** 3
(C) 4 **(D)** 5

32. I buy 40 animals consisting of goats at ₹4, pigs at ₹2, and oxen at ₹17. If I spend ₹301, how many of each do I buy, respectively?
(A) 28, 1, 11 **(B)** 13, 14, 13
(C) 12, 15, 13 **(D)** Either (A) or (B)

PART – II KVPY Level Problems

1. If $(x^2 - x - 1)^n = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \ldots + a_1x^2 + a_1x + a_0$, find the value of $a_0 + a_2 + a_4 + \ldots + a_{2n}$.

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(A) $\dfrac{1+(-1)^n}{2} = \begin{cases} 1, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$

(B) $\dfrac{1+(-1)^n}{2} = 1$

(C) $\dfrac{1+(-1)^2}{2} = \begin{cases} 1, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$

(D) $\dfrac{1+(-1)^n}{2} = 0$

2. Given that $f(x)$ is a polynomial of degree 3, and its remainders are $2x - 5$ and $-3x + 4$ when divided by $x^2 - 1$ and $x^2 - 4$, respectively, find the $f(x)$.

(A) $f(x) = -\dfrac{5}{3}x^3 + 3x^2 + \dfrac{11}{3}x - 8$

(B) $f(x) = -\dfrac{5}{3}x^3$

(C) $3x^2 + \dfrac{11}{3}x - 8$

(D) $f(x) = -\dfrac{5}{3}x^3 + 3x^2$

3. Suppose $x - y = 1$. Find the value of $x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4$.

(A) 02 **(B)** 03
(C) 04 **(D)** 01

4. If $x^2 + x - 1 = 0$, find the value of $x^4 - 3x^2 + 3$.

(A) 01 **(B)** 03
(C) 02 **(D)** 04

5. If a, b, c, and d are real numbers such that

$$\frac{b+c+d}{a} = \frac{a+c+d}{b} = \frac{a+b+d}{c} = \frac{a+b+c}{d} = r,$$

find the sum of all the possible values of r.

(A) 03 **(B)** 02
(C) 01 **(D)** 02

6. Let the value(s) of x is such that $8xy - 12y + 2x - 3 = 0$ is true for all values of y. Then find $(16x)$.

(A) 21 **(B)** 22
(C) 23 **(D)** 24

7. If $x^2 + x + 1 = 0$, find the value of $x^{1999} + x^{2000}$.

(A) -2 **(B)** -3
(C) -1 **(D)** -4

8. Ram and Rahul are 20 kilometers a part. They bike toward one another with Ram travelling three times as fast as Rahul, and the distance between them decreasing at a rate of 1 kilometers per minute. After 5 minutes, Ram stops biking because of a flat tire and waits for Rahul. After how many minutes from the time they started to bike does Rahul reach Ram?

(A) 62 **(B)** 63
(C) 64 **(D)** 65

9. Ram, Mohan, and Hari were on a bike and were wondering how far away the nearest town was. When Ram said, 'We are at least 6 km away', Mohan replied, 'We are at most 5 km away'. Hari then remarked, 'Actually the nearest town is at most 4 km away'. It turned out that none of the three statements were true. Let d be the distance in km to the nearest town. Find the sum of all possible values of d?

(A) 9 **(B)** 8
(C) 11 **(D)** 12

10. A shopkeeper sold a TV set for ₹17,940 with a discount of 8% and earned a profit of 19.6%. What would have been the percentage of profit earned if no discount was offered?

(A) 24.8% **(B)** 25%
(C) 30% **(D)** None of these

11. Three containers have their volumes in the ratio 3 : 4 : 5. They are full of mixtures of milk and water. The mixtures contain milk and water in the ratio of (4 : 1), (3 : 1), and (5 : 2), respectively. The contents of all these three containers are poured into a fourth container. The ratio of milk and water in the fourth container is:

(A) 4 : 1 **(B)** 151 : 48
(C) 157 : 53 **(D)** 5 : 2

12. Let $Q = (x/y)$ where x and y are real numbers. If both x and y are increased equally, then:

(A) Q will increase
(B) Q will decrease
(C) Q will remain the same
(D) None of the above

13. If $a/b = c/d$, then:

(A) $(a + b)/a = (c + d)/c$
(B) $(a - b)/a = (c - d)/c$
(C) $(a^2 + b^2)/(a^2 - b^2) = (ac + bd)/(ac - bd)$
(D) all of these

14. 'A' varies directly as the sum of two quantities 'B' and 'C'. 'B' varies directly as 'x' and 'C' varies inversely as 'x'. When $x = 1$ or 2, $A = 3$. Find the value of A when $x = 4$.

(A) 5 **(B)** 4.5
(C) 6 **(D)** 5.5

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15. A and B can do a job in 12 days; B and C can do it in 16 days. After A has worked for 5 days and B has worked for 7 days, C can finish the rest in 13 days. In how many days can C do the work alone?

(A) 16 days **(B)** 24 days
(C) 36 days **(D)** 48 days

16. A is thrice as efficient as B, and B is twice as efficient as C. If A, B, and C work together, how long will they take to complete a job which B completes in 10 days?

(A) $\frac{20}{9}$ days **(B)** $\frac{11}{9}$ days
(C) 3 days **(D)** None of these

17. A thief escaped from police custody. Since he was sprinter he could clock 40 km/hr. The police realized it after 3 hr. and started chasing him in the same direction at 50 km/hr. The police had a dog which could run at 60 km/hr. The dog could run to the thief and then return to the police and then would turn back towards the thief. It kept on doing so till the police caught the thief. Find the total distance travelled by the dog in the direction of the thief.

(A) 720 km **(B)** 600 km
(C) 660 km **(D)** 360 km

18. If $4x - 17y = 1$ and $x, y \leq 500$. Find how many positive integer solutions are possible?

(A) 29 **(B)** 28
(C) 27 **(D)** Cannot be determined

19. Grass in lawn grows equally thick and in a uniform rate. It takes 24 days for 70 cows and 60 days for 30 cows to eat the whole of the grass. How many cows are needed to eat the grass in 96 days?

(A) 20 cows **(B)** 24 cows
(C) 28 cows **(D)** 32 cows

20. The marks obtained by a student of Class X in first and second unit test are 35 and 21, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 30 marks.

(A) $x \leq 34$ **(B)** $x \geq 34$
(C) $x > 34$ **(D)** $x < 34$

21. If a, b, and c are distinct real numbers such that $a : (b + c) = b : (c + a)$ then

(A) a, b, c are all positive
(B) a, b, c are all negative
(C) $a + b + c = 0$
(D) $ab + bc + ca + 1 = 0$

22. The wealth of a person A equals the sum of that of B and C. If he distributes half of his wealth between B and C in the ratio 2 : 1 then the wealth of B equals the sum of that sum of that of A and C. Then the fraction of the wealth that A should distribute between B and C in the ratio 1 : 2 so that the wealth of C equals the sum of that of A and B is

(A) 1/2 **(B)** 2/3
(C) 3/4 **(D)** 1

23. You have a measuring cup with capacity 25 ml and another with capacity 110 ml, the cups have no markings showing intermediate volumes. Using large container and as much tap water as you wish. What is the smallest amount of water you can measure accurately?

(A) 1 ml **(B)** 5 ml
(C) 10 ml **(D)** 25 ml

24. A person travels in a car from his village to a town at a speed of 50 km/hr for four hours. There he spends an hour moving about 20 kms. He then returns to his village at a speed 40 kms/hr. His average speed over the whole journey (in kms/hr) is

(A) 44 **(B)** 42
(C) $44\frac{4}{9}$ **(D)** $46\frac{2}{3}$

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25. A certain school has 300 students. Every student reads 5 newspapers and every newspaper is read by 60 students. Then the number of newspapers.

(A) is at least 30
(B) is at most 20
(C) is exactly 25
(D) cannot be determined by the data

26. We want to find a polynomial $f(x)$ of degreen n such that $f(1) = \sqrt{2}$ and $f(3) = \pi$. Which of the following is true?

(A) There does not exist such a polynomial
(B) There is exactly one such polynomial and it has degree 1
(C) There are infinitely many such polynomials for each $n \geq 1$
(D) There are infinitely many such polynomials for each $n \geq 2$ but not infinitely many for $n = 1$

27. A polynomial $p(x)$ when divided by $x^2 - 3x + 2$ leaves remainder $2x - 3$. Then:

(A) $p(x)$ must have a root between 0 and 3
(B) $p(x)$ cannot have a root between 0 and 3
(C) $p(x)$ must have a real root but may or may not be between 0 and 3
(D) $p(x)$ need not have a real root

28. A large tanker can be filled by two pipes A and B in 60 minutes and 40 minutes, respectively. How many minutes will it take to fill the empty tanker if only B is used in the first-half of the time and A and B are both used in the second-half of the time?

(A) 15 **(B)** 20
(C) 27.5 **(D)** 30

29. Five real numbers x_1, x_2, x_3, x_4, x_5 are such that:

$$\sqrt{x_1,1} + 2\sqrt{x_2 - 4} + 3\sqrt{x_3 - 9} + 4\sqrt{x_4 - 16} + 5\sqrt{x_5 - 25} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{2}$$

The value of $\frac{x_1 + x_2 + x_3 + x_4 + x_5}{2}$ is

(A) not uniquely determined
(B) 55
(C) 110
(D) 210

30. If $x = 2 + 2^{2/3} + 2^{1/3}$, then the value of $x^3 - 6x^2 + 6x$ is

(A) 3 **(B)** 2
(C) 1 **(D)** −2

31. $P(x)$ is a polynomial with integral coefficients such that for four distinct integers a, b, c, d : $Pa = P(a) = P(b) = P(c) = P(d) = 3$. If $P(e) = 5$ (e is an integer), then

(A) $e = 1$
(B) $e = 3$
(C) $e = 4$
(D) no real valueofe

32. The value of the expression $x^4 - 8x^3 + 18x^2 - 8x + 2$, when $x = 2 + \sqrt{3}$

(A) 2 **(B)** 1
(C) 0 **(D)** 3

PART – III PRMO Level Problems

1. It is given that $x = \frac{1}{2 - \sqrt{3}}$. Find the value of $x^6 - 2\sqrt{3}\,x^5 - x^4 + x^3 - 4x^2 + 2x - \sqrt{3}$.

2. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where a_1 are non-negative integers for $I = 0, 1, 2, \ldots, n$. If $f(1) = 21$ and $f(25) = 78357$, find the value of $\frac{f(10) + 3}{100}$.

3. Suppose that a, b, x, and y are real numbers such that $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$ and $ax^4 + by^4 = 42$. Find the value of $ax^5 + by^5$.

4. Let a, b, c, d, e be five numbers satisfying the following conditions: $a + b + c + d + e = 0$, and $abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde = 33$. Find the value of $a^3 + b^3 + c^3 + d^3 + e^3$.

5. For certain real numbers a, b, and c, the polynomial $g(x) = x^3 + ax^2 + x + 10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial $f(x) = x^4 + x^3 + bx^2 + 100x + c$. Find the value of $[F(1) + 7100]$.

6. How many ordered pairs of integers (x, y) satisfy the equation $x^2 + y^2 = 2(x + y) + xy$?

7. Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

8. If x and y can take only natural number values, find the number of (x, y) pairs satisfying the equation $2x + 5y = 100$.

9. If a, b, c are real numbers such that $a + \left(\frac{1}{b}\right) = \frac{7}{3}$; $b + \left(\frac{1}{c}\right) = 4$; $c + \left(\frac{1}{a}\right) = 1$, find abc.

10. If a, b, c, d satisfy the equations $a + 7b + 3c + 5d = 0$, $8a + 4b + 6c + 2d = -16$, $2a + 6b + 4c + 8d = 16$, $5a + 3b + 7c + d = -16$, then the value of $|(a + d)(b + c)| =$

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11. If positive numbers a, b, c satisfy $abc = 1$, solve the equation in x $\frac{2ax}{ab + a + 1} + \frac{2bx}{bc + b + 1} + \frac{2cx}{ca + c + 1} = 1$.

12. Solve the system of equations

$$\begin{cases} x - y + z = 1 \\ y - z + u = 2 \\ z - u + v = 3 \\ u - v + x = 4 \\ v - x + y = 5 \end{cases}$$

13. Suppose that x and y are non-zero real numbers such that $\frac{3x + y}{x - 3y} = -2$. What is the value of $\left(\frac{x + 3y}{3x - y}\right)^4$?

14. Let a, b and c be real numbers such that $a + b + c = 2$, and $a^2 - b^2 - c^2 = 12$. The difference between the maximum and minimum possible values of c is $\frac{p}{q}$ with coprime p and q. What is $(p + q^2)$?

15. The number of positive integral solutions (a, b, c, d) satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$ with the condition that $a < b < c < d$ is

16. Akash and Rahul run for 30 minutes on a circular track. Akash runs clockwise at 250 m/min and uses the inner lane with a radius of 50 meters. Rahul runs counterclockwise at 300 m/min and uses the outer lane with a radius of 60 meters, starting on the same radius line as Akash. How many times after the start do they pass each other?

17. On a 50 question multiple choice math contest, students receive 4 points for a correct answer, 0 points for an answer left blank, and −1 point for an incorrect answer. Jasvinder's total score on the contest was 99. What is the maximum number of questions that Jasvinder could have answered correctly?

18. Mohini rides on a flat road at 20 kilometers per hour (kph), downhill at 30 kph, and uphill at 5 kph. Honey rides on a flat road at 30 kph, downhill at 40 kph. Mohini goes from town A to town B, a distance of 10 km all uphill, then from town B to town C, a distance of 15 km all downhill, and then back to town A, a distance of 20 km on the flat. Honey goes the other way around using the same route. How many mare minutes does it take Mohini to complete the 45-km ride than it takes Honey?

19. Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

20. Let the sum of the coefficients of the polynomial is S. Find $\frac{S}{16}$.

$$(4x^2 - 4x + 3)^4 (4 + 3x - 3x^2)^2.$$

21. Suppose that $a + x^2 = 2006$, $b + x^2 = 2007$ and $c + x^2 = 2008$ and $abc = 3$. Find the value of $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}$

22. Find the value of $\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}$ when $x = \sqrt{19 - 8\sqrt{3}}$.

23. If $a^2x^3 + b^2y^3 + c^2z^3 = p^n$, $ax^2 = by^2 = cz^2$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{p}$, then $\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{p}$. Find value of n.

24. Let $p(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that $p(0) = 2007$, $p(1) = 2006$, $p(2) = 2005$, …, $p(2007) = 0$ determine $\frac{p(2008) + 1}{2008!}$.

25. If $p(x)$ denotes a polynomial of degree n, such that $p(k) = \frac{1}{k}$ for $k = 1, 2, 3, \ldots, n + 1$, determine $P(2019)$ for $n = 2017$.

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. If p, q, and r are positive real numbers, then the quantity $(p + r)/(q + r)$ is **[NSEJS-2008]**

(A) $> (p/q)$ if $p > q$ **(B)** $< (p/q)$ if $p > q$

(C) $> (p/q)$ if $p < q$ **(D)** $< (p/q)$ if $p < q$

2. The expression $(5a - 3b)^3 + (3b - 7c)^3 - (5a - 7c)^3$ is divisible by **[NSEJS-2008]**

(A) $(5a + 3b + 7c)$ **(B)** $(5a - 3b - 7c)$

(C) $(3b - 7c)$ **(D)** $(7c - 5a)$

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3. If $a^2 + 2b = 7$, $b^2 + 4c = -7$ and $c^2 + 6a = -14$, then the value of $(a^2 + b^2 + c^2)$ is: **[NSEJS-2009]**

(A) 14 **(B)** 25

(C) 36 **(D)** 47

4. In a rectangle $ABCD$ the lengths of sides AB, BC, CD, and DA are $(5x + 2y + 2)$ cm, $(x + y + 4)$ cm, $(2x + 5y - 7)$ cm, and $(3x + 2y - 11)$ cm, respectively. Which of the following statements is/are true? **[NSEJS-2009]**

(A) One of the sides of the rectangle is 15 cm long.

(B) Each diagonal of the rectangle is 39 cm long.

(C) Perimeter of the rectangle is 102 cm

(D) Area of the rectangle is 560 cm^2

5. If $4a - 18b + 13c = 0$ and $3a + 3b - 4c = 0$, then the ration $a : b : c$ will be: **[NSEJS-2010]**

(A) 3 : 6 : 5 **(B)** 5 : 3 : 6

(C) 3 : 5 : 6 **(D)** 5 : 6 : 3

6. If a, b, c are in continued proportion, the expression $\frac{a^2 + ab + b^2}{b^2 + bc + c^2}$ can be simplified to: **[NSEJS-2010]**

(A) $\frac{a + c}{a + b}$ **(B)** $\frac{a}{c}$

(C) $\frac{c}{a}$ **(D)** $\frac{a - c}{b}$

7. If $a^2 + b^2 + c^2 = 1$ and $p = ab + bc + ca$, then **[NSEJS-2010]**

(A) $\frac{1}{2} \le p \le 2$ **(B)** $-\frac{1}{2} \le p \le \frac{1}{2}$

(C) $-\frac{1}{2} \le p \le 1$ **(D)** $-1 \le p \le \frac{1}{2}$

8. If $x^2 - 5x + 1 = 0$, then $\frac{x^{10} + 1}{x^5}$ has the value **[NSEJS-2010]**

(A) 2524 **(B)** 2525
(C) 2424 **(D)** 2010

9. When the polynomial $(6x^4 + 8x^3 + 17x^2 + 21x + 7)$ is divided by $(3x^2 + 4x + 1)$, the remainder is $(ax - b)$ therefore, **[NSEJS-2011]**

(A) $a = 1, b = 2$ **(B)** $a = 1, b = -2$
(C) $a = 2, b = 1$ **(D)** $a = -1, b = -2$

10. If a, b, c are positive, $\frac{a+c}{b+c}$ is **[NSEJS-2011]**

(A) always smaller than $\frac{a}{b}$

(B) always greater than $\frac{a}{b}$

(C) greater than $\frac{a}{b}$ only if $a > b$

(D) greater than $\frac{a}{b}$ only if $a < b$

11. Find $x^2 + y^2 + z^2$ if $x^2 + xy + xz = 135$, $y^2 + yz + yx = 351$ and $z^2 + zx + zy = 243$. **[NSEJS-2012]**

(A) 225 **(B)** 250
(C) 275 **(D)** 300

12. Y varies inversely as x. If x is increased by 25%, then the value of percentage change to y is **[NSEJS-2012]**

(A) 80% **(B)** 75%
(C) 60% **(D)** 62.5%

13. If $\sqrt[x]{75} = \sqrt[y]{45} = \sqrt[z]{15} = 0$, then which of the statement is true: **[NSEJS-2012]**

(A) $x + y = 2z$ **(B)** $x + y = 3z$
(C) $x - y = 2z$ **(D)** $x - y = 3z$

14. The sixty sixth Independence Day was on Wednesday. After how many years the Independence Day will be on Wednesday? **[NSEJS-2012]**

(A) 6 years **(B)** 7 years
(C) 11 years **(D)** 28 years

15. If $xy^2 = a^2$, $yz^2 = b^3$ and $zx^2 = c^3$, then z^3 equals **[NSEJS-2013]**

(A) $\frac{bc^4}{a^2}$ **(B)** $\frac{b^4c}{a^2}$

(C) $\frac{b^2c^4}{b^2}$ **(D)** $\frac{ab^4}{c^2}$

16. At what time (to the nearest second) immediately after 4 O' clock will angle between the hands of the clock be the same as that at 4 O' clock? **[NSEJS-2013]**

(A) 4 h 42 m 50 s **(B)** 4 h 43 m 38 s
(C) 4 h 43 m 40 s **(D)** 5 h 5 m 27 s

17. Radha purchases x pencils at ₹x each, y pens at ₹y each, and z notebooks at ₹z each. She purchases altogether 50 items and pays ₹1000. The cost of y pencils, z pens and x notebooks is **[NSEJS-2013]**

(A) ₹600 **(B)** ₹750
(C) ₹500 **(D)** ₹350

18. A certain principal becomes ₹96800 in 2 years if compounded annually at a certain rate of interest. The same principal becomes ₹79240 in two years if compounded half yearly at the same rate of interest. The rate of interest is **[NSEJS-2013]**

(A) 8% **(B)** $8\frac{1}{3}$
(C) 10% **(D)** 12 ½%

19. A pen costs ₹13 and a notebook costs ₹35 Let m be the maximum number of items that can be bought for ₹1000 and n be the minimum number of items that can be bought for the same amount. Then $m + n$ is **[NSEJS-2013]**

(A) 76 **(B)** 88
(C) 96 **(D)** 98

20. If $x^3 = a + 1$ and $x + (b/x) = a$, then x equals **[NSEJS-2013]**

(A) $\frac{a(b+1)}{a^2 - b}$ **(B)** $\frac{ab + 1}{a^2 - b}$

(C) $\frac{ab + a + 1}{a^2 - b}$ **(D)** $\frac{ab - a - 1}{a^2 - b}$

21. Let a, b, c be positive real numbers such that $abc \neq 1$, $(ab)^2 = (bc)^4 = (ca)^x = abc$. Then x equals **[NSEJS-2013]**

(A) 1 **(B)** 2
(C) 3 **(D)** 4/5

22. A water filter advertisement claims to provide 8 liters of water per hour. How much time does it take to fill four bottles of 1.5 liters each? **[NSEJS-2014]**

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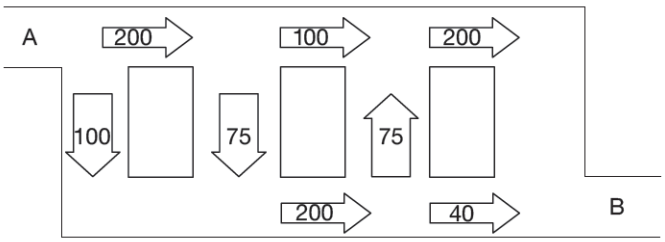
(A) 2 hr **(B)** 1 hr
(C) 30 min **(D)** 45 min

23. The diagram shows a road network. All vehicles drive in one direction from *A* to *B*. Numbers represent the maximum flow rate (capacity of roads) in vehicles per hour. The maximum number of vehicles that can drive through the network every hour is **[NSEJS-2014]**

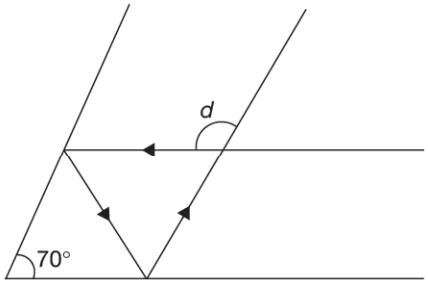


(A) 315 **(B)** 240
(C) 215 **(D)** 340

24. Scienctists in an *R* and *D* company made three design improvements on a car : the first saves 50% of fuel, the second saves 30% of fuel and the third saves 20%. If the company implements all three design changes at once, the new car will consume fuel that is _____% of the fuel consumption of normal car **[NSEJS-2014]**

(A) 50% **(B)** 100%
(C) 28% **(D)** 20%

25. A ray of light is incident on system of mirror as shown in the adjacent figure. What is the total deflection (d) of the ray when it emerges out after two reflections? **[NSEJS-2014]**



(A) 220° **(B)** 180°
(C) 120° **(D)** 140°

26. If $a : b = c : d$ then how many of the following statements are true? **[NSEJS-2015]**

(i) $c(a + b) = a(c + d)$
(ii) $d(a - b) = b(c - d)$
(iii) $(a^2 + b^2)(ac - bd) = (a^2 - b^2)(ac + bd)$
(iv) $\left(\frac{a^2}{b^2}\right) + \left(\frac{c^2}{d^2}\right) = \left(\frac{2ac}{bd}\right)$

(A) 1 **(B)** 2
(C) 3 **(D)** All

27. $(41)^{16} - (14)^{16}$ is a multiple of **[NSEJS-2015]**

(A) 1485 **(B)** 1584
(C) 1845 **(D)** 1854

28. The angle between the hour arm and the minute arm of a clock at 2:10 a.m. is: **[NSEJS-2016]**

(A) zero **(B)** 4°
(C) 5° **(D)** 6°

29. A train is running at a speed of 54 km/hr. It is not stopping at a certain station. It crosses the person showing green flag in 20 seconds and crosses the platform in 36 seconds. What is the length of the train? **[NSEJS-2017]**

(A) 240 m **(B)** 300 m
(C) 320 m **(D)** 360 m

30. If $(a + b + c + d) = 4$, then

$$\frac{1}{(1-a)(1-b)(1-c)} + \frac{1}{(1-b)(1-c)(1-d)} + \frac{1}{(1-c)(1-d)(1-a)} + \frac{1}{(1-d)(1-a)(1-b)} = _____.$$

[NSEJS-2017]

(A) 0 **(B)** 0.25
(C) 1 **(D)** 4

31. If $x^2 + xy + xz = 135$, $y^2 + yz + xy = 351$, and $z^2 + xz + yz = 243$, then $x^2 + y^2 + z^2 =$ _____. **[NSEJS-2017]**

(A) 225 **(B)** 250
(C) 275 **(D)** 300

32. On seventy first 'Independence Day', there was Tuesday. After how many years there will be Tuesday on 'Independence Day'? **[NSEJS-2017]**

(A) 4 yrs. **(B)** 5 yrs.
(C) 6 yrs. **(D)** 7 yrs.

33. If $\left(x + \frac{1}{x}\right) = 5$, then $\left(x^3 + \frac{1}{x^3}\right) - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) =$ _____. **[NSEJS-2017]**

(A) 0 **(B)** 5
(C) − 5 **(D)** 10

34. If $x^2 - 3x + 2$ is a factor of $x^4 - px^2 + q$, then p, q are: **[NSEJS-2017]**

(A) 2, 3 **(B)** 4, 5
(C) 5, 4 **(D)** 0, 0

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35. If a, b, c are distinct real number such that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}$ evaluate abc. **[NSEJS-2019]**

(A) $\pm\sqrt{2}$ **(B)** $\sqrt{2}-1$

(C) $\sqrt{3}$ **(D)** ± 1

36. Find the remainder when x^{51} is divided by x^2-3x+2 **[NSEJS-2019]**

(A) x **(B)** $(2^{51}-2)x+2-2^{51}$

(C) $(2^{51}-1)x+2-2^{51}$ **(D)** 0

PART – II Previous Years' KVPY Problems

1. Let $P(x)=1+x+x^2+x^3+x^4+x^5$. What is the remainder when $P(x^{12})$ is divided by $P(x)$? **[KVPY-2009]**

(A) 0 **(B)** 6

(C) $1+x$ **(D)** $1+x+x^2+x^3+x^4$

2. At what time between 10 O'clock and 11 O'clock are the two hands of a clock symmetric with respect to the vertical line (give the answer to the nearest second)? **[KVPY-2009]**

(A) 10 h 9 m 13 s **(B)** 10 h 9 m 14 s

(C) 10 h 9 m 22 s **(D)** 10 h 9 m 50 s

3. I carried 1000 kg of watermelon in summer by train. In the beginning the water content was 99%. By the time I reached the destination, the water content had dropped to 98%. The reduction in the weight of the watermelon was: **[KVPY-2010]**

(A) 10 kg **(B)** 50 kg

(C) 100 kg **(D)** 500 kg

4. If x, y are real numbers such that $3^{(x/y)+1}-3^{(x/y)-1}=24$, then the value of $(x+y)/(x-y)$ is: **[KVPY-2010]**

(A) 0 **(B)** 1

(C) 2 **(D)** 3

5. Three taps A, B, C fill up a tank independently in 10 h, 20 h, 30 h, respectively. Initially, the tank is empty and exactly one pair of taps is open during each hour and every pair of taps is open at least for 1 hour. What is the minimum number of hours required to fill the tank? **[KVPY-2011]**

(A) 8 **(B)** 9

(C) 10 **(D)** 11

6. Two friends A and B are 30 km apart and they start simultaneously on motorcycles to meet each other. The speed of A is 3 times that of B. The distance between them decreases at the rate of 2 km per minute. Ten minutes after they start, A's vehicle breaks down and A stops and waits for B to arrive. After how much time (in minutes) A started riding, does B meet A? **[KVPY-2011]**

(A) 15 **(B)** 20

(C) 25 **(D)** 30

7. There are 30 questions in a multiple-choice test. A student gets 1 mark for each unattempted question, 0 mark for each wrong answer, and 4 marks for each correct answer. A student answered x questions correctly and scored 60. Then, the number of possible value of x is **[KVPY-2011]**

(A) 15 **(B)** 10

(C) 6 **(D)** 5

8. A certain 12-hour digital clock displays the hour and the minute of a day. Due to a defect in the clock whenever the digit 1 is supposed to be displayed, it displays 7. What fraction of the day will the clock show the correct time? **[KVPY-2011]**

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$

(C) $\frac{3}{4}$ **(D)** $\frac{5}{6}$

9. A man wants to reach a certain destination. One-sixth of the total distance is muddy while half the distance is tar road. For the remaining distance, he takes a boat. His speed of travelling in mud, in water, on tar road is in the ratio 3 : 4 : 5. The ratio of the durations he requires to cross the patch of mud, stream, and tar road is: **[KVPY-2011]**

(A) $\frac{1}{2}:\frac{4}{3}:\frac{5}{2}$ **(B)** 3: 8: 15

(C) 10 : 15 : 18 **(D)** 1 : 2 : 3

10. The number of distinct prime divisors of the number $(512)^3-(253)^3-(259)^3$ is **[KVPY-2011]**

(A) 4 **(B)** 5

(C) 6 **(D)** 7

11. Below are four equations in x. Assume that $0<r<4$. Which of the following equations has the largest solution for x? **[KVPY-2011]**

(A) $5\left(1+\frac{r}{\pi}\right)^x=9$ **(B)** $5\left(1+\frac{r}{17}\right)^x=9$

(C) $5(1+2r)^x=9$ **(D)** $5\left(1+\frac{1}{r}\right)^x=9$

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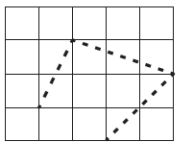
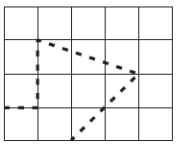
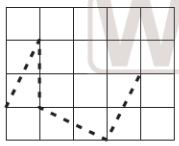
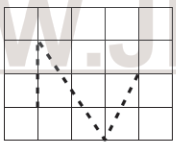
12. If $x + \frac{1}{x} = a$, $x^2 + \frac{1}{x^3} = b$, then $x^3 + \frac{1}{x^2}$ is **[KVPY-2011]**

(A) $a^3 + a^2 - 3a - 2 - b$
(B) $a^3 - a^2 - 3a + 4 - b$
(C) $a^3 - a^2 + 3a - 6 - b$
(D) $a^3 + a^2 + 3a - 16 - b$

13. Let a, b, x, y be real numbers such that $a^2 + b^2 = 81$, $x^2 + y^2 = 121$, and $ax + by = 99$. Then, the set of all possible values of $ay - bx$ is: **[KVPY-2011]**

(A) $\left(0, \frac{9}{11}\right]$ **(B)** $\left(0, \frac{9}{11}\right)$
(C) $\{0\}$ **(D)** $\left[\frac{9}{11}, \infty\right)$

14. In the given figures, which has the shortest path? **[KVPY-2012]**

(A)  **(B)** 
(C)  **(D)** 

15. A train leaves Pune at 7:30 am and reaches Mumbai at 11:30 am. Another train leaves Mumbai at 9:30 am and reaches Pune at 1:00 pm. Assuming that the two trains travel at constant speeds, at what time do the two trains cross each other? **[KVPY-2012]**

(A) 10:20 am **(B)** 11:30 am
(C) 10:26 am **(D)** data not sufficient

16. How many ordered pairs of (m, n) integers satisfy $\frac{m}{12} = \frac{12}{n}$? **[KVPY-2012]**

(A) 30 **(B)** 15
(C) 12 **(D)** 10

17. When a bucket is half full, the weight of the bucket and the water is 10 kg. When the bucket is two-third full, the total weight is 11 kg. What is the total weight (in kg), when the bucket is completely full? **[KVPY-2012]**

(A) 12 **(B)** $12\frac{1}{2}$
(C) $12\frac{2}{3}$ **(D)** 13

18. Two workers A and B are engaged to do a piece of work. Working alone, A takes 8 h more to completer the work than, if both worked together. On the other hand, working alone, B would need $4\frac{1}{2}$ h more to complete the work than if both worked together. How much time would they take to complete the job working together? **[KVPY-2012]**

(A) 4 h **(B)** 5 h
(C) 6 h **(D)** 7 h

19. Observe that, at any instant, the minute and hour hands of a clock make two angles between them whose sum is 360°. At 6 : 15 the difference between these two angles is **[KVPY-2012]**

(A) 165° **(B)** 170°
(C) 175° **(D)** 180°

20. If a, b, c, d are positive real numbers, such that $\frac{a}{3} = \frac{a+b}{4} = \frac{a+b+c}{5} = \frac{a+b+c+d}{6}$, then $\frac{a}{b+2c+3d}$ **[KVPY-2012]**

(A) $\frac{1}{2}$ **(B)** 1
(C) 2 **(D)** not determinable

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21. The square root of $\frac{(0.75)^3}{1-(0.75)} + [0.75 + (0.75)^2 + 1]$ is **[KVPY-2012]**

(A) 1 **(B)** 2
(C) 3 **(D)** 4

22. There are three kinds of liquids X, Y, Z. Three jars J_1, J_2, J_3 contain 100 ml of liquids X, Y, Z, respectively. By and operation we mean three steps in the following order: **[KVPY-2013]**
– stir the liquid in J_1 and transfer 10 ml from J_1 into J_2,
– stir the liquid in J_2 and transfer 10 ml form J_2 into J_3,
– stir the liquid in J_3 and transfer 10 ml form J_3 into J_1.

(A) $x > y > z$ **(B)** $x > z > y$
(C) $y > x > z$ **(D)** $z > x > y$

23. In a city, to total income of all people with salary below ₹1000 per annum is less than the total income of all people with salary above ₹10000 per annum. If the salaries of people in the first group increases by 5% and the salaries of people in the second group decreases by 5%, then the average income of all people **[KVPY-2013]**

(A) increases
(B) decreases
(C) remains the same
(D) cannot be determined form the data

24. A person X is running around a circular track completing one round every 40 s. Another person Y running in the opposite direction meets X every 15 s. The time, expressed in seconds, taken by Y to completer one round is **[KVPY-2013]**

(A) 12.5 **(B)** 24
(C) 25 **(D)** 55

25. Let x, y, z be non-zero real numbers such that $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=7$ and $\frac{y}{x}+\frac{z}{y}+\frac{x}{z}=9$, then $\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}-3$ is equal to: **[KVPY-2013]**

(A) 152 **(B)** 153
(C) 154 **(D)** 155

26. If a, b are natural numbers such that $2013 + a^2 + b^2$, then the minimum possible value of ab is: **[KVPY-2013]**

(A) 671 **(B)** 668
(C) 658 **(D)** 645

27. Let a, b, c be non-zero real numbers, such that $a + b + c = 0$, Let $q = a^2 + b^2 + c^2$ and $r = a^4 + b^4 + c^4$. Then **[KVPY-2014]**

(A) $q^2 < 2r$ always
(B) $q^2 = 2r$ always
(C) $q2 > 2r$ always
(D) $q^2 - 2r$ can take both positive and negative values

28. The population of cattle in a farm increases so that the difference between the population in year $n + 2$ and that in year n is proportional to the population in year $n + 1$. If the populations in years 2010, 2011 and 2013 were 39, 60 and 123, respectively, then the population in 2012 was: **[KVPY-2014]**

(A) 81 **(B)** 84
(C) 87 **(D)** 90

29. In a cinema hall, the charge per person is ₹200. On the first day, only 60% of the seats were filled. The owner decided to reduce the price by 20% and there was an increase of 50% in the number of spectators on the next day. The percentage increase in the revenue on the second day was: **[KVPY-2014]**

(A) 50 **(B)** 40
(C) 30 **(D)** 20

30. Let $p(x) = x^2 - 5x + a$ and $q(x) = x^2 - 3x + b$, where a and b are positive integers. Suppose HCF $(p(x), q(x)) = x - 1$ and $k(x) = 1$ cm $(p(x), q(x))$ If the coefficient of the highest degree term of $k(x)$ is 1, then sum of the roots of $(x - 1) + k(x)$ is **[KVPY-2014]**

(A) 4 **(B)** 5
(C) 6 **(D)** 7

31. Let x and y be two 2-digit numbers such that y is obtained by reversing the digits of x. Suppose they also satisfy $x^2 - y^2 = m^2$ for some positive integer m. The value of $x + y + m$ is **[KVPY-2014]**

(A) 88 **(B)** 112
(C) 144 **(D)** 154

32. A man standing on a railway platform noticed that a train took 21 s to cross to platform (this means the time elapsed from the moment the engine enters the platform till the last compartment leaves the platform) which is 88 m long, and that it look 9 s to pass him. Assuming that the train was moving with uniform speed, what is the length of the train in meters? **[KVPY-2015]**

(A) 55 **(B)** 60
(C) 66 **(D)** 72

33. Let x, y, z be positive real. Which of the following implies $x = y = z$? **[KVPY-2015]**

I. $x^3 + y^3 + z^3 = 3xyz$
II. $x^3 + y^2z + yz^2 = 3xyz$
III. $x^3 + y^2z + z^2x = 3xyz$
IV. $(x + y + z)^3 = 27\ xyz$

(A) I, IV only **(B)** I, II, and IV only
(C) I, II, and III only **(D)** All of them

34. There are two candles of same length and same size. Both of them burn at uniform rate. The first one burns in 5 hr and the second one burns in 3 hr. Both the candles are lit together. After how many minutes the length of the first candle is 3 times that of the other? **[KVPY-2016]**

(A) 90 **(B)** 120
(C) 135 **(D)** 150

35. Consider the equation $(1 + a + b)^2 = 3(1 + a^2 + b^2)$, where a, b are real numbers. Then, **[KVPY-2016]**

(A) there is no solution pair (a, b)
(B) there are infinitely many solution pairs (a, b)
(C) there are exactly two solution pairs (a, b)
(D) there is exactly one solution pair (a, b)

36. Let t be real number such that $t^2 = at + b$ for some positive integers a and b. Then, for any choice of positive integers a and b, t^3 is never equal to **[KVPY-2016]**

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(A) $4t + 3$ **(B)** $8t + 5$
(C) $10t + 3$ **(D)** $6t + 5$

37. Let $r(x)$ be the remainder when the polynomial $x^{135} + x^{125} - x^{115} + x^5 + 1$ is divided by $x^3 - x$. Then **[KVPY-2017]**

(A) $r(x)$ is the zero polynomial
(B) $r(x)$ is a non-zero constant
(C) degree of $r(x)$ is one
(D) degree of $r(x)$ is two

38. If $72^x \cdot 48^y = 6^{xy}$, where x and y are non-zero rational numbers, then $x + y$ equals **[KVPY-2017]**

(A) 3 **(B)** $\frac{10}{3}$
(C) -3 **(D)** $-\frac{10}{3}$

39. Let S be the set of all ordered pairs (x, y) of positive integers satisfying the condition $x^2 - y^2 = 12345678$. Then, **[KVPY-2017]**

(A) S is an infinite set
(B) S is the empty set
(C) S has exactly one element
(D) S is a finite set and has at least two elements

40. Let a_1, a_2, a_3, a_4 be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$ and $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Then the smallest possible value of the expression $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4) + (a_4 - a_1)^2$ lies in the interval: **[KVPY-2017]**

(A) (0, 1.5) **(B)** (1.5, 2.5)
(C) (2.5, 3) **(D)** (3, 3.5)

41. In a classroom, one-fifth of the boys leave the class and the ration of the remaining boys to girls is 2 : 3. If further 44 girls leave the class, then class the ratio of boys to girls is 5 : 2. How many more boys should leave the class so that the number of boys equals that of girls? **[KVPY-2017]**

(A) 16 **(B)** 24
(C) 30 **(D)** 36

42. There are exactly twelve Sunday in the period from January 1 to March 31 in a certain year. Then, the day corresponding to February 15 in that year is **[KVPY-2017]**

(A) Tuesday
(B) Wednesday
(C) Thursday
(D) Not possible to determine from the given data

43. If a, b, c are real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$, then $(3a + 5b - 8c)^2 + (-8a + 3b + 5c)^2 + (5a - 8b + 3c)^2$ is equal to **[KVPY-2017]**

(A) 49 **(B)** 98
(C) 147 **(D)** 294

44. Consider the following two statements **[KVPY-2018]**

i. Any pair of consistent liner equations in two variables must have a unique solution.
ii. There do not exist two consecutive integers, the sum of whose squares is 365.

(A) both I and II are true
(B) both I and II are false
(C) I is true and II is false
(D) I is false and II is true

45. The number of cubic polynomials $P(x)$ satisfying $P(1) = 2, P(2) = 4, P(3) = 6, P(4) = 8$ is **[KVPY-2019]**

(A) 0
(B) 1
(C) more than one but finitely many
(D) infinitely many

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PART – III Previous Years' PRMO Problems

1. For how may pairs of positive integers (x, y) is $x + 3y = 100$? **[PRE-RMO-2012]**

2. The letters R, M, and O represent whole numbers If $R \times M \times O = 240$, $R \times O + M = 46$ and $R + M \times O = 64$, what is the value of $R + M + O$? **[PRE-RMO-2012]**

3. Suppose that $4^{x_1} = 5, 5^{x_2} = 6, 6^{x_3} = 7, \ldots 126^{x_{123}} = 127$, $127^{x_{124}} = 128$. What is the value of the product $X1$, $X2$, …, $X124$? **[PRE-RMO-2012]**

4. If $a = b - c, b - c, b = c - d, c = d - a$ and $abcd \neq 0$ then what is the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$? **[PRE-RMO-2012]**

5. How many integer pairs (x, y) satisfy $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$? **[PRE-RMO-2012]**

6. Three points X, Y, Z are on a straight line such that $XY = 10$ and $XZ = 3$. What is the product of all possible values of YZ? **[PRE-RMO-2013]**

7. Let Akbar and Birbal together have n marbles, where $n > 0$.
Akbar says to Birbal, " If I give you some marbles then you will have thrice as many marbles as I will have".

What is the minimum possible value of n for which the above statements are true? **[PRE-RMO-2013]**

8. Carol was given three numbers and was asked to add the largest of three to the product of the other two. Instead, she multiplied the largest with the sum of the other two, but still got the right answer. What is the sum of the three numbers. **[PRE-RMO-2013]**

9. Three real number x, y, z are such that $x^2 6y = -17, y^2 + 4x = 1$, and $z^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$? **[PRE-RMO-2013]**

10. If real number $a, b, c, d\ e$ satisfy $a + 1 = b + 2 = c + 3 = d + 4 = e + = a + b + c + d + e + 3$, what is the value of $a^2 + b^2 + c^2 + d^2 + e^2$. **[PRE-RMO-2014]**

11. If $x^{(x^4)} = 4$, what is the value of $x^{(x^2)} + x^{(x^8)}$. **[PRE-RMO-2014]**

12. One morning, each member of Manjul's Family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup. but were never zero. Manjul drank 1/7th of total amount of milk and $2/17^{th}$ of the total amount of coffee. How many people are there in Manjul's family? **[PRE-RMO-2014]**

13. Let $x_1, x_2, \ldots, x_{2014}$ be real number different from 1, such that $x_1 + x_2 + \ldots + x_{2014} = 1$ and $\frac{x_1}{1+x_1} + \frac{x_2}{1-x_2} + \ldots + \frac{x_{2014}}{1-x_{2014}} = 1$. What is the value of: $\frac{x_1^2}{1-x_1} + \frac{x_2^2}{1-x_2} + \frac{x_3^2}{1-x_3} + \ldots + \frac{x_{2014}^2}{1-x_{2014}}$? **[PRE-RMO-2014]**

14. A man walks a certain distance and rides back in $3\frac{3}{4}$ hours; he could ride both ways in $2\frac{1}{2}$ hours. How many hours would it take him to walk both ways? **[PRE-RMO-2015]**

15. Positive integers a and b are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$? **[PRE-RMO-2015]**

16. Let $P(x)$ be a non-zero polynomial with integer coefficients. If $P(n)$ is divisible by n for each positive integer n, what is the value of $P(0)$? **[PRE-RMO-2015]**

17. Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. What is the value of $a^2 - b^2 + c^2$? **[PRE-RMO-2015]**

18. If $3^x + 2^y = 985$ and $3^x - 2^y = 473$, what is the value of xy? **[PRE-RMO-2015]**

19. Let a, b and c be such tat $a + b + c = 0$ and $P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$ is defined. What is the value of P? **[PRE-RMO-2015]**

20. A contractor has two team of worker: team A and team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joints team After four days. The team A withdraws after two more days. For how many more days should team B work to complete the job ? **[PRE-RMO-2017]**

21. A pen costs ₹11 and a notebook costs ₹13. Find the number of ways in which a person can spend exactly ₹1000 to buy pens and notebooks. **[PRE-RMO-2017]**

22. In a class, the total number of boys and girls are in the ratio 4 : 3. On one day it was found that 8 boys and 14 girls were absent from the class and that the number of boys was the square of the number of girls. What is the total number of student s in the class ? **[PRE-RMO-2017]**

23. If the real number x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and $xy + 4yx + 2zx = 24$, what is the value of $x^2 + y^2 + z^2$? **[PRE-RMO-2017]**

24. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n ? **[PRE-RMO-2018]**

25. If $a, b, c \geq 4$ are integers, not all equal, and $4abc = (a + 3)(b + 3)(c + 3)$, then what is the value of $a + b + c$? **[PRE-RMO-2018]**

26. Let $P(x) = a_0 + a_1 + a_2x^2 + \ldots + a_nx^n$ be a polynomial in which is non-negative integer for each $i \in \{0, 1, 2, 3, \ldots, n\}$. If $P(1) = 4$ and $P(5) = 136$, what is the value of $P(3)$? **[PRE-RMO-2018]**

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27. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as $a + \frac{a}{b}$, where a, b, c are positive integers, with $b < c$ and b/c in the reduced from. What is the value of $a + b + c$? **[PRE-RMO-2019]**

28. How many positive integers n are there such that $3 \leq n \leq 100$ and $x^{2^n} + x + 1$ is divisible by $x_2 + x + 1$? **[PRE-RMO-2019]**

29. Each of the numbers $x_1, x_2 \ldots\ldots x_{10}$ is ± 1. What is the smallest positive value of $\sum_{1 \leq i \leq j \leq 101} x_i x_j$? **[PRE-RMO-2019]**

30. Find the smallest positive integer $n \geq 10$ such that $n + 6$ is a prime and $9n + 7$ is a perfect square. **[PRE-RMO-2019]**

31. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two digit number. When I returned, it was y minutes past 6'O clock, and I noticed that

(i) I walked exactly for x minutes and

(ii) y was a 2 digit number obtained by reversing the digits of x. How many minutes did I walk? **[PRE-RMO-2019]**

32. Find the largest value of such that the positive integers a, $b > 1$ satisfy. $a^b b^a + a^b + b^a = 5329$. **[PRE-RMO-2019]**

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ANSWER KEYS

Know Your Progress - 1

1. 4 : 5 **2.** 21 : 22 **3.** ₹20000 **4.** 16 : 56 **5.** 25 **6.** ab **10.** $y = 2x - \frac{8}{x}$ **11.** 59520

Know Your Progress - 2

1. -35/8 **2.** $a = 1$ **5.** $-\frac{53}{8}$ **6.** $\frac{18}{127}$ **7.** $a = 0$ and $b = 2$ **8.** $k = -90$

Know Your Progress - 3

1. $(x^2 + 5x + 7)(x^2 + 5x + 3)$

2. (iv) $(x^{53} + x^{52}y \ldots + xy^{52} + y^{53})$ (i) $(x^2 - x + 3)(x^2 + x + 3)$ (iii) $5(2p + q)(3p + 7q)$

3. (i) $25x^2 + 16y^2 + 40xy$ (ii) $16x^2 + 25y^2 - 40xy$ (iii) $4x^2 + \frac{1}{x^2} - 4$

4. ±36 **5.** 189 **6.** 180

10. (i) $a^2b + b^2c + c^2a$ (ii) $bc(b-c) + ca(c-a) + ab(a-b)$ (iii) $a(b-c)^2 + b(c-a)^2 + c(a-b)^2$

11. (i) $2x + 2y + 2z$ (ii) 0 (iii) 0

12. (i) $\sum_{a,b,c} a^2bc$ (ii) $\sum_{x,y,z} (x-y)^2$

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13. (i) $(x-y)(y-z)(z-x)$ (ii) $(a+b)(b+c)(c+a)$

(iii) $(a-b)(b-c)(c-a)(a+b+c)$ (iv) $5(a + b)(b + c)(c + a)(a^2 + b^2 + c^2 + ab + bc + ca)$

Know Your Progress - 4

1. 100, 50 (car and bus) **2.** 30 hrs, 15 hrs **4.** (a) $x = 1, y = 2$ (b) $x = -1, -y = -2$ **5.** $x = 0.6, y = 1.5$
6. 75 **7.** 20, 120 **8.** Table = ₹700, Chair = ₹600 **9.** 300 km/hr, 200 km/hr **10.** 720 km
11. Infinite solution **12.** $k \in R$ **13.** $k = 6$ **14.** $k = -4$ **15.** $a = 5; b = 1$
16. Number of cows = 13 and Number of horses = 11 **17.** $x = 29, 21, 13, 5$ and $y = 2, 5, 8, 11$
18. 3 **19.** 56, 25 or 16, 65 **20.** 37, 99 or 77, 59 or 117, 19

Know Your Progress - 5

1. 35 **2.** 18 **3.** 27 **4.** 51 **5.** 65 **6.** 15 **7.** 06 **8.** 93 **9.** 12 **10.** 25
11. 18 **12.** 50 **13.** 72 **14.** 11 **15.** 02

Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. B **2.** D **3.** A **4.** B **5.** D **6.** B **7.** D **8.** A **9.** D **10.** C
11. B **12.** C **13.** A **14.** A **15.** B **16.** C **17.** A **18.** C **19.** C **20.** A
21. A **22.** A **23.** A **24.** A **25.** D **26.** B **27.** D **28.** C **29.** D **30.** C
31. D **32.** D

PART – II KVPY Level Problems

1. A	2. A	3. D	4. C	5. D	6. D	7. C	8. D	9. C	10. C
11. C	12. D	13. D	14. B	15. B	16. A	17. C	18. A	19. A	20. B
21. C	22. D	23. B	24. B	25. C	26. D	27. A	28. D	29. B	30. B
31. D	32. B								

PART – III PRMO Level Problems

1. 2	2. 51	3. 20	4. 99	5. 93	6. 6	7. 1	8. 9	9. 1	10. 16
11. 1/2	12. –1	13. 16	14. 25	15. 2	16. 47	17. 29	18. 64	19. 1	20. 81
21. 1	22. 5	23. 5	24. 1	25. 0					

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. A, D	2. C, D	3. A	4. A, B, C	5. C	6. B	7. C	8. B	9. B	10. D
11. C	12. A	13. B	14. A	15. B	16. B	17. B	18. C	19. D	20. C
21. D	22. D	23. C	24. C	25. D	26. D	27. A	28. C	29. B	30. A
31. C	32. C	33. A	34. C	35. D	36. C				

PART – II Previous Years' KVPY Problems

1. B	2. B	3. D	4. D	5. A	6. D	7. C	8. A	9. C	10. C
11. B	12. A	13. C	14. D	15. C	16. A	17. D	18. C	19. A	20. A
21. B	22. B	23. B	24. B	25. C	26. C	27. B	28. B	29. D	30. D
31. D	32. C	33. B	34. D	35. D	36. B	37. C	38. D	39. B	40. B
41. B	42. C	43. C	44. B	45. A					

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PART – III Previous Years' PRMO Problems

1. 33	2. 20	3. 3.5	4. 0.5	5. 6	6. 91	7. 12	8. 3	9. 4	10. 10
11. 258	12. 8	13. 0	14. 2	15. 2	16. 0	17. 1	18. 48	19. 1	20. 16
21. 7	22. 42	23. 21	24. 17	25. 16	26. 34	27. 51	28. 49	29. 10	30. 53
31. 42	32. 81								

Chapter

2 Quadratic Equations

Learning Outcomes

After reading this chapter, you will be able to understand

- Linear Inequations
- Range of Quadratic Expressions
- Method of Solving Linear Equations
- Condition of Common Root
- Nature of Roots
- Theory of Equation

Inequations and Modulus

A statement involving variable(s) and the sign of inequality viz, $>$, $<$, $\geq$, or $\leq$ is called an inequation. An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

(a) $3x - 2 < 0$ (Linear)

(b) $2x^2 + 3x + 4 > 0$ (Quadratic)

(c) $2x + 5y \geq 4$ (Linear in two variable)

Properties of Inequalities

Let x, y and a be real number

(a) Addition property: If $x < y$, then $x + a < y + a$.

(b) Subtraction property: If $x < y$, then $x - a < y - a$.

(c) Multiplication property: If $x < y$ and $a > 0$, then $xa < ya$. If $x < y$ and $a < 0$, then $xa > ya$.

(d) Division property: If $x < y$ and $a > 0$, then $\frac{x}{a} < \frac{y}{a}$. If $x < y$ and $a < 0$, then $\frac{x}{a} > \frac{y}{a}$.

(e) Transitive property: If $x < y$ and $y < a$, then $x < a$.

(f) Comparison property: If $x = y + a$ and $a > 0$, then $x > y$.

Types of Intervals

(a) Closed interval: Let a and b be two given real numbers such that $a < b$. Then, the set of all real numbers x such that $a \leq x \leq b$ is called closed interval and is denoted by $[a, b]$ may be graphed as:

(b) Open interval: If a and b are two real numbers such that $a < b$, then the set of all real numbers x satisfying $a < x < b$ is called an open interval and is denoted by (a, b) or $]a, b[$ and may be graphed as:

(c) Semi-closed or semi-open interval: If a and b are two real numbers such that $a < b$, then the sets $(a, b] = \{x \in R: a < x \leq b\}$ and $[a, b) = \{x \in R : a \leq x < b\}$ are known as semi-open or semi-closed intervals.

Linear Inequations in One Variable

Let a be a non-zero real number and x be a variable. Then inequations of the form $ax + b < 0$, $ax + b \leq 0$, $ax + b > 0$, and $ax + b \geq 0$ are known as linear inequations in one variable x.

For example: $9x - 15 > 0$, $5x - 4 \geq 0$, $3x + 2 < 0$, and $2x - 3 \leq 0$ are linear inequations in one variable.

Solving Linear Inequations in One Variable

Rule 1: Same number may be added to (or subtracted from) both side of an inequation without changing the sign of inequality.

Rule 2: Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is

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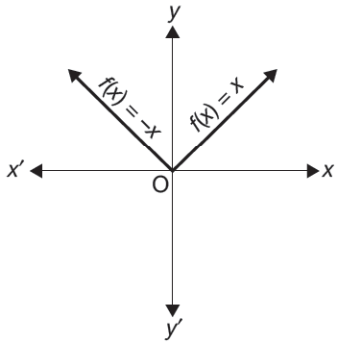
reversed when both sides of an inequation are multiplied or divided by a negative number.

Rule 3: Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.

Absolute Value of a Real Number

The function $f(x)$ defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function. It is also called the absolute value function.



Solution of Rational Inequalities

Let $y = \dfrac{f(x)}{g(x)}$ be an expression in x where $f(x)$ and $g(x)$ are polynomials in x. Now, if it is given that $y > 0$ (or < 0 or ≥ 0 or ≤ 0), this calls for all the values of x for which y satisfies the constraint. This solution set can be found by following steps:

Step 1: Factorize $f(x)$ and $g(x)$ and generate the form:

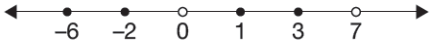
$$y = \frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\ldots(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_p)^{m_p}}$$

where $n_1, n_2 \ldots n_k, m_1, m_2 \ldots m_p$ are natural numbers and $a_1, a_2 \ldots a_k, b_1, b_2 \ldots b_p$ are real numbers. Clearly, here $a_1, a_2 \ldots a_k$ are roots of $f(x) = 0$ and $b_1, b_2, \ldots b_p$ are roots of $g(x) = 0$.

Step 2: Here y vanishes (becomes zero) for $a_1, a_2, \ldots\ldots a_k$. These points are marked on the number line with a black dot. They are solution of $y = 0$.

If $g(x) = 0$, $y = \dfrac{f(x)}{g(x)}$ attains an undefined form, hence $b_1, b_2 \ldots\ldots b_k$ are excluded from the solution. These points are marked with white dots.

e.g., $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$



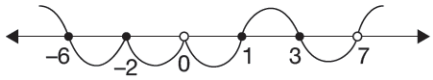
Step 3: Check the value of y for any real number greater than the right most marked number on the number line. If it is positive, then y is positive for all the real numbers greater than the right most marked number and vice versa.

Step 4: If the exponent of a factor is odd, then the point is called simple point and if the exponent of a factor is even, then the point is called double point

$$\frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$

Here, 1, 3, –6, and 7 are simple points and –2 and 0 are double points. From right to left, beginning above the number line (if y is positive in step 3 otherwise from below the line), a wavy curve should be drawn, which passes through all the marked points so that when passing through a simple point, the curve intersects the number line and when passing through a double point, the curve remains on the same side of number line.

$$f(x) = \frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$



As exponents of $(x+2)$ and x are even, the curve does not cross the number line. This method is called wavy curve method.

Step 5: The intervals where the curve is above number line, y will be positive and the intervals where the curve is below the number line, y will be negative. The appropriate intervals are chosen in accordance with the sign of inequality and their union represents the solution of inequality.

Notes

(a) Points where denominator is zero will never be included in the answer.

(b) If you are asked to find the intervals where $f(x)$ is non-negative or non-positive, then make the intervals closed corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.

(c) Normally, we cannot cross-multiply in inequalities. But we cross multiply if we are sure that quantity in denominator is always positive.

(d) Normally, we cannot square in inequalities. But we can square if we are sure that both sides are non-negative.

(e) We can multiply both sides with a negative number by changing the sign of inequality.

(f) We can add or subtract equal quantity to both sides of inequalities without changing the sign of inequality.

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Inequalities Involving Modulus Function

Properties of Modulus Function

(a) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$, where a is positive.
(b) $|x| \leq a \Rightarrow x \in [-a, a]$, where a is positive
(c) $|x| > |y| \Rightarrow x^2 > y^2$
(d) $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$
(e) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$
(f) $|x - y| = |x| + |y| \Rightarrow xy \leq 0$

Example 1

Find x such that $3x^2 - 7x + 6 < 0$

Solution

$$D = 49 - 72 < 0$$

As $D < 0$, $3x^2 - 7x + 6$ will always be positive. Hence $x \in \phi$. **Ans.**

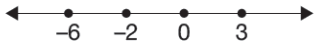
Example 2

$$(x^2 - x - 6)(x^2 + 6x) \geq 0$$

Solution

$$(x - 3)(x + 2)(x)(x + 6) \geq 0$$

Consider $E = x(x - 3)(x + 2)(x + 6)$, $E = 0 \Rightarrow x = 0, 3, -2, -6$ (all are simple points)



For $x \geq 3$ $\quad E = \underset{+ve}{x}\underbrace{(x-3)}_{+ve}\underbrace{(x+2)}_{+ve}\underbrace{(x+6)}_{+ve}$

= positive

Hence for $x(x - 3)(x + 2)(x + 6) \geq 0$

$$x \in (-\infty, -6] \cup [-2, 0] \cup [3, \infty)$$ **Ans.**

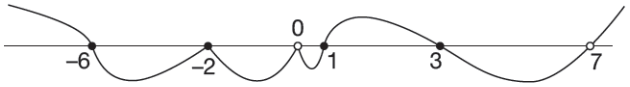
Example 3

Let $f(x) = \dfrac{(x-1)^3 (x+2)^4 (x-3)^5 (x+6)}{x^2 (x-7)^3}$. Solve the following inequality

(a) $f(x) > 0$ **(b)** $f(x) \geq 0$
(c) $f(x) < 0$ **(d)** $f(x) \leq 0$

Solution

We mark on the number line zeros of the function: 1, –2, 3, and –6 (with black circles) and the points of discontinuity 0 and 7 (with white circles), isolate the double points: –2 and 0 and draw the wavy curve:



From graph, we get

(a) $x \in (-\infty, -6) \cup (1, 3) \cup (7, \infty)$
(b) $x \in (-\infty, -6] \cup \{-2\} \cup [1, 3] \cup (7, \infty)$
(c) $x \in (-6, -2) \cup (-2, 0) \cup (0, 1) \cup (3, 7)$
(d) $x \in [-6, 0) \cup (0, 1] \cup [3, 7)$ **Ans.**

Example 4

Solve the inequality $ax > a$.

Solution

This inequality has the parameter a that needs to be investigated further.

If $a > 0$, then $x > 1$
If $a < 0$, then $x < 1$ **Ans.**

Example 5

Solve: $24\ x < 100$ when

(a) x is a natural number
(b) x is an integer.

Solution

We are given $24\ x < 100$

$$\Rightarrow \quad \frac{24x}{24} < \frac{100}{24}$$

$$\Rightarrow \quad x < \frac{25}{6}$$

(a) When x is a natural number, the following values of x make the statement true. 1, 2, 3, 4.
(b) When x is an integer, the solutions of the given equations are ...., –3, –2, –1, 0, 1, 2, 3, 4.
The solution set of the equation is:

$$\{ ..., -3, -2, -1, 0, 1, 2, 3, 4\}.$$ **Ans.**

Example 6

Solve and graph the solution set of $3x + 6 \geq 9$ and $-5x > -15$, $x \in R$.

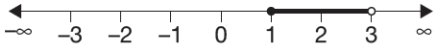
Solution

$$3x + 6 \geq 9 \text{ and } -5x > -15$$

$$\Rightarrow \quad 3x \geq 3 \text{ and } -x > -3$$

$$\Rightarrow \quad x \geq 1 \text{ and } x < 3$$

Combining the solution



So, the solution is $x \in [1, 3)$. **Ans.**

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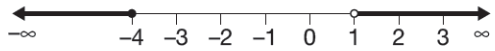
Example 7

Solve and graph the solution set of $-2 < 2x - 4$ and $-2x + 5 \geq 13$, $x \in R$.

Solution



$$-2 < 2x - 4$$

$\Rightarrow \quad 2x - 4 > -2 \quad \text{and} \quad -2x + 5 \geq 13$

$\Rightarrow \quad 2x > 2 \quad \text{and} \quad -2x \geq 13 - 5$

$\Rightarrow \quad x > 1 \quad \text{and} \quad -2x \geq 8$

$\Rightarrow \quad x > 1 \quad \text{and} \quad -x \geq 4$

$\Rightarrow \quad x > 1 \quad \text{and} \quad x \leq -4$

$\therefore \quad x > 1$ and $x \leq -4$ or $x \in (-\infty, -4] \cap (1, \infty)$.

$\therefore \quad x \in \Phi \Rightarrow \{\text{No value satisfies}\}$ **Ans.**

Example 8

Solve the following equation: $2(2x + 3) - 10 < 6(x - 2)$

Solution

We have,

$\Rightarrow \quad 2(2x + 3) - 10 \leq 6(x - 2)$

$\Rightarrow \quad 4x + 6 - 10 \leq 6x - 12$

$\Rightarrow \quad 4x - 4 \leq 6x - 12$

$\Rightarrow \quad 4x - 6x \leq -12 + 4$

[Transposing -4 to RHS and $6x$ to LHS]

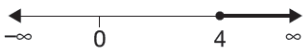
$\Rightarrow \quad -2x \leq -8$

$\Rightarrow \quad \frac{-2x}{-2} \geq \frac{-8}{-2}$

$\Rightarrow \quad x \geq 4$

$\Rightarrow \quad x \in [4, \infty)$

Hence, the solution set of the given inequation is $[4, \infty)$ which can be graphed on real line as shown in Figure.



Ans.

Example 9

Solve the following inequations: $\frac{5x-2}{3} - \frac{7x-3}{5} > \frac{x}{4}$.

Solution

$$\frac{5x-2}{3} - \frac{7x-3}{5} > \frac{x}{4}$$

$$\frac{5(5x-2) - 3(7x-3)}{15} > \frac{x}{4}$$

$$\frac{25x - 10 - 21x + 9}{15} > \frac{x}{4}$$

$$\frac{4x-1}{15} > \frac{x}{4}$$

$4(4x - 1) > 15x$ [Multiplying both sides by 60 i.e., LCM of 15 and 4]

$16x - 4 > 15x$

$16x - 15x > 4$ [Transposing $15x$ to LHS and -4 to RHS]

$x > 4$

$x \in (4, \infty)$

Hence, the solution set of the given inequation is $(4, \infty)$. This can be graphed on the real number line as shown in figure.

Ans.

Example 10

Solve the following in equations: $\frac{2x+4}{x-1} \geq 5$.

Solution

We have, $\frac{2x+4}{x-1} \geq 5$

$\Rightarrow \quad \frac{2x+4}{x-1} - 5 \geq 0$

$\Rightarrow \quad \frac{2x + 4 - 5(x-1)}{x-1} \geq 0$

$\Rightarrow \quad \frac{2x + 4 - 5x + 5}{x-1} \geq 0$

$\Rightarrow \quad \frac{-3x+9}{x-1} \geq 0$ [Multiplying both sides by -1]

$\Rightarrow \quad \frac{3x-9}{x-1} \leq 0$

$\Rightarrow \quad \frac{3(x-3)}{x-1} \leq 0$ [Dividing both sides by 3]

$\Rightarrow \quad \frac{x-3}{x-1} \leq 0$

$\Rightarrow \quad 1 < x \leq 3$

$\Rightarrow \quad x \in (1, 3]$

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Hence, the solution set of the given inequations is (1, 3].

Ans.

Example 11

Solve $5x - 3 < 3x + 1$ when

(a) x is an integer,

(b) x is a real number.

Solution

We have, $5x - 3 < 3x + 1$

$\Rightarrow \quad 5x - 3 + 3 < 3x + 1 + 3$

$\Rightarrow \quad 5x < 3x + 4$

$\Rightarrow \quad 5x - 3x < 3x + 4 - 3x$

$\Rightarrow \quad 2x < 4$

$\Rightarrow \quad x < 2$

(a) When x is an integer, the solutions of the given inequality are $\{\ldots\ldots, -4, -3, -2, -1, 0, 1\}$

(b) When x is a real number, the solutions of the inequality are given by $x < 2$, i.e., all real number x that are less than 2. Therefore, the solution set of the inequality is $x \in (-\infty, 2)$.

Example 12

Solve: $|x - 4| = 7$

Solution

$$|x-4| = \begin{cases} x-4, & \text{where } x-4 \geq 0 \Rightarrow x \geq 4 \\ -(x-4), & \text{where } x-4 < 0 \Rightarrow x < 4 \end{cases}$$

$\therefore \quad x - 4 = 7$ and $-(x - 4) = 7$

$x = 11$ and $-x + 4 = 7$

$\Rightarrow \quad -x = 3 \Rightarrow x = -3 \qquad \therefore$ **Ans.** $\begin{cases} x = -3 \\ x = 11 \end{cases}$

Example 13

Evaluate: $|3| + |-2 - 3| - 3 - |-7|$.

Solution

$|3| + |-2 - 3| - 3 - |7 = 3 + |5| - 3 - \{-(-7)\}$

$= 3 + \{-(-5)\} - 3 - 7$

$= 3 + 5 - 10 = 8 - 10 = -2.$

Ans.

Inequations Involving Absolute Value

Result 1. If a is a positive real number, then

(1) $|x| < a \Leftrightarrow -a < x < a$ i.e. $x \in (-a, a)$

(2) $|x| \leq a \Leftrightarrow -a \leq x \leq a$ i.e. $x \in [-a, a]$

Result 2. If a is a positive real number, then

(1) $|x| > a \Leftrightarrow x < -a$ or $x > a$

$x \in (-\infty, -a) \cup (a, \infty)$

(2) $|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$

$x \in (-\infty, -a] \cup [a, \infty)$

Example 14

Find x from $1 < |x| < 2$ and represent it on number line.

Solution

$1 < |x| \Rightarrow |x| > 1 \Rightarrow x > 1$ or $x < -1$

$\therefore \quad x \in (-\infty, -1) \cup (1, \infty)$ ...(i)

also $\quad |x| < 2 \Rightarrow x < 2$ or $x > -2$

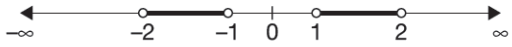
$\therefore \quad x$ lies between -2 and 2

$\Rightarrow \quad x \in (-2, 2)$ ...(ii)

Combining the two results, we get

$1 < |x| < 2 \Rightarrow \{-2 < x < -1\} \cup \{1 < x < 2\}$

i.e., $\quad x \in (-2, -1) \cup (1, 2)$



Ans.

Example 15

Find x satisfying $|x - 5| \leq 3$.

Solution

as $|x - a| \leq r \Leftrightarrow a - r \leq x \leq a + r$, i.e., $x \in [a - r, a + r]$

$\therefore \quad |x - 5| \leq 3 \Leftrightarrow 5 - 3 \leq x \leq 5 + 3$, i.e., $2 \leq x \leq 8$

i.e., $\quad x \in [2, 8]$

Ans.

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Example 16

If x satisfies $|x-1|+|x-2|+|x-3| \geq 6$, then

(A) $0 \leq x \leq 4$ **(C)** $x \leq -2$ or $x \geq 4$

(B) $x \leq 0$ or $x \geq 4$ **(D)** none of these

Solution

Case 1: $x \leq 1$, then

$$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow x \leq 0$$

Hence $x \leq 0$ ...(i)

Case 2: $1 < x \leq 2$, then

$$x - 1 + 2 - x + 3 - x \geq 6 \Rightarrow x \leq -2$$

But $1 < x < 2 \Rightarrow$ No solution ...(ii)

Case 3: $2 < x \leq 3$, then

$$x - 1 + x - 2 + 3 - x \geq 6 \Rightarrow x \geq 6$$

But $2 < x < 3 \Rightarrow$ No solution ...(iii)

Case 4: $x > 3$, then

$$x - 1 + x - 2 + x - 3 \geq 6 \Rightarrow x \geq 4$$

Hence $x \geq 4$ ...(iv)

From (i), (ii), (iii), and (iv), the given inequality holds for $x \leq 0$ or $x \geq 4$. **Ans (A)**

Example 17

Solve for x:

(a) $||x-1|+2| \leq 4$.

(b) $\dfrac{x-4}{x+2} \leq \left|\dfrac{x-2}{x-1}\right|$

Solution

(a) $||x-1|+2| \leq 4 \Rightarrow -4 \leq |x-1|+2 \leq 4$

$\Rightarrow \quad -6 \leq |x-1| \leq 2$

$\Rightarrow \quad |x-1| \leq 2 \quad \Rightarrow \quad -2 \leq x-1 \leq 2$

$\Rightarrow \quad -1 \leq x \leq 3 \quad \Rightarrow \quad x \in [-1, 3]$

(b) **Case 1:** Given inequation will be satisfied for all x such that

$$\frac{x-4}{x+2} \leq 0 \quad \Rightarrow \quad x \in (-2, 4] - \{1\} \quad ...(i)$$

(Note: $\{1\}$ is not in domain of RHS)

Case 2: $\dfrac{x-4}{x+2} > 0 \quad \Rightarrow \quad x \in (-\infty, -2) \cup (4, \infty)$ ...(ii)

Given inequation becomes

$$\frac{x-2}{x-1} \geq \frac{x-4}{x+2} \quad \text{or} \quad \frac{x-2}{x-1} \leq -\frac{x-4}{x+2}$$

on solving we get $x \in (-2, 4/5) \cup (1, \infty)$ ...(iii)

on solving we get $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii), we get $x \in \phi$

Hence, solution of the original in equation: $x \in (-2, \infty) - \{1\}$ (taking union of (i) and (iii)) **Ans.**

Example 18

The equation $|x| + \left|\dfrac{x}{x-1}\right| = \dfrac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ **(B)** $(1, \infty)$

(C) $(-1, 1)$ **(D)** $(-\infty, \infty)$

Solution

$$\frac{x^2}{|x-1|} = \left|x + \frac{x}{x-1}\right|$$

$\because |x| + \left|\dfrac{x}{x-1}\right| = \left|x + \dfrac{x}{x-1}\right|$ is true only if $\left(x \cdot \dfrac{x}{x-1}\right) \geq 0$

$\Rightarrow x \in \{0\} \cup (1, \infty)$ **Ans. (A, B)**

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Know Your Progress - 1

1. $\dfrac{x^2+2x-3}{x^2+1} < 0$

2. $\dfrac{(x-1)(x+2)^2}{-1-x} < 0$

3. $x^4 - 2x^2 - 63 \leq 0$

4. $\dfrac{x+1}{(x-1)^2} < 1$

5. $\dfrac{x^2-7x+12}{2x^2+4x+5} > 0$

6. $\frac{x^2+6x-7}{x^2+1} \le 2$

7. $\frac{x^4+x^2+1}{x^2-4x-5} < 0$

8. $\frac{x+7}{x-5}+\frac{3x+1}{2} \ge 0$

9. $\frac{1}{x+2} < \frac{3}{x-3}$

10. $\frac{14x}{x+1}-\frac{9x-30}{x-4} < 0$

11. $\frac{x^2-5x+12}{x^2-4x+5} > 3$

12. $\frac{x^2+2}{x^2-1} < -2$

13. $\frac{\left(2-x^2\right)(x-3)^3}{(x+1)\left(x^2-3x-4\right)} \ge 0$

14. $\frac{5-4x}{3x^2-x-4} < 4$

15. $\frac{(x+2)(x^2-2x+1)}{4+3x-x^2} \ge 0$

16. $\frac{x^4-3x^3+2x^2}{x^2-x-30} > 0$

17. $\frac{2x}{x^2-9} \le \frac{1}{x+2}$

18. $\frac{1}{x-2}+\frac{1}{x-1} > \frac{1}{x}$

19. $\frac{20}{(x-3)(x-4)}+\frac{10}{x-4}+1 > 0$

20. $\frac{(x-2)(x-4)(x-7)}{(x+2)(x+4)(x+7)} > 1$

21. $(x^2-2x)(2x-2)-9\frac{2x-2}{x^2-2x} \le 0$

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Quadratic Equation

A polynomial of degree two of the form $P(x) = ax^2 + bx + c (a \neq 0)$ is called a quadratic expression in variable x, and $P(x) = 0$ is known as a quadratic equation.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

Roots of a Quadratic Equation

The value of x which satisfies the given quadratic equation is known as its root. The roots of the given equation are known as its solution.

Methods of Solving a Quadratic Equation

Factorization Method

Step 1: Factorize the constant term of the given quadratic equation.

Step 2: Express the coefficient of middle term as the sum or difference of the factors obtained in step 1.

Clearly, the product of these two factors will be equal to the product of the coefficient of x^2 and the constant term.

Step 3: Split the middle term in two parts obtained in step 2.

Step 4: Factorize the quadratic equation obtained in step 3.

Completing the Square Method

Step 1: Let the quadratic equation be $ax^2 + bx + c = 0$, $a \neq 0$.

Step 2: Make the coefficient of x^2 unity, if it is not unity. i.e., obtain $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Step 3: Shift the constant term $\frac{c}{a}$ on R.H.S. to get $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 4: Add square of half of the coefficient of x. i.e. $\left(\frac{b}{2a}\right)^2$ on both sides to obtain

$$x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Step 5: Write L.H.S. as the perfect square of a binomial expression and simplify R.H.S., to get:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Step 6: Taking square root both the sides, we get:

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Step 7: By shifting the constant term $\frac{b}{2a}$ on R.H.S. and simplifying we obtain a formula for values of x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is also known as Quadratic Formula.

Hence, the roots of the quadratic equation

$ax^2 + bx + c = 0$ are: $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

A quadratic equation is satisfied by exactly two values of 'x' which may be real or imaginary. The equation $a\,x^2 + b\,x + c = 0$ is:

- A quadratic equation if $a \neq 0$. [Two roots]
- A linear equation if $a = 0$, $b \neq 0$. [One root]
- A contradiction if $a = b = 0$, $c \neq 0$. [No root]
- An identity if $a = b = c = 0$. [Infinite roots]
- A quadratic equation cannot have more than two roots.
- It follows from the above statement that if a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity. In fact, $a \neq 0$ is necessary condition for equation to be quadratic. Hence, identity can never be quadratic equation.

Example 19

If $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = 2\frac{1}{6}$, then value of x are:

(A) $\frac{5}{13}, \frac{7}{13}$ **(B)** $\frac{7}{13}, \frac{9}{13}$

(C) $\frac{9}{13}, \frac{4}{13}$ **(D)** none

Solution

Let $\sqrt{\frac{x}{1-x}} = y$

$$y + \frac{1}{y} = \frac{13}{6}$$

$$\frac{y^2 + 1}{y} = \frac{13}{6}$$

$$6y^2 + 6 = 13y$$

$$6y^2 - 13y + 6 = 0$$

$$6y^2 - 9y - 4y + 6 = 0$$

$$3y(2y - 3) - 2(2y - 3) = 0$$

$$(3y - 2)(2y - 3) = 0$$

$$y = \frac{2}{3}, \frac{3}{2}$$

When $y = \frac{3}{2}$

$$\sqrt{\frac{x}{1-x}} = \frac{3}{2}$$

$$\frac{x}{1-x} = \frac{9}{4}$$

$$4x = 9 - 9x$$

$$13x = 9$$

$$\boxed{x = \frac{9}{13}}$$

When $y = \frac{2}{3}$

$$\sqrt{\frac{x}{1-x}} = \frac{2}{3}$$

$$9x = 4 - 4x$$

$$13x = 4$$

$$\boxed{x = \frac{4}{13}}$$

Ans (C)

Example 20

Find the roots of the equation $x^2 - x - 3 = 0$. Also comment on nature of roots.

Solution

$$x^2 - x - 3 = 0$$

From the quadratic formula we can find the value of x,

$$x = \frac{+1 \pm \sqrt{1 - 4 \times 1(-3)}}{2 \times 1} = \frac{1 \pm \sqrt{13}}{2}$$

So, $x = \frac{1 + \sqrt{13}}{2}, \frac{1 - \sqrt{13}}{2}$

Hence, the roots are irrational. **Ans.**

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Example 21

Determine the value of K for which the $x = -a$ is a solution of the equation:

$$x^2 - 2(a + b)\, x + 3K = 0.$$

Solution

Putting $x = -a$ in the given equation, we have

$$(-a)^2 - 2(a + b)(-a) + 3K = 0$$

$$\Rightarrow \quad a^2 + 2a^2 + 2ab + 3K = 0$$

$$\Rightarrow \quad 3K = -3a^2 - 2ab$$

$$\Rightarrow \quad K = -\frac{a}{3}(3a + 2b). \qquad \textbf{Ans.}$$

Example 22

Solve the following quadratic equation by factorization method: $x^2 - 2ax + a^2 - b^2 = 0$.

Solution

Here, factors of constant term $(a^2 - b^2)$ are $(a - b)$ and $(a + b)$.

Also, coefficient of the middle term $= -2a = -[(a - b) + (a + b)]$

$$\therefore \quad x^2 - 2ax + a^2 - b^2 = 0$$

$$\Rightarrow \quad x^2 - \{(a - b) + (a + b)\}x + (a - b)(a + b) = 0$$

$$\Rightarrow \quad x^2 - (a - b)x - (a + b)x + (a - b)(a + b) = 0$$

$$\Rightarrow \quad x\{x - (a - b)\} - (a + b)\{x - (a - b)\} = 0$$

$$\Rightarrow \quad \{x - (a - b)\}\{x - (a + b)\} = 0$$

$$\Rightarrow \quad x - (a - b) = 0 \text{ or, } x - (a + b) = 0$$

$$\Rightarrow \quad x = a - b \text{ or } x = a + b \qquad \textbf{Ans.}$$

Aliter: Completing the square method may be more useful.

Example 23

The sum of the squares of two consecutive positive integers is 545. Find the integers.

Solution

Let x be one of the positive integers. Then the other integer is $x + 1$, where $x \in z^+$.

Since the sum of the squares of the integers is 545, we get $x^2 + (x + 1)^2 = 545$

$$\Rightarrow \quad 2x^2 + 2x - 544 = 0$$

$$\Rightarrow \quad x^2 + x - 272 = 0$$

$$\Rightarrow \quad x^2 + 17x - 16x - 272 = 0$$

$$\Rightarrow \quad x(x + 17) - 16(x + 17) = 0$$

$$\Rightarrow \quad (x - 16)(x + 17) = 0.$$

Here, $x = 16$ or $x = -17$. But, x is a positive integer. Therefore, reject $x = -17$ and take $x = 16$. Hence, two consecutive positive integers are 16 and (16 + 1), i.e., 16 and 17. **Ans.**

Example 24

The sum of two numbers is 48 and its product is 432. Find the numbers.

Solution

Let the two numbers be x and $48 - x$ so that their sum is 48. It is given that the product of the two numbers is 432. Hence, we have

$$x(48 - x) = 432$$

$$\Rightarrow \quad 48x - x^2 - 432 = 0$$

$$\Rightarrow \quad x^2 - 48x + 432 = 0$$

$$\Rightarrow \quad x^2 - 36x - 12x + 432 = 0$$

$$\Rightarrow \quad x(x - 36) - 12(x - 36) = 0$$

$$\Rightarrow \quad (x - 36)(x - 12) = 0$$

$\therefore$ Either $x - 36 = 0$

$$\Rightarrow \quad x = 36 \text{ or } x - 12 = 0 \Rightarrow x = 12.$$

$\therefore$ When one number is 12 another number is $48 - 12 = 36$ and when one number is 36, another number is $48 - 36 = 12$. **Ans.**

Example 25

Out of a group of swans, $\frac{7}{2}$ times the square root of the total number are playing on the shore of a tank. The two remaining ones are playing, in deep water. What is the total number of swans?

Solution

Let us denote the number of swans by x.

Then, the number of swans playing on the shore of the tank $= \frac{7}{2}\sqrt{x}$.

There are two remaining swans.

Therefore, $x = \frac{7}{2}\sqrt{x} + 2 \quad \Rightarrow \quad x - 2 = \frac{7}{2}\sqrt{x}$

$$\Rightarrow (x - 2)^2 = \left(\frac{7}{2}\right)^2 x \quad \Rightarrow \quad x \quad \Rightarrow \quad 4(x^2 - 4x + 4) = 49x$$

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$\Rightarrow \quad 4x^2 - 65x + 16 = 0 \quad \Rightarrow \quad 4x^2 - 64x - x + 16 = 0$

$\Rightarrow \quad 4x(x - 16) - 1(x - 16) = 0 \quad \Rightarrow \quad (x - 16)(4x - 1) = 0$

This gives $x = 16$ or $x = \frac{1}{4}$.

We reject $x = \frac{1}{4}$ and take $x = 16$.

Hence, the total number of swans is 16. **Ans.**

Know Your Progress - 2

1. Solve the following quadratic equation by factorization method: $x^2 + 2ax + a^2 - b^2 = 0$.

2. Solve: $9x^2 - 30x + 25 = 0$

3. Solve the following quadratic equation by completing the square method: $x^2 + 9x + 1 = 0$

4. Solve: $\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(x-3)(2x+3)} = 0$

5. Solve the following equation for x: $9x^2 - 9(a + b)\,x + (2a^2 + 5ab + 2b^2) = 0$.

6. If the roots of $4x^2 + 5k = (5k + l)x$ differ by unity, then find the values of k.

7. Find the roots of equation $9x^2 - 9\,(a + b)\,x + 2a^2 + 5ab + 2b^2 = 0$

(A) $\frac{2a+b}{3}, \frac{a+2b}{3}$

(B) $\frac{2a+b}{9}, \frac{a+2b}{9}$

(C) $\frac{a+3b}{9}, \frac{3a+b}{9}$

(D) None of these

8. Find the roots of the following equation $(x - 3)(x - 4) = \frac{34}{33^2}$

(A) $\frac{51}{33}, \frac{82}{33}$ **(B)** $\frac{98}{33}, \frac{133}{33}$

(C) $\frac{122}{33}, \frac{53}{33}$ **(D)** $\frac{89}{33}, \frac{67}{33}$

9. Solve the equation for x: $\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}$

(A) $\frac{9}{2}$ **(B)** $\frac{3}{2}$

(C) $\frac{7}{2}$ **(D)** $\frac{1}{2}$

10. Solve the equation for x: $\frac{x-a}{x-b} + \frac{x-b}{x-a} = \frac{a}{b} + \frac{b}{a}$

(A) $0, a - b$ **(B)** $0, a + b$

(C) $0, 1$ **(D)** $a + b, 2$

11. If $2x + 9 + x = 13$, then $x =$

(A) 8, 10 **(B)** 8, 20

(C) 10, 20 **(D)** 8

12. How many values of x are there which satisfy the equation:

$\sqrt{x^2 + x - 6} - x + 2 = \sqrt{x^2 - 7x + 10}$, $x \in R$

(A) 2 **(B)** 3

(C) 4 **(D)** 5

13. Solve for x: $3^{x+2} + 3^{-x} = 10$

(A) 2, 0 **(B)** 0, 3

(C) −2, −1 **(D)** −2, 0

14. How many real solutions are there for equation $(x + 1)(x + 2)(x + 3)(x + 4) = 24$

(A) 2 **(B)** 3

(C) 4 **(D)** 5

15. If $\left(2+\sqrt{3}\right)^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{2}{2-\sqrt{3}}$, then x is equal to:

(A) 0 **(B)** 1

(C) 2 **(D)** both (A) and (C)

16. Find the harmonic mean of the roots of $\left(3+\sqrt{2}\right)x^2 - \left(2+\sqrt{3}\right)x + \left(2\sqrt{2}+\sqrt{6}\right) = 0$

(A) 2 **(B)** $2\sqrt{2}$

(C) $\sqrt{2}$ **(D)** 1

17. Solve the equation for x:

$\frac{1}{(x-1)(x-1)} + \frac{1}{(x-2)(x-3)} + \frac{1}{(x-3)(x-4)} = \frac{3}{40}$

(A) 9, 4 **(B)** − 9, − 4

(C) − 4, 9 **(D)** 4, − 9

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Relation Between Roots and Coefficients

If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are α and β then,

$$\alpha+\beta = \left(\frac{-b+\sqrt{D}}{2a}\right)+\left(\frac{-b-\sqrt{D}}{2a}\right) \Rightarrow \alpha+\beta = \frac{-2b}{2a} = \frac{-b}{a}$$

and $$\alpha\beta = \left(\frac{-b+\sqrt{D}}{2a}\right)\times\left(\frac{-b-\sqrt{D}}{2a}\right) \Rightarrow \alpha\beta = \frac{c}{a}$$

- **Some more relation between α and β**

Note

Students are advised to prove all these formulas

(i) $(\alpha-\beta) = \sqrt{(\alpha+\beta)^2-4\alpha\beta} = \pm\frac{\sqrt{b^2-4ac}}{a}$

$= \frac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = \frac{b^2-2ac}{a^2}$

(iii) $\alpha^2-\beta^2 = (\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

$= \frac{b\sqrt{b^2-4ac}}{a^2}$

(iv) $\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$= -\frac{b(b^2-3ac)}{a^3}$

(v) $\alpha^3-\beta^3 = (\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

$= \sqrt{(\alpha+\beta)^2-4\alpha\beta}\{(\alpha+\beta)^2-\alpha\beta\}$

$= \frac{(b^2-ac)\sqrt{b^2-4ac}}{a^3}$

(vi) $\alpha^4+\beta^4 = \{(\alpha+\beta)^2-2\alpha\beta\}^2-2\alpha^2\beta^2$

$= \left(\frac{b^2-ac}{a^2}\right)^2-2\frac{c^2}{a^2}$

(vii) $\alpha^4-\beta^4 = (\alpha^2-\beta^2)(\alpha^2+\beta^2)$

$= \frac{-b(b^2-2ac)\sqrt{b^2-4ac}}{a^4}$

(viii) $\alpha^2+\alpha\beta+\beta^2 = (\alpha+\beta)^2-\alpha\beta$

(ix) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

(x) $\alpha^2\beta+\beta^2\alpha = \alpha\beta(\alpha+\beta)$

(xi) $\left(\frac{\alpha}{\beta}\right)^2+\left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^4+\beta^4}{\alpha^2\beta^2}$

$= \frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}$

(xii) $nb^2 = ac(1+n)^2$ which one root is n times of another.

Theory of Equations and Nature of Roots

If $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_n$ are the roots of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots a_{n-1}x + a_n = 0$$

where $a_0, a_1, a_2 \ldots\ldots a_n$ are all real, $a_0 \neq 0$ then

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots a_{n-1}x + a_n$$
$$= a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n)$$
$$= a_0[x^n - x^{n-1}(\Sigma\alpha_1) + x^{n-2}(\Sigma\alpha_1\alpha_2) + \ldots\ldots$$
$$+ (-1)^n\alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n$$

Now comparing the coefficients from above identity we get,

$\Sigma\alpha_1 = -\frac{a_1}{a_0}$, [Sum of roots taken one at a time]

$\Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}$, [Sum of roots taken two at a time]

$\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \ldots\ldots\ldots$ [Sum of roots taken three at a time]

$\Pi\alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n = (-1)^n\frac{a_n}{a_0}$ [Product of roots]

e.g., If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$ then,

$$\alpha+\beta+\gamma = -\frac{b}{a}, \ \beta\gamma+\gamma\alpha+\alpha\beta = \frac{c}{a}$$

and $$\alpha\beta\gamma = -\frac{d}{a}$$

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Formation of an Equation with Given Roots

A quadratic equation whose roots are α and β is given by $(x-\alpha)(x-\beta)=0$

$\therefore \quad x^2-\alpha x-\beta x+\alpha\beta=0$

$\therefore \quad x^2-(\alpha+\beta)x+\alpha\beta=0$

i.e., x^2 – (sum of roots)x + product of roots = 0

$$\therefore x^2-Sx+P=0$$

Equation in Terms of the Roots of Another Equation (Transformation of Roots)

If α, β are roots of the equation $ax^2+bx+c=0$, then the equation whose roots are

(i) $-\alpha, -\beta \Rightarrow ax^2-bx+c=0$ (replace x by $-x$)

(ii) $1/\alpha, 1/\beta \Rightarrow cx^2+bx+a=0$ (replace x by $1/x$)

(iii) $\alpha^n, \beta^n; n\in N \Rightarrow a(x^{1/n})^2+b(x^{1/n})+c=0$ (replace x by $x^{1/n}$)

(iv) $k\alpha, k\beta \Rightarrow ax^2+kbx+k^2c=0$ (replace x by x/k)

(v) $k+\alpha, k+\beta \Rightarrow a(x-k)^2+b(x-k)+c=0$

(vi) $\frac{\alpha}{k}, \frac{\beta}{k} \Rightarrow k^2ax^2+kbx+c=0$ (replace x by kx)

(vii) $\alpha^{1/n}, \beta^{1/n}; n\in N \Rightarrow a(x^n)^2+b(x^n)+c=0$ (replace x by x^n)

Note

This method can also be applied to higher degree equations.

Example 26

If α, β be the roots of $x^2+px-q=0$ and γ, δ be the roots of $x^2+px+r=0$, then $\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}=$

(A) 1 **(B)** q

(C) r **(D)** $q+r$

Solution

Here

$$\left.\begin{aligned}\alpha+\beta&=-p\\ \gamma+\delta&=-p\end{aligned}\right\}\Rightarrow \alpha+\beta=\gamma+\delta$$

and $\alpha\beta=-q,\ \gamma\delta=r$

Now, $(\alpha-\gamma)(\alpha-\delta)=\alpha^2-\alpha(\gamma+\delta)+\gamma\delta$

$=\alpha^2-\alpha(\alpha+\beta)+r$

$=-\alpha\beta+r$

$=-(-q)+r=q+r$

By symmetry of the results,

$$(\beta-\gamma)(\beta-\delta)=q+r$$

Hence, the ratio is 1. **Ans (A)**

Example 27

If tan q and sec q are the roots of the quadratic equation $ax^2+bx+c=0$, then

(A) $a^4=b^2(b^2-4ac)$ **(B)** $b^4=a^2(b^2-4ac)$

(C) $a^4=b^2(b^2+4ac)$ **(D)** $b^4=a^2(b^2-4ac)$

Solution

$$ax^2+bx+c=0$$

equation has two roots tan q, sec q

Then,

$$a\tan^2q+b\tan q+c=0 \quad \text{.....(i)}$$

$$a\sec^2q+b\sec q+c=0 \quad \text{.....(ii)}$$

$$\Rightarrow \quad \sec q+\tan q=-\frac{b}{a} \quad \text{.....(iii)}$$

$$\Rightarrow \quad \sec q\tan q=\frac{c}{a}$$

$$(\sec q-\tan q)^2=(\sec q+\tan q)^2-4\sec q\tan q$$

$$=\left(\frac{-b}{a}\right)^2=-4\frac{c}{a}=\frac{b^2-4ac}{a^2}$$

$$(\sec q-\tan q)=\frac{\sqrt{b^2-4ac}}{a} \quad \text{.....(iv)}$$

multiplying eq.(iii) and (iv), we get

$$\Rightarrow \quad 1=\frac{-b}{a}\frac{\sqrt{b^2-4ac}}{a}$$

$$\Rightarrow \quad (a^2)^2=b^2(b^2-4ac)$$

$$\Rightarrow \quad a^4=b^2(b^2-4ac)$$ **Ans (A)**

Example 28

If α, β are the roots of $x^2+ax+b=0$, find the equation for which $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$ are the roots.

Solution

$$\alpha+\beta=-a,\ \alpha\beta=b$$

$$\alpha^2+\beta^2=(\alpha+\beta)^{-2}\alpha\beta=a^2-2b$$

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$$\alpha^{-2}+\beta^{-2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{a^2-2b}{b^2}$$

$a-2b$

Required equation is

$$x^2-Sx+P=0$$

$$x^2-x\left\{(a^2-2b)+\frac{a^2-2b}{b^2}\right\}+(a^2-2b)\frac{(a^2-2b)}{b^2}=0$$

$$\Rightarrow\quad x^2-x\left\{\frac{b^2(a^2-2b)+a^2-2b}{b^2}\right\}+\frac{(a^2-2b)(a^2-2b)}{b^2}=0$$

$$\Rightarrow\quad b^2x^2-x\{b^2(a^2-2b)+a^2-2b\}+(a^2-2b)^2=0 \quad \textbf{Ans.}$$

Example 29

If α, β are the roots of a quadratic equation $x^2-3x+5=0$, then the equation whose roots are

$$(\alpha^2-3\alpha+7) \text{ and } (\beta^2-3\beta+7) \text{ is}$$

(A) $x^2+4x+1=0$ **(B)** $x^2-4x+4=0$
(C) $x^2-4x-1=0$ **(D)** $x^2+2x+3=0$

Solution

Since α, β are the roots of equation $x^2-3x+5=0$

So $\alpha^2-3\alpha+5=0$

$$\beta^2-3\beta+5=0$$

$$\therefore\qquad \alpha^2-3\alpha=-5$$

$$\beta^2-3\beta=-5$$

Putting in (α^2-3a+7) and $(\beta^2-3\beta+7)$ .....(i)

$$-5+7, -5+7$$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is

$$x^2-4x+4=0. \quad \textbf{Ans. (B)}$$

Example 30

If α and β are the roots of $ax^2+bx+c=0$, find the value of $(a\alpha+b)^{-2}+(a\beta+b)^{-2}$.

Solution

Take $b=-(\alpha+\beta)\,a$

$$(a\alpha+b)^{-2}+(a\beta+b)^{-2}=\frac{1}{a^2}\left[\frac{1}{(\alpha-\alpha-\beta)^2}+\frac{1}{(\beta-\alpha-\beta)^2}\right]$$

$$=\frac{1}{a^2}\left[\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right]=\frac{1}{a^2}\left[\frac{b^2-2ac}{a^2.\frac{c^2}{a^2}}\right]=\frac{b^2-2ac}{a^2c^2} \quad \textbf{Ans.}$$

Example 31

If two roots of $4x^3+20x^2-23x+6=0$ are equal, find all roots.

Solution

Let roots be α, α, and β.

$$\therefore\qquad \alpha+\alpha+\beta=-\frac{20}{4}\quad\Rightarrow\quad 2\alpha+\beta=-5 \quad \text{.....(i)}$$

$$\therefore \alpha\cdot\alpha+\alpha\beta+\alpha\beta=-\frac{23}{4}\ \Rightarrow\ \alpha^2+2\alpha\beta=- \text{ and } \alpha^2\beta=-\frac{6}{4}$$

from equation (i)

$$\alpha^2+2\alpha(-5-2\alpha)=-\frac{23}{4}\ \Rightarrow\ \alpha^2-10\alpha-4\alpha^2=-\frac{23}{4}$$

$$\Rightarrow\qquad 12\alpha^2+40\alpha-23=0$$

$$\therefore\qquad \alpha=1/2, -\frac{23}{6}$$

when $\alpha=\frac{1}{2}$

$$\alpha^2\beta=\frac{1}{4}(-5-1)=-\frac{3}{2}$$

when $\alpha=-\frac{23}{6}$

$$\Rightarrow\qquad \alpha^2\beta=\frac{23\times23}{36}\left(-5-2x\left(-\frac{23}{6}\right)\right)\neq-\frac{3}{2}$$

$$\Rightarrow\qquad \alpha=\frac{1}{2}\ \beta=-6$$

Hence, roots of equation $=\frac{1}{2}, \frac{1}{2}, -6$ **Ans.**

Example 32

If α, β, γ are the roots of $x^3-px^2+qx-r=0$, find:

(i) $\Sigma\alpha^3$

(ii) $\alpha^2(\beta+\gamma)+\beta^2(\gamma+\alpha)+\gamma^2(\alpha+\beta)$

Solution

We know that $\alpha+\beta+\gamma=p$

$$\alpha\beta+\beta\gamma+\gamma\alpha=q$$

$$\alpha\beta\gamma=r$$

(i) $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma+(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)\}$

$$=3r+p\{p^2-3q\}=3r+p^3-3pq$$

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(ii) $\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta)$

$= \alpha^2\{p-\alpha)+\beta^2(p-\beta)+\gamma^2(p-\gamma)$

$= p(\alpha^2+\beta^2+\gamma^2)-3r-p^3+3pq$

$= p(p^2-2q)-3r-p^3+3pq = pq-3r$ **Ans.**

Example 33

If q, r, s are positive, show that the equation $f(x) = x^4 + qx^2 + rx - s = 0$ has one positive, one negative, and two imaginary roots.

Solution

Product $= -s < 0$

let roots be $\alpha, \beta, \gamma, \delta$

$\Rightarrow \quad \alpha\beta\gamma\delta < 0$

this is possible when

(i) one root is negative and three are positive
(ii) three roots are negative and one is positive
(iii) one root negative, one positive and two roots imaginary.

$$f(x) \equiv x^4 + qx^2 + rx - s$$

As there is one change of sign, the equation has almost one positive root.

$$f(-x) \equiv x^4 + qx^2 - rx - s$$

Again there is only one change of sign, the equation has almost only one negative root.

so (i), (ii) are not possible.

Hence, there is only one negative root, one positive root, and two imaginary roots.

Example 34

If the roots of $ax^2 + bx + c = 0$ are α and β, then find the equation whose roots are:

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

(c) α^2, β^2

Solution

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$

put, $y = \frac{-2}{\alpha} \Rightarrow \alpha = \frac{-2}{y}$

$$a\left(-\frac{2}{y}\right)^2 + b\left(\frac{-2}{y}\right) + c = 0$$

$\Rightarrow \quad cy^2 - 2by + 4a = 0$

required equation is $cx^2 - 2bx + 4a = 0$ **Ans.**

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

put, $y = \frac{\alpha}{\alpha+1} \Rightarrow \alpha = \frac{y}{1-y}$

$$\Rightarrow \quad a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0$$

$\Rightarrow \quad (a+c-b)y^2 + (-2c+b)y + c = 0$ **Ans.**

Required equation is $(a+c-b)x^2 + (b-2c)x + c = 0$

(c) α^2, β^2

put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$

$\Rightarrow \quad a^2y^2 + (2ac - b^2)\, y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)\, x + c^2 = 0$

Ans.

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Example 35

If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ, then find equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$.

Solution

Put $y = \frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = -\frac{a\gamma}{d}$ $\left(\because\ \alpha\beta\gamma = -\frac{d}{a}\right)$

Put $x = -\frac{dy}{a}$

$$\Rightarrow \quad a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$$

Required equation is $d^2x^3 - bdx^2 + acx - a^2 = 0$

Example 36

If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$. In terms of p, q, r, and s. Deduce the condition that the equations have a common root.

Solution

α, β are the roots of $x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \ \alpha\beta = q$ .....(1)

and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \quad \gamma+\delta=-r,\ \gamma\delta=s$

Now, $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$ .....(2)

$= [\alpha^2-\alpha(\gamma+\delta)+\gamma\delta]\ [\beta^2-\beta(\gamma+\delta)+\gamma\delta]$

$= (\alpha^2+r\alpha+s)\ (\beta^2+r\beta+s)$

$= \alpha^2\beta^2+r\alpha\beta(\alpha+\beta)+r^2\alpha\beta+s(\alpha^2+\beta^2)+sr(\alpha+\beta)+s^2$

$= \alpha^2\beta^2+r\alpha\beta(\alpha+\beta)+r^2\alpha\beta+s((\alpha+\beta)^2-2\alpha\beta)+sr(\alpha+\beta)+s^2$

$= q^2-pqr+r^2q+s(p^2-2q)+sr\,(-p)+s^2$

$= (q-s)^2-rpq+r^2q+sp^2-prs$

$= (q-s)^2-rq(p-r)+sp(p-r)$

$= (q-s)^2+(p-r)\ (sp-rq)$

For a common root (Let $\alpha=\gamma$ or $\beta=\delta$) .....(3)

then $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)=0$ .....(4)

from (3) and (4), we get

$(q-s)^2+(p-r)(sp-rq)=0$

$\Rightarrow \quad (q-s)^2=(p-r)(rq-sp),$

which is the required condition. **Ans.**

Know Your Progress - 3

1. If α, β are the roots of $x^2+ax+1=0$ and γ, δ are roots of $x^2+bx+1=0$, find the value of $(\delta-\alpha)(\delta-\beta)(\gamma-\alpha)(\gamma-\beta)$.

(A) $(a+b)^2$ **(B)** $(a-b)^2$
(C) (a^2+b^2) **(D)** (a^2-b^2)

2. If α, β be the roots of $x^2+px-q=0$ and γ, δ be the roots of $x^2+px+r=0$, then $\dfrac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}=$

(A) 1 **(B)** q
(C) r **(D)** $q+r$

3. Given that α, β are the roots of $x^2+bx+c=0$, find the value of $(\alpha+b)^{-2}+(\beta+b)^{-2}$.

(A) $\dfrac{b^2+2c}{c^2}$ **(B)** $\dfrac{b^2-2c}{c^2}$

(C) $\dfrac{b^2-2c}{c}$ **(D)** None of these

4. If α, β are the roots of $x^2+x+1=0$ and γ, δ are the roots of $x^2+3x+1=0$, then

$(\alpha-\gamma)(\beta+\delta)(\alpha+\delta)(\beta-\gamma)=$

(A) 2 **(B)** 4
(C) 6 **(D)** 8

5. α, β are the roots of $x^2+ax+1=0$ and γ, δ are roots of $x^2+bx+1=0$. Find the value of

$(\gamma-\alpha-\beta+\delta)(\gamma+\alpha+\beta+\delta)$.

(A) a^2+b^2 **(B)** a^2-b^2
(C) $a+b$ **(D)** b^2-a^2

6. If the roots of the equation $px^2+qx+r=0$ are in the ratio $\ell : m$, then:

(A) $(\ell+m)^2pq=\ell mr^2$ **(B)** $(\ell+m)^2\,pr=\ell mq$
(C) $(\ell+m)^2pr=\ell mq^2$ **(D)** None of these

7. If the roots of $ax^2+bx+b=0$ are in the ratio $m : n$, then find the value of $\sqrt{\dfrac{m}{n}}+\sqrt{\dfrac{n}{m}}$.

(A) $-\sqrt{\dfrac{b}{a}}$ **(B)** $\sqrt{\dfrac{b}{a}}$

(C) $\sqrt{\dfrac{a}{b}}$ **(D)** $-\sqrt{\dfrac{a}{b}}$

8. If α, β are the roots of $x^2-p(x+1)-c=0$, then $\dfrac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\dfrac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$ is equal to

(A) 0 **(B)** 1
(C) 2 **(D)** None of these

9. If r_1 and r_2 are the roots of $x^2+bx+c=0$ and $S_0=r_1^0+r_2^0$, $S_1=r_1^1+r_2^1$ and , $S_1={r_1}^2+{r_2}^2$, then the value of $S_2+bS_1+cS_0$ is:

(A) depends on b only
(B) depends on c only
(C) depends on both b and c.
(D) does not dependent on b and c.

10. If α, β, γ are the roots of the equation $x^3+a_0x^2+a_1x+a^2=0$, then $(1-\alpha^2)(1-\beta^2)(1-\gamma^2)$ is equal to:

(A) $(1-a_1)^2+(a_0-a^2)^2$
(B) $(1+a_1)^2-(a_0+a^2)^2$
(C) $(1+a_1)^2+(a_0+a^2)^2$
(D) None of these

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11. If α, β are the roots of the equation $8x^2 - 3x + 27 = 0$, then the value of $[(\alpha^2/\beta)^{1/3} + (\beta^2/\alpha)^{1/3}]$ is:

(A) 1/3 **(B)** 1/4
(C) 1/5 **(D)** 1/6

12. If α, β be the roots of the equation $x^2 - 2x + 3 = 0$, then the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ is:

(A) $x^2 + 2x + 1 = 0$ **(B)** $9x^2 + 2x + 1 = 0$
(C) $9x^2 - 2x + 1 = 0$ **(D)** $9x^2 + 2x - 1 = 0$

13. Given that α, β are the roots of $\ell x^2 + mx + n = 0$, find the equation with roots $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$:

(A) $\ell 4x^2 - 2\ell 2\ (m^2 - 2n\ell)x + m^2(m^2 + 4n\ell)$
(B) $\ell 4x^2 - 2\ell 2\ (m^2 - 2n\ell)x + m^2(m^2 - 4n\ell)$
(C) $\ell 4x^2 + 2\ell 2\ (m^2 - 2n\ell)x + m^2(m^2 + 4n\ell)$
(D) $\ell 4x^2 - 2\ell 2\ (m^2 + 2n\ell)x + m^2(m^2 + 4n\ell)$

14. A quadratic equation the product of whose roots x_1 and x_2 is equal to 4 and satisfying the relation $\frac{x_1}{x_1 - 1} + \frac{x_2}{x_2 - 1} = 2$, is:

(A) $x^2 - 2x + 4 = 0$ **(B)** $x^2 - 4x + 4 = 0$
(C) $x^2 + 2x + 4 = 0$ **(D)** $x^2 + 4x + 4 = 0$

Nature of Roots

The term $b^2 - 4ac$ is called **discriminant** of the equation. It is denoted by Δ or D.

Case 1: Suppose $a, b, c \in R$ and $a \neq 0$ then

(i) If $D > 0 \Rightarrow$ roots are real and unequal
(ii) If $D = 0 \Rightarrow$ roots are real and equal and each equal to $-b/2a$
(iii) If $D < 0 \Rightarrow$ roots are imaginary and unequal or complex conjugate.

i.e., if one root is, $\alpha = x + iy$ then the other root will be $\beta = x - iy$

Case 2: Suppose $a, b, c \in Q$, $a \neq 0$ then

(i) If $D > 0$ and D is a perfect square
$\Rightarrow$ roots are unequal and rational
(ii) If $D > 0$ and D is not a perfect square
$\Rightarrow$ roots are irrational and unequal and occurs in conjugate pairs, i.e., if one root is, $\alpha = x + \sqrt{y}$ then the other root will be $\beta = x - \sqrt{y}$ where $x, y \in Q$

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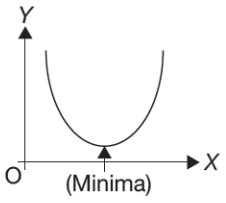
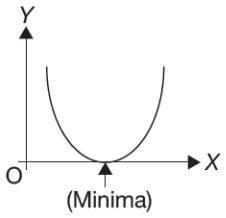
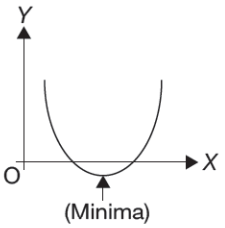
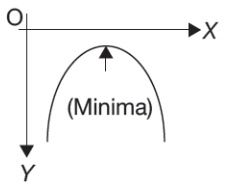
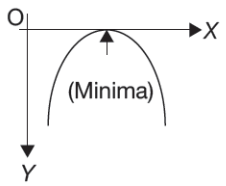
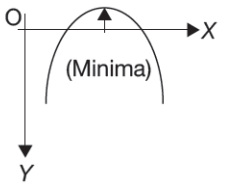
Characteristics of the Function	$b^2 - 4ac < 0$	$b^2 - 4ac = 0$	$b^2 - 4ac > 0$
When a is positive i.e., $a > 0$			
When a is negative i.e., $a < 0$			

Roots Under Particular Cases

For the quadratic equation $ax^2 + bx + c = 0$

(a) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign.
(b) If $c = 0 \Rightarrow$ one root is zero and the other root is $-b/a$.
(c) If $b = c = 0 \Rightarrow$ both roots are zero.
(d) If $a = c \Rightarrow$ roots are reciprocal to each other.
(e) If sign of $a \neq$ sign of $c \Rightarrow$ both roots are of opposite signs.
(f) If sign of a = sign of b = sign of $c \Rightarrow$ both roots are negative.
(g) If sign of a = sign of $c \neq$ sign of $b \Rightarrow$ both roots are positive.

(h) If sign of a = sign of $b \neq$ sign of $c \Rightarrow$ greater root in magnitude is negative.

(i) If sign of b = sign of $c \neq$ sign of $a \Rightarrow$ Greater root in magnitude is positive.

(j) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

(k) If $a = b = c = 0$ then equation will become an identity and will be satisfied by every value of x.

Condition for Common Roots

Let quadratic equations are $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

(a) If only one root is common:

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - b_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

(b) If both roots are common:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$\therefore$ The condition for only one root common is

$$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$$

Note

Two different quadratic equations with rational coefficients cannot have single common root, which is complex or irrational, as imaginary and surd roots always occur in conjugate pairs.

Example 37

Show that the roots of $x^2 - 2ax + a^2 - b^2 - 2bc - c^2 = 0$ are rational.

Solution

When the roots of a quadratic equation are rational, then the discriminant is a perfect square. The discriminant for $x^2 - 2ax + a^2 - b^2 - 2bc - c^2 = 0$ is $(-2a)^2 - 4(1)(a^2 - b^2 - 2bc - c^2)$

$$= 4a^2 - 4(a^2 - b^2 - 2bc - c^2)$$
$$= 4\{a^2 - a^2 + b^2 + 2bc + c^2\}$$
$$= 4(b^2 + 2bc + c^2)$$
$$= 4(b + c)^2,$$

which is a perfect square. Hence, the roots are rational. Proved **Ans.**

Example 38

If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, then show that $a : b : c = 1 : 2 : 9$

Solution

$$x^2 + 2x + 9 = 0 \quad \ldots\ldots(i)$$

$$9x^2 + bx + c = 0 \quad \ldots\ldots(ii)$$

Clearly, roots of equation (i) are imaginary (as $D < 0$).

Since equation (i) and (ii) have a common root, therefore common root must be imaginary and hence both roots will be common (since imaginary roots occur in conjugate pairs), therefore eq. (i) and (ii) are identical.

$$\Rightarrow \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{9} \Rightarrow \therefore a : b : c = 1 : 2 : 9 \quad \textbf{Ans.}$$

Example 39

If $x^2 + 2x + c = 0$ and $x^2 + 4x - 5 = 0$ have a common root, then the value of 'c' is

(A) 3 **(B)** -3

(C) 5 **(D)** -5

Solution

Let α be the common root, then it must satisfy both the equations:

$$\therefore \quad \alpha^2 + 2\alpha + c = 0 \quad \ldots\ldots(1)$$

$$\alpha^2 + 4\alpha - 5 = 0 \quad \ldots\ldots(2)$$

Aliter: Put $x = -5$ or

$$x = 1$$

in first equation to get value of c.

By cross multiplication,

$$\frac{\alpha^2}{-10 - 4c} = \frac{\alpha}{c + 5} = \frac{1}{4 - 2}$$

(i) (ii) (iii)

By (i) and (ii)

$$\alpha = \frac{-10 - 4c}{c + 5} \quad \ldots\ldots(4)$$

By (ii) and (iii)

$$\alpha = \frac{c + 5}{2} \quad \ldots\ldots(5)$$

from (4) and (5)

$$\alpha = \frac{-10 - 4c}{c + 5} = \frac{c + 5}{2}$$

$$\Rightarrow \quad -20 - 8c = (c + 5)^2$$

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$$\Rightarrow \quad -20 - 8c = c^2 + 10c + 25$$

$$\Rightarrow \quad c^2 + 18c + 45 = 0$$

$$\Rightarrow \quad c(c + 15) + 3(c + 15) = 0$$

$$\Rightarrow \quad (c + 15)(c + 3)$$

So, $c = -15$ or $c = -3$ **Ans. (B)**

Example 40

Determine the nature of roots of the equation $px^2 + 2\sqrt{pq}\,x + q = 0$ $(p \neq 0)$

(A) Imaginary and Unqual roots
(B) Rational and Equal roots
(C) Real and and Unqual roots
(D) None of these

Solution

$$px^2 + 2\sqrt{pq}\;x + q = 0$$

$$a = p,\ b = 2\sqrt{pq},\ c = q$$

$$D = b^2 - 4ac = \left(2\sqrt{pq}\right)^2 - 4 \times p \times q$$

$$D = 0$$

Since $D = 0$

Hence, roots are real and equal. **Ans. (D)**

Example 41

Show that the equation $x^2 + ax - 4 = 0$ has real and distinct roots for all real values of a.

Solution

The given equation is

$$x^2 + ax - 4 = 0$$

The discriminant of the given equation is given by

$$\therefore \quad D = a^2 - 4 \times (-4) = a^2 + 16$$

Clearly, $D = a^2 + 16 > 0$ for all $a \in R$.

Hence, the given equation has real and distinct roots. **Ans.**

Example 42

If p, q, r are real and $p \neq q$, then show that the roots of the equation $(p - q)x^2 + 5(p + q)x - 2(p - q) = 0$ are real and unequal.

Solution

The given equation is

$$(p - q)x^2 + 5(p + q)x - 2(p - q) = 0$$

Hence $a = p - q$, $b = 5(p + q)$ and $c = -2(p - q)$

$$\therefore \quad D = b^2 - 4ac = 25(p + q)^2 - 4(p - q) \times (-2(p - q))$$

$$\Rightarrow \quad D = 25(p + q)^2 + 8(p - q)^2$$

Clearly, $25(p + q)^2 > 0$ and $8(p - q)^2 > 0$

$$\therefore \quad D = 25(p + q)^2 + 8(p - q)^2 > 0.$$

Hence, roots of the given equation are real and unequal. **Ans.**

Example 43

Find the values of k for which the equation $x^2 + 5kx + 16 = 0$ has no real roots.

Solution

The given equation is $x^2 + 5kx + 16 = 0$

Here, $a = 1$, $b = 5k$ and $c = 16$

$$\therefore \quad D = b^2 - 4ac = (5k)^2 - 4 \times 1 \times 16 = 25k^2 - 64$$

The given equation will have no real roots, if $D < 0$

$$\Rightarrow \quad 25k^2 - 64 < 0$$

$$\Rightarrow \quad 25\left(k^2 - \frac{64}{25}\right) < 0$$

$$\Rightarrow \quad k^2 - \frac{64}{25} < 0 \quad [\text{Q } ab < 0 \text{ and } a > 0 \Rightarrow b < 0]$$

$$\Rightarrow \quad -\frac{8}{5} < k < \frac{8}{5} \quad [\text{Q } x^2 - a^2 < 0 \Rightarrow -a < x < a] \quad \textbf{Ans.}$$

Example 44

Show that the roots of $(x - k)(x - l) = a^2$ are always real.

Solution

For real roots, the discriminant should be ≥ 0

Now, $(x - k)(x - l) - a^2 = 0$

$$\Rightarrow \quad x^2 - x(k + l) + (kl - a^2) = 0$$

Discriminant $= (k + 1)^2 - 4(kl - a^2)$

$$= k^2 + 2kl + l^2 - 4kl + 4a^2$$

$$= k^2 + l^2 - 2kl + 4a^2 = (k - l)^2 + 4a^2$$

Since, this is the sum of two squares, its value should be ≥ 0.

Hence, the roots are real. **Ans.**

Example 45

If the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are equal, then prove that $2b = a + c$.

Solution

If the roots of the given equation are equal, then discriminant is zero, i.e.,

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$$(c-a)^2-4(b-c)(a-b)=0$$

$\Rightarrow$ $c^2+a^2-2ac+4b^2-4ab+4ac-4bc=0$

$\Rightarrow$ $c^2+a^2+4b^2+2ac-4ab-4bc=0$

$\Rightarrow$ $(c+a-2b)^2=0 \Rightarrow c+a=2b.$ **Ans.**

Note

Some of coefficients = 0 $\Rightarrow$ $x=1$ is one root.

Example 46

If the roots of the equation $x^2-8x+a^2-6a=0$ are real and distinct, then find all possible values of a.

Solution

Since the roots of the given equation are real and distinct, we must have $D>0$

$\Rightarrow$ $64-4(a^2-6a)>0 \Rightarrow 4[16-a^2+6a]>0$

$\Rightarrow$ $-4(a^2-6a-16)>0 \Rightarrow a^2-6a-16<0$

$\Rightarrow$ $(a-8)(a+2)<0$

$\Rightarrow$ $-2<a<8.$

Hence, the roots of the given equation are real if 'a' lies between -2 and 8. **Ans.**

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Example 47

Find all the integral values of a for which the quadratic equation $(x-a)(x-10)+1=0$ has integral roots.

Solution

Here, the equation is $x^2-(a+10)x+10a+1=0$. Since integral roots will means D should be a perfect square.

From (i) $D=a^2-20a+96.$

$\Rightarrow$ $D=(a-10)^2-4 \Rightarrow 4=(a-10)^2-D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a-10)^2=4$ and $D=0$.

$\Rightarrow$ $(a-10)=\pm 2 \Rightarrow a=12, 8$ **Ans.**

Example 48

In equation $\frac{x^2-bx}{ax-c}=\frac{k-1}{k+1}$ has roots equal in magnitude and opposite in sign, then the value of k is –

(A) $\frac{a+b}{a-b}$ **(B)** $\frac{a-b}{a+b}$

(C) $\frac{a}{b}+1$ **(D)** $\frac{a}{b}-1$

Solution

Let the roots are a and $-a$.

Given equation is

$$(x^2-bx)(k+1)=(k-1)(ax-c)$$

{Considering, $x \neq c/a$ and $k \neq -1$}

$\Rightarrow$ $x^2(k+1)-bx(k+1)=ax(k-1)-c(k-1)$

$\Rightarrow$ $x^2(k+1)-bx(k+1)-ax(k-1)+c(k-1)=0$

Now sum of roots = 0 $(\because \alpha-\alpha=0)$

$\therefore$ $b(k+1)+a(k-1)=0 \Rightarrow k=\frac{a-b}{a+b}$ **Ans. (B)**

Example 49

If the equation $(\lambda^2-5\lambda+6)x^2+(\lambda^2-3\lambda+2)x+(\lambda^2-4)=0$ has more than two roots, then find the value of λ?

Solution

As the equation has more than two roots, so it becomes an identity. Hence,

$\lambda^2-5\lambda+6=0 \Rightarrow \lambda=2, 3$

and $\lambda^2-3\lambda+2=0 \Rightarrow \lambda=1, 2$

and $\lambda^2-4=0 \Rightarrow \lambda=2, -2$

So $\lambda=2$ **Ans.**

Example 50

If roots of the equation $(a-b)x^2+(c-a)x+(b-c)=0$ are equal, then $a, b,$ care in

(A) AP. **(B)** H.P.

(C) G.P. **(D)** none of these

Solution

$$(a-b)x^2+(c-a)x+(b-c)=0$$

As roots are equal so

$$B^2-4AC=0\text{v}$$

$\Rightarrow$ $(c-a)^2-4(a-b)(b-c)=0$

$\Rightarrow$ $(c-a)^2-4ab+4b^2+4ac-4bc=0$

$\Rightarrow$ $(c-a)^2+4ac-4b(c+a)+4b^2=0$

$\Rightarrow$ $(c+a)^2-2\cdot(2b)(c+a)+(2b)^2=0$

$\Rightarrow$ $[c+a-2b]^2=0$

$\Rightarrow$ $c+a-2b=0$

$\Rightarrow$ $c+a=2b$

Hence a, b, c are in A P.

Alternative method:

$\because$ Sum of the coefficients = 0

Hence, one root is 1 and other root is $\frac{b-c}{a-b}$.

Given that both roots are equal, so

$$1 = \frac{b-c}{a-b}$$

$$\Rightarrow \quad a-b = b-c$$

$$\Rightarrow \quad 2b = a+c$$

Hence a, b, c are in AP. **Ans. (B)**

Example 51

The equations $5x^2 + 12x + 13 = 0$ and $ax^2 + bx + c = 0$ $(a, b, c \in R)$ have a common root, where a, b, c are the sides of the ΔABC. Then find $\angle C$.

(A) 45° **(B)** 60°
(C) 90° **(D)** 30°

Solution

As we can see discriminant of the equation $5x^2 + 12x + 13 = 0$ is negative so roots of the equation are imaginary. We know that imaginary roots always occurs in pair. So this equation can not have single common roots with any other equation having real coefficients. So both roots are common of the given equations.

Hence, $\frac{a}{5} = \frac{b}{12} = \frac{c}{13} = \lambda$(let)

then $a = 5\lambda$, $b = 12\lambda$, $c = 13\lambda$

$\Rightarrow$ a, b, c forms pythagoras triplet

$\therefore \quad \angle C = 90°$ **Ans. (C)**

Example 52

Find the values of a for which the expression $\frac{ax^2 + 3x - 4}{3x + 4x^2 + a}$ assumes all real values for real values of x.

Solution

Let $y = \frac{ax^2 + 3x - 4}{3x + 4x^2 + a}$

$$x^2(a + 4y) + 3(1 - y)x - (4 + ay) = 0$$

If $x \in R$, $D \geq 0$

$$\Rightarrow \quad 9(1-y)^2 + 4(a+4y)(4+ay) \geq 0$$

$$\Rightarrow \quad (9+16a)y^2 + (4a^2+46)y + (9+16a) \geq 0$$

for all $y \in R$, $(9 + 16a) > 0$ and $D \leq 0$

$$\Rightarrow \quad (4a^2+46)^2 - 4(9+16a)(9+16a) \leq 0$$

$$\Rightarrow \quad 4(a^2 - 8a + 7)(a^2 + 8a + 16) \leq 0$$

$$\Rightarrow \quad a^2 - 8a + 7 \leq 0 \Rightarrow 1 \leq a \leq 7$$

$$9 + 16a > 0 \text{ and } 1 \leq a \leq 7$$

Taking intersection, $a \in$ [l, 7]

Now, checking the boundary values of a

For $a = l$

$$y = \frac{x^2 + 3x - 4}{3x + 4x^2 + 1} = -\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$\because \quad x \neq 1 \Rightarrow y \neq -1$

$\Rightarrow a = 1$ is not possible

if $a = 7$

$$y = \frac{7x^2 + 3x - 4}{3x - 4x^2 + 7} = \frac{(7x-4)(x+4)}{(7-4x)(x+1)} \quad \because x \neq -1 \Rightarrow y \neq -1$$

So y will assume all real values for some real values of x.

So $a \in (1, 7)$ **Ans.**

Example 53

If a. is a root of the equation $ax^2 + bx + c = 0$ and β is a root of the equation $-ax^2 + bx + c = 0$, then prove a that there will be a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ lying between α and β.

Solution

Let $f(x) = \frac{a}{2}x^2 + bx + c$

$$f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c = a\alpha^2 + b\alpha + c - \frac{a}{2}a^2$$

$$= -\frac{a}{2}a^2 \qquad (\text{As } \alpha \text{ is a root of } ax^2 + bx + c = 0)$$

And

$$f(\beta) = \frac{a}{2}\beta^2 + b\beta + c = -a\beta^2 + b\beta + c + \frac{3a}{2}\beta^2$$

$$= \frac{3a}{2}\beta^2 \qquad (\text{As } \beta \text{ is a root of } -ax^2 + bx + c = 0)$$

Now $f(\alpha) \cdot f(\beta) = \frac{-3}{4}a^2\alpha^2\beta^2 < 0$

$\Rightarrow f(x) = 0$ has one real root between α and β. **Ans.**

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Know Your Progress - 4

1. Find the value of k for which quadratic equation $(k-2)x^2 + 2(2k-3)x + 5k - 6 = 0$ has equal roots.

2. If the roots of the equations $px^2 + 2qx + r = 0$ and $qx^2 - 2\sqrt{pr}\ x + q = 0$, then prove that $q^2 = pr$.

3. If the roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in G.P., then find the roots.

4. If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real and distinct, then find all possible values of a .

5. For what value of k, $(4-k)x^2 + (2k+4)x + (8k+1)$ is a perfect square.

6. For the following equations, find the nature of the roots (real and distinct, real, and coincident or imaginary).
   (a) $x^2 - 6x + 10 = 0$
   (b) $x^2 - (7 + \sqrt{3})x + 6(1 + \sqrt{3}) = 0$
   (c) $4x^2 + 28x + 49 = 0$

7. If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

8. Consider $f(x) = x^2 + bx + c$.
   (a) Find c if $x = 0$ is a root of $f(x) = 0$.
   (b) Find c if $\alpha, \frac{1}{\alpha}$ are roots of $f(x) = 0$.
   (c) Comment on sign of b and c, if $\alpha < 0 < \beta$ and $|\beta| > |\alpha|$, where α, β are roots of $f(x) = 0$.

9. If $x^2 + bx + c = 0$ and $2x^2 + 9x + 10 = 0$ have both roots common, then find b and c.

10. If $x^2 - 7x + 10 = 0$ and $x^2 - 5x + c = 0$ have a common root, find c.

11. The roots of $x^2 - x + k^2 = 0$ are not real. Find k.
   (A) $k < -\frac{1}{2}, k > \frac{1}{2}$
   (B) $k > -\frac{1}{2}, k < \frac{1}{2}$
   (C) $k < -\frac{3}{2}, k > \frac{1}{2}$
   (D) None of these

12. For how many positive integers n between 1 to 100 expression $x^2 + x - n$ can be factorize into two linear factors with integer coefficient -
   (A) 7 **(B)** 8
   (C) 9 **(D)** None of these

13. The roots of $a^2x^2 + abx = b^2$, $a, b \neq 0$ are:
   (A) equal **(B)** non real
   (C) unequal **(D)** none of these

14. The number of integral values of 'm' less than 50, so that the roots of the quadratic equation $mx^2 + (2m-1)x + (m-2) = 0$ are rational, are
   (A) 6 **(B)** 7
   (C) 8 **(D)** None of these

15. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$ where $(a \neq b \neq c)$
   (A) 0 **(B)** positive
   (C) negative **(D)** none of these

16. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + (c^2 + d^2) = 0$ are equal, then:
   (A) $\frac{a}{b} = \frac{c}{d}$ **(B)** $\frac{a}{c} = \frac{b}{d} = 0$
   (C) $ab = cd$ **(D)** $a^2 + b^2 = c^2 + d^2$

17. Which of the following quadratic expression can be expressed as a product of real factors?
   (A) $x^2 - 2x + 3$ **(B)** $3x^2 - 2x + 3$
   (C) $2x^2 + 3x - 4$ **(D)** $2x^2 - 5x + 3$

18. The quadratic in t, such that A.M. of its roots is A and G.M. is G, is
   (A) $t^2 - 2At + G^2 = 0$ **(B)** $t^2 - 2At - G^2 = 0$
   (C) $t^2 + 2At + G^2 = 0$ **(D)** None of these

19. For what values of 'a', the equations $1998x^2 + ax + 8991 = 0$, and $8991x^2 + ax + 1998 = 0$ have a common root
   (A) ± 10989 **(B)** ± 1
   (C) 0 **(D)** None of these

20. If $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ has equal roots, find the relation among a, b, and c.
   (A) A.P. **(B)** G.P.
   (C) H.P. **(D)** None of these

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Maximum and Minimum Value of Quadratic Expression

Range of Quadratic Expression

$$ax^2 + bx + c$$

(a) If $a > 0$, quadratic expression has least value at $x = -\frac{b}{2a}$. This least value is given by

$$\frac{4ac - b^2}{4a} = -\frac{D}{4a}$$

(b) If $a < 0$, quadratic expression has greatest value at $x = -\frac{b}{2a}$. This greatest value is given by

$$\frac{4ac - b^2}{4a} = -\frac{D}{4a}$$

Range of an Algebraic Expression

Let $$y = \frac{ax^2 - bx + c}{px^2 + qx + r} \qquad (1)$$

To find the range of the given expression, we first solve it for x.

$$x^2(py - a) + x(qy - b) + (ry - c) = 0 \qquad \ldots (2)$$

For real value of x, D of (2) should be greater than or equal to zero.

$$D = (qy - b)^2 - 4(py - a)(ry - c) \geq 0 \qquad \ldots (3)$$

Those values of y which will satisfy (3) are called range of the function.

Some Results to Remember

(i) $ax - b > 0 \Rightarrow x > b/a$ if $a > 0$ and $x < b/a$ if $a < 0$
(ii) $x^2 - a^2 > 0 \Rightarrow x < -a$ or $x > a$
(iii) $x^2 - a^2 \geq 0 \Rightarrow x \leq -a$ or $x \geq a$
(iv) $x^2 - a^2 < 0 \Rightarrow -a < x < a$
(v) $(x - a)(x - b) > 0,\ a < b \Rightarrow x <$ a or $x > b$
(vi) $(x - a)(x - b) < 0,\ a < b \Rightarrow a < x < b$

Example 54

Find the minimum value of $3x^2 - 2x + 1$.

Solution

If $a > 0$, the minimum value of $ax^2 + bx + c$ is

$$\frac{4ac - b^2}{4a} \text{ at } x = \frac{-b}{2a}$$

Here $a = 3$, $b = -2$, $c = 1$

$$\text{Minimum value} = \frac{4(3)(1) - (-2)^2}{4(3)} = \frac{12 - 4}{12} = \frac{2}{3}$$ **Ans.**

Example 55

If x be real, find the maximum value of $\frac{x + 2}{2x^2 + 3x + 6}$

Solution

Let $y = \frac{x + 2}{2x^2 + 3x + 6}$

then, $2x^2y + (3y - 1)x + 6y - 2 = 0$

for x to be real, $(3y - 1)^2 - 8y(6y - 2) \geq 0$

or $$(1 + 13y)(1 - 3y) \geq 0$$

or $$(13y + 1)(3y - 1) \leq 0$$

Putting each factor equal to zero, we get

$$y = \frac{-1}{13}, \frac{1}{3}$$

If, $y < \frac{-1}{13}$, $(1 + 13y)(1 - 3y) < 0$

If, $\frac{-1}{13} < y < \frac{1}{3}$, $(1 + 13y)(1 - 3y) > 0$

If, $y > \frac{1}{3}$ $(1 + 13y)(1 - 3y) < 0$

Thus, y will lie between $-\frac{1}{13}$ and $\frac{1}{3}$.

Hence, the maximum value of y is $\frac{1}{3}$ and minimum value is $\frac{-1}{13}$. **Ans.**

Example 56

The value of the expression $x^2 + 2bx + c$ will be positive for all real x if:

(A) $b^2 - 4c > 0$ **(B)** $b^2 - 4c < 0$
(C) $c^2 < b$ **(D)** $b^2 < c$

Solution

As $a > 0$, so this expression will be positive if $D < 0$
so $4b^2 - 4c < 0$

$$b^2 < c$$ **Ans. (D)**

Example 57

The minimum value of the expression $4x^2 + 2x + 1$ is -

(A) 1/4 **(B)** 1/2
(C) 3/4 **(D)** 1

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Solution

Since $a = 4 > 0$, therefore its minimum value is $= \frac{4(4)(1)-(2)^2}{4(4)} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$ **Ans. (C)**

Example 58

If $y = x^2 - 2x - 3$, then find the range of y when:

(i) $x \in R$

(ii) $x \in [0, 3]$

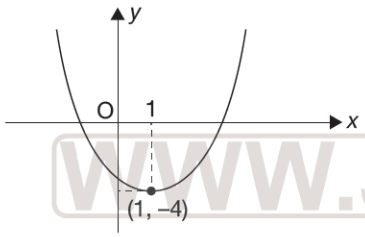
(iii) $x \in [-2, 0]$

Solution

We know that minimum value of y will occur at

$$x = -\frac{b}{2a} = -\frac{(-2)}{2\times 1} = 1$$

$$y_{\min} = -\frac{D}{4a} = -\frac{-(4+3\times 4)}{4} = -4$$



(i) $x \in R$;

$$y \in [-4, \infty)$$

(ii) $x \in (0, 3)$

$$f(0) = -3, f(1) = -4, f(3) = 0$$

$\because$ $f(3) > f(0)$

$\therefore$ y will take all the values from minimum to $f(3)$.

$$y \in [-4, 0]$$

(iii) $x \in [-2, 0]$

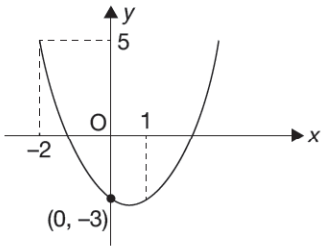
This interval does not contain the minimum value of y for $x \in R$.

y will take values from $f(0)$ to $f(-2)$

$$f(0) = -3$$

$$f(-2) = 5$$

$$y \in [-3, 5]$$ **Ans.**



Example 59

For $x \in R$, find the set of values attainable by $\frac{x^2-3x+4}{x^2+3x+4}$.

Solution

Let $y = \frac{x^2-3x+4}{x^2+3x+4}$

$$x^2(y-1) + 3x(y+1) + 4(y-1) = 0$$

Case 1: $y \neq 1$

For $y \neq 1$ above equation is a quadratic equation.

So for $x \in R$, $D \geq 0$

$$\Rightarrow \quad 9(y+1)^2 - 16(y-1)^2 \geq 0$$

$$\Rightarrow \quad 7y^2 - 50y + 7 \leq 0$$

$$\Rightarrow \quad (7y - 1)(y - 7) \leq 0$$

$$\Rightarrow \quad y \in \left[\frac{1}{7}, 7\right] - \{1\}$$

Case 2: when $y = 1$

$$\Rightarrow \quad 1 = \frac{x^2-3x+4}{x^2+3x+4}$$

$$\Rightarrow \quad x^2 + 3x + 4 = x^2 - 3x + 4$$

$$\Rightarrow \quad x = 0$$

Hence $y = 1$ for real value of x.

so range of y is $\left[\frac{1}{7}, 7\right]$. **Ans.**

Example 60

If $b^2 < 2ac$ and $a, b, c, d \in R$, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution

Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$\alpha^2 + \beta^2 + \gamma^2 = (a + b + g)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 < 0,$$

which is not possible if all a, p, y are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots. **Ans.**

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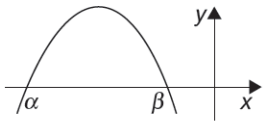
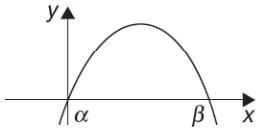
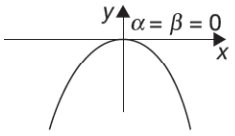
Know Your Progress - 5

1. Find the minimum value of:
   (a) $y = x^2 + 2x + 2$
   (b) $y = 4x^2 - 16x + 15$

2. For following graphs of $y = ax^2 + bx + c$ with $a, b, c \in R$, comment on the sign of:
   (i) a (ii) b (iii) c
   (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$

   (1)
   

   (2)
   

   (3)
   

3. Given the roots of equation $ax^2 + bx + c = 0$ are real and distinct, where $a, b, c \in R^+$, then the vertex of the graph will lie in which quadrant.

4. Prove that the expression $\dfrac{8x-4}{x^2+2x-1}$ cannot have values between 2 and 4, in its domain.

5. Find the range of $\dfrac{x^2+2x+1}{x^2+2x+7}$, where x is real.

Miscellaneous Equations

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Miscellaneous equations will be dealt with following examples and theory:

Example 60

Solve $(x-5)(x-7)(x+6)(x+4) = 504$.

Solution

We have $(x^2 - x - 20)(x^2 - x - 42) = 504$;

Which, being arranged as a quadratic in $x^2 - x$, given.

$$(x^2 - x)^2 - 62(x^2 - x) + 336 = 0;$$

$$\therefore \quad (x^2 - x - 6)(x^2 - x) + 336 = 0;$$

$$\therefore \quad x^2 - x - 6 = 0, \text{ or } x^2 - x - 56 = 0;$$

whence $x = 3, -2, 8, -7$.

$$x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$$

Add 3 to each side ; then $x^2 - 5x + 3 + 2\sqrt{x^2 - 5x + 3} = 15$.

Putting $\sqrt{x^2 - 5x + 3} = y$, we obtain $y^2 + 2y - 15 = 0$; whence $y = 3$ or -5.

Thus, $\sqrt{x^2 - 5x + 3} = +3$

or $\sqrt{x^2 - 5x + 3} = -5$.

Squaring, and solving the resulting quadratics, we obtain from the first $x = 6$ or -1; and from the second $x = \dfrac{5 \pm \sqrt{113}}{2}$. The first pair of values satisfies the given equation, but the second pair satisfies the equation.

$$x^2 - 5x - 2\sqrt{x^2 - 5x + 3} = 12.$$

Before clearing an equations of radicals it is advisable to examine whether any common factor can be remove by division. **Ans.**

Example 61

Solve $\sqrt{x^2 - 7ax + 10a^2} - \sqrt{x^2 + ax - 6a^2} = x - 2a$.

Solution

We have $\sqrt{(x-2a)(x-5a)} - \sqrt{(x-2a)(x+3a)} = x - 2a$.

The factor $\sqrt{x-2a}$ can now be removed from every term;

$$\therefore \quad \sqrt{x-5a} - \sqrt{x+3a} = \sqrt{x-2a};$$

$$x - 5a + x + 3a - 2\sqrt{(x-5a)(x+3a)} = x - 2a;$$

$$x = 2\sqrt{x^2 - 2ax - 15a^2};$$

$$3x^2 - 8ax - 60a^2 = 0;$$

$$(x - 6a)(3x + 10a) = 2;\ x = 6a,\ \text{or}\ -\frac{10a}{3}.$$

Also by equating to zero the factor $\sqrt{x - 2a}$, we obtain $x = 2a$.

On trial it will be found that $x = 6a$ does not satisfy the equations; thus the roots are $-\frac{10a}{3}$ and $2a$. The student may compare a similar questions discussed in the Elementary algebra. Art 281.

The following artifice is sometimes useful. **Ans.**

Example 62

Solve $\sqrt{3x^2 - 4x + 34} + \sqrt{3x^2 - 4x - 11} = 9$ ...(1)

Solution

We have identically

$$(3x^2 - 4x + 34) - (3x^2 - 4x - 11) = 45 \quad ...(2)$$

Divide each member of (2) by the corresponding member of (1); thus

$$\sqrt{3x^2 - 4x + 34} - \sqrt{3x^2 - 4x - 11} = 5 \quad ...(3)$$

Now (2) is an identical equation true for all values of x, whereas (1) is an equations, which is true only for certain values of x; hence also equations (3) is only true for these values of x.

From (1) and (3) by addition, $\sqrt{3x^2 - 4x + 34} = 7$;

Whence $$x = 3,\ \text{or}\ -\frac{5}{3}.$$

The solution of an equation of the form

$$ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0,$$

in which the coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of quadratic. Equations of this type are known as reciprocal equations, and are some named because they are not altered when x is changed into its reciprocal. $\frac{1}{x}$. For a more complete discussion of reciprocal equations, the student is referred to Arts. 568-570. **Ans.**

Example 63

Solve $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

Solution

Dividing by x^2 and rearranging.

$$12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0.$$

Put $$x + \frac{1}{x} = z;\ \text{then}\ x^2 + \frac{1}{x^2} = z^2 - 2;$$

$$\therefore \quad 12(z^2 - 2) - 56z + 89 = 0;$$

whence we obtain

$$z = \frac{5}{2},\ \text{or}\ \frac{13}{6}.$$

$$\therefore \quad x + \frac{1}{x} = \frac{5}{2},\ \text{or}\ \frac{13}{6}.$$

By solving these equations, we find that $x = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}$.

When one root of a quadratic equation is obvious by inspection, the other root my often by readily obtained by making use of the properties of the roots of quadratic equations proved in Art. 114. **Ans.**

Example 64

Solve $(1 - a^2)(x + a) - 2a(1 - x^2) = 0$.

Solution

This is a quadratic, one of whose roots is clearly a.

Also, since the equation may be written

$$2ax^2 + (1 - a^2)x - a(1 + a^2) = 0.$$

the product of the roots is $-\frac{1 + a^2}{2}$; and therefore the other root is $-\frac{1 + a^2}{2a}$.

We shall now discuss some simultaneous equations of two unknown quantities. **Ans.**

Example 65

Solve $x + 2 + y + 3 + \sqrt{(x + 2)(y + 3)} = 39$.

Solution

$$(x + 2)^2 + (y + 3)^2 + (x + 2)(y + 3) = 741.$$

Put $x + 2 = u$, and $y + 3 = v$; then

$$u + v + \sqrt{uv} = 39, \quad ...(1)$$

$$u^2 + v^2 + uv = 741, \quad ...(2)$$

Hence , from (1) and (2), we obtain by division,

$$u + v - \sqrt{uv} = 19 \quad ...(3)$$

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From (1) and (3) $u+v=29$;

and $\sqrt{uv}=10$ or $uv=100$;

whence $u=25$, or 4; v =4, or 25;

thus $x=53$, or 2; $y=1$, or 22 **Ans.**

Example 66

Solve $x^4+y^4=82$.

Solution

$$x-y=2.$$

Put $x=u+v$, and $y=u-v$;

then from (2) we obtain $v=1$.

Substituting in (1),

$$(u+1)^4+(u-1)^4=82;$$

$$\therefore \quad 2(u^4+6u^2+1)=82;$$

$$u^4+6u^2-40=0;$$

whence $u^2=4$, or -10;

and $u=\pm 2$, or $\pm\sqrt{-10}$.

Thus, $x=3,-1\pm\sqrt{-10}$;

$y=1,-3,-1\pm\sqrt{-10}$. **Ans.**

Example 67

Solve $\dfrac{2x+y}{3x-4}-\dfrac{x-y}{x+y}=2\dfrac{8}{15}$. ...(1)

Solution

$$7x+5y=29. \quad ...(2)$$

From (1),

$$15(2x^2-3xy+y^2-3x^2+4xy-y^2)=38(3x^2+2xy-y^2);$$

$$\therefore \quad 129x^2\,29xy-38y^2=0;$$

$$\therefore \quad (3x-2y)(43x+19y)=0.$$

Hence $3x=2y$. ...(3)

or $43x=-19y$, ...(4)

From (3), $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{7z+5y}{29}=1$, by equation (2)

$$\therefore \quad x=2, y=3.$$

Again, From (4), $\dfrac{x}{19}=\dfrac{y}{-43}=\dfrac{7x+5y}{-82}=\dfrac{29}{82}$, by equation (2)

$$\therefore \quad x=-\frac{551}{82}, y=\frac{1247}{82}.$$

Hence, $x=2, y=3$; or $x=-\dfrac{551}{82}, y=\dfrac{1247}{82}$. **Ans.**

Example 68

Solve $4x^3+3x^2y+y^3=8$.

Solution

$$2x^3-2x^2y+xy^2=1.$$

Put $y=mx$, and substitute in both equations. Thus,

$$x^3(4+3m+m^3)=8 \quad ...(1)$$

$$x^3(2-2m+m^2)=1 \quad ...(2)$$

$$\therefore \quad \frac{4+3m+m^3}{2-2m+m^2}=8;$$

$$m^3-8m^2+19m-12=0;$$

That is, $(m-1)(m-3)(m-4)=0$;

$$\therefore \quad m=1, \text{ or } 3, \text{ or } 4.$$

(i) Take $m=1$, and substitute in either (1) or (2).

From (2), $x^3=1$;

$\therefore\ x=1$; and $y=mx=x=1$.

(ii) Take $m=3$; and substitute in (2);

thus $5x^3=1$;

$$\therefore \quad x=\sqrt[3]{\frac{1}{5}}; \text{ and } y=mx=3x=3\sqrt[3]{\frac{1}{5}}.$$

(iii) Take $m=4$; we obtain, $10x^3=1$;

$$\therefore \quad x=\sqrt[3]{\frac{1}{10}}; \text{ and } y=mx=4x=4\sqrt[3]{\frac{1}{10}}.$$

Hence the complete solution is, $x=1, \sqrt[3]{\frac{1}{5}}, \sqrt[3]{\frac{1}{10}}$.

$y=1, 3\sqrt[3]{\frac{1}{5}}, 4\sqrt[3]{\frac{1}{10}}$. **Ans.**

Example 69

Solve $31x^2y^2-7y^4-112xy+64=0$ ...(1)

Solution

$$x^2-7xy+4y^2+8=0 \quad ...(2)$$

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From (2) we have $-8 = x^2 - 7xy + 4y^2$; and substituting in (1),

$$31x^2y^2 - 7y^4 + 14xy(x^2 - 7xy + 4y^2) + (x^2 7xy + 4y^2)^2 = 0;$$

$$\therefore\ 31x^2y^2 - 7y^4 + (x^2 - 7xy + 4y^2)(14xy + x^2 - 7xy + 4y^2) = 0$$

$$\therefore\quad 31x^2y^2 - 7y^4 + (x^2 + 4y^2)^2 - (7xy)^2 = 0;$$

that is, $$x^4 - 10x^2y^2 + 9y^4 = 0 \qquad ...(3)$$

$$\therefore\quad (x^2 - y^2)(x^2 - 9y^2) = 0;$$

Hence $$x = \pm y, \text{ or } x = \pm 3y.$$

Taking these cases in succession and substituting in (2), we obtain

$$x = y = \pm 2, = -y = \pm\sqrt{-\frac{2}{3}};$$

$$x = \pm 3,\ y = \pm 1;$$

$$x = \pm 3\sqrt{-\frac{4}{17}},\ y = \pm\sqrt{\frac{4}{17}}. \qquad \textbf{Ans.}$$

Note

It should be observed that equation (3) is homogeneous. The method here employed by which one equation is made homogeneous by a suitable combination with the other is a valuable artifice. It is especially useful in Analytical Geometry.

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Example 70

Solve $(x+y)^{\frac{2}{3}} + 2(x-y)^{\frac{2}{3}} = 3(x^2 - y^2)^{\frac{1}{3}}$ ...(1)

Solution

$$3x - 2y = 13 \qquad ...(2)$$

Divide each term of (1) by $(x^2 - y^2)^{\frac{1}{3}}$, or, $(x+y)^{1/3}(x-y)^{\frac{1}{3}}$;

$$\therefore\quad \left(\frac{x+y}{x-y}\right)^{\frac{1}{3}} + 2\left(\frac{x-y}{x+y}\right)\frac{1}{3} = 3.$$

This equation is a quadratic in $\left(\frac{x+y}{x-y}\right)^{\frac{1}{3}}$, from which we easily find,

$$\left(\frac{x+y}{x-y}\right)^{\frac{1}{3}} = 2 \text{ or } 1; \text{ whence } \frac{x+y}{x-y} = 8 \text{ or } 1;$$

$$\therefore\quad 7x = 9y, \text{ or } y = 0.$$

Combining these equations with (2), we obtain

$$x = 9,\ y = 7; \text{ or } x = \frac{13}{3}, y = 0.$$

Equation involving three or more unknown quantities can only be solved in special cases. We shall here consider some of the most useful methods of solution. **Ans.**

Example 71

Solve $x + y + z = 13.$ ...(1)

Solution

$$x^2 + y^2 + z^2 = 65 \qquad ...(2)$$

$$xy = 10 \qquad ...(3)$$

From (2) and (3),

$$(x+y)^2 + z^2 = 85.$$

Put u for $x + y$; then this equation becomes

$$u^2 + z^2 = 85.$$

Also from (1) $u + z = 13;$

whence we obtain $u = 7$ or 6; $z = 6$ or 7.

Thus, we have $\left.\begin{matrix} x+y=7 \\ xy=10 \end{matrix}\right\}$ and $\left.\begin{matrix} x+y=6, \\ xy=10 \end{matrix}\right\}$

Hence, the solution are

$$\left.\begin{matrix} x = 5, \text{ or } 2, \\ y = 2, \text{ or } 5, \\ z = 6 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x = 3 \pm = 6, \\ y = 3 \pm \sqrt{-1}, \\ z = 7. \end{matrix}\right\} \qquad \textbf{Ans.}$$

Example 72

Solve $y^2 - x^2 + z(y - x) = 30.$ ...(1)

Solution

$$z^2 + zx + x^2 = 19 \qquad ...(2),$$

$$x^2 + xy + y^2 = 39 \qquad ...(3).$$

Subtracting (2) from (1)

$$y^2 - x^2 + z(y - x) = 30;$$

that is, $$(y - x)(x + y + z) = 30 \qquad ...(4)$$

Similarly, from (1) and (3) $(z - x)(x + y + z) = 10$ ...(5)

Hence, from (4) and (5), by division $\frac{y - x}{z - x} = 3;$

whence $y = 3z - 2x$.

Substituting in equation (3), we obtain

$$x^2 - 3xz + 3z^2 = 13.$$

From (2),

$$x^2 + xz + z^2 = 19.$$

Solving these homogeneous equations as in Example 4, Art. 136, we obtain

$$x = \pm 2,\ z = \pm 3;\ \text{and therefore}\ \ y = \pm 5;$$

$$x = \pm\frac{11}{\sqrt{7}},\ z = \pm\frac{1}{\sqrt{7}};\ \text{and therefore}\ \ y = \mp\frac{19}{\sqrt{7}}.\ \textbf{Ans.}$$

Example 73

Solve $x^2 - yz = a^2,\ y^2 - zx = b^2,\ z^2 - xy = c^2$

Solution

Multiply the equations by y, z, x, respectively, and add; then

$$c^2x + a^2y + b^2z = 0 \qquad \ldots(1)$$

Multiply the equations by z, x, y, respectively, and add; then

$$b^2x + c^2y + a^2z = 0 \qquad \ldots(2)$$

From (1) and (2) by by cross multiplication,

$$\frac{x}{a^4 - b^2c^2} = \frac{y}{b^4 - c^2a^2} = \frac{z}{c^4 - a^2b^2} = \text{suppose.}$$

Substitute in any one of the given equations, then

$$k^2(a^6 + b^6 + c^6 - 3a^2b^2c^3) = 1;$$

$$\therefore \quad \frac{x}{a^4 - b^2c^2} = \frac{y}{b^4 - c^2a^2} = \frac{z}{c^4 - a^2b^2} = \pm\frac{1}{\sqrt{a^6 + b^6 + c^6 - 3a^bb^2c^3}}.\quad \textbf{Ans.}$$

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Know Your Progress - 6

1. If a, b, c are real numbers such that $a + \left(\frac{1}{b}\right) = \frac{7}{3};\ b + \left(\frac{1}{c}\right) = 4;\ c + \left(\frac{1}{a}\right) = 1$, find abc.

2. If a, b, c, d, satisfy the equations $a + 7b + 3c + 5d = 0$, $8a + 4b + 6c + 2d = -16$, $2a + 6b + 4c + 8d = 16$, $5a + 3b + 7c + d = -16$, then the value of $(a + d)(b + c) =$

3. If $x + y = 5xy,\ y + z = 6yz,\ z + x = 7zx$, find the value of $x + y + z$.

4. Solve the system of equations

$$\begin{cases} x - y + z = 1 \\ y - z + u = 2 \\ z - u + v = 3 \\ u - v + x = 4 \\ v - x + y = 5 \end{cases}$$

5. Let f be a function with the following properties:
   (i) $f(1) = 1$, and
   (ii) $f(2n) = n \times f(n)$. for any positive integer n.
   If the value of $f(2^{100}) = 2^{49xy}$, then the digits x and y, respectively?

6. R nonnegative integers can be written in the form
   $a_7.3^7 + a_6.3^6 + a_5.3^5 + a_4.3^4 + a_3.3^3 + a_2.3^2 + a_1.3^1 + a_0.3^0$,
   where $a_1 \in \{-1, 0, 1\}$ for $\le I \le 7$. What is the last two digits of R?

7. Let a, b, and c be real numbers such that $a + b + c = 2$, and $a^2 - b^2 - c^2 = 12$.
   The difference between the maximum and minimum possible values of c is $\frac{p}{q}$ with coprime p and q.
   What is $(p + q^2)$?

8. For integers $a_1, \ldots, a_n \in \{1, 2, 3, \ldots, 9\}$, we use the notation $\overline{a_1a_2 \ldots a_n}$ to denote the number $10^{n-1}\,a_1 + 10^{n-2}\,a_2 + \ldots + 10a_{n-1} + a_n$. For example, when $a = 2$ and $b = 0$, $\overline{ab} = b2$ and $\overline{acbc} = (\overline{ba})^2$. Find the value of $a + b + c$.

Functional Equations

A functional equation is an equation whose variables are ranging over functions and our aim is to find all possible functions satisfying the equation. There is no fixed method to solve a functional equation few standard approaches as follows.

Substitution of Variable/Function

This is most common method for solving functional equations. By substitution, we get simplified form or some time some additional information regarding equation. We replace old variable with new variable by keeping domain of old variable unchanged. See the following examples.

Example 74

Let $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be such that $f\left(1+\frac{1}{x}\right) = x^2+\frac{1}{x^2}$ $\forall x \in \mathbb{R}\setminus\{0\}$, find $f(x)$.

Solution

Let $y = 1+\frac{1}{x} \Rightarrow x = \frac{1}{y-1}$

$\Rightarrow \quad f(y) = \left(\frac{1}{y-1}\right)^2 + (y-1)^2 \quad \forall y \in \mathbb{R}-\{1\}.$ **Ans.**

Example 75

$f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ such that $2f(x)+3f\left(\frac{1}{x}\right) = x;\ \forall \in \mathbb{R}\setminus\{0\}$, find $f(x)$.

Solution

Replace x by $\frac{1}{x}$, we get $2f\left(\frac{1}{x}\right)+3f(x) = \frac{1}{x}$

Now by eliminating $f\left(\frac{1}{x}\right)$ from the two equations, we get

$$(9-4)f(x) = \frac{3}{x} - 2x$$

$\Rightarrow \quad f(x) = \frac{3-2x^2}{5x}$ **Ans.**

Example 76

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $x^2f(x) + f(1-x) = 2x - x^4$; $\forall x \in \mathbb{R}$

Solution

Replace x by $(1-x)$, we get

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4$$

Now elimination $f(1-x)$ from the two equations, we get $f(x) = 1-x^2$. **Ans.**

Example 77

$f : \mathbb{R}-\{0,1\} \to \mathbb{R}, f(x) + f\left(\frac{x+1}{x}\right) = 1+x$ find $f(x)$

Solution

Replacing x by $\frac{x-1}{x}$, we get

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{\frac{x-1}{x}-1}{\frac{x-1}{x}}\right) = 1+\frac{x-1}{x}$$

or $$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = \frac{2x-1}{x} \qquad ...(1)$$

Again replacing x by $\frac{1}{1-x}$ in parent equation, we get

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}\right) = 1+\frac{1}{1-x} = \frac{2-x}{1-x}$$

$$f\left(\frac{1}{1-x}\right) + f(x) = \frac{2-x}{1-x} \qquad ...(2)$$

By adding parent equation + Eq. (2) and subtracting Eq. (1), we get

$$2f(x) = 1+x+\frac{2-x}{1-x} - \frac{2x-1}{x} \Rightarrow f(x) = \frac{x^3-x^2-1}{2x(x-1)}$$

Ans.

Isolation of Variables

We try to bring all functions of x to one side and all functions of y on other side. For some particular type of problems this works wonderfully. See the following examples.

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Example 78

Find $f(x)$ such that $xf(y) = yf(x) \forall x, y \in \mathbb{R} - \{0\}$.

Solution

$$xf(y) = yf(x)$$

$$\Rightarrow \qquad \Rightarrow \frac{f(x)}{x} \frac{f(y)}{y}$$

As x, y are independent of each other

$$\Rightarrow \qquad \frac{f(x)}{x} = \text{Constant} = c$$

$$\Rightarrow \qquad f(x) = cx. \qquad \textbf{Ans.}$$

Know Your Progress - 7

1. Find $f : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$, such that
$$f\left(\frac{x}{x-1}\right) = 2f\left(\frac{x-1}{x}\right); \forall x \in \mathbb{R} \setminus \{0,1\}.$$
2. Find $f : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$, such that
$$f(x) + f\left(\frac{1}{1-x}\right) = x; \forall x \in \mathbb{R} \setminus \{0,1\}.$$
3. $f(x^2 + x) + 2f(x^2 - 3x + 2) = 9x^2 - 15x; \forall x \in \mathbb{R}$, find $f(2016)$.
4. Find $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) + xf(1 - x) = 1 + x; \forall x \in \mathbb{R}$.
5. $f : \mathbb{R} \to \mathbb{R}, f(x + y) + f(x - y) = 2f(x)\cos y; \forall x, y \in \mathbb{R}$, find all such functions.
6. Find al functions $f : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$, such that
$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}; \forall x \in \mathbb{R} = \{0,1\}.$$
7. Find all functions $f : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$, such that
$$f(x) + 2f\left(\frac{1}{x}\right) + 3f\left(\frac{x}{x-1}\right) = x.$$

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. If p and q are the roots of the equation $x^2 + px + q = 0$, then:
(A) $p = 1$ **(B)** $p = 1$ or 0
(C) $p = -2$ **(D)** $p = -2$ or 0

2. If the equation $\frac{a}{x-a} + \frac{b}{x-b} = 1$ has roots equal in magnitude but opposite in sign, then the value of $a + b$ is:
(A) -1 **(B)** 0
(C) 1 **(D)** none of these

3. The roots of the quadratic equation $(a + b - 2c)\,x^2 - (2a - b - c)\,x + (a - 2b + c) = 0$ are:
(A) $a + b + c$ and $a - b + c$
(B) $1/2$ and $a - 2b + c$
(C) $a - 2b + c$ and $1/(a + b - 2c)$
(D) none of these

4. The least integral value of 'm' for which the expression $m\,x^2 - 4\,x + 3m + 1$ is positive for every $x \in R$ is:
(A) 1 **(B)** -2
(C) -1 **(D)** 2

5. If the quadratic equations, $ax^2 + 2cx + b = 0$ and $ax^2 + 2bx + c = 0$ $(b \neq c)$ have a common root, then $a + 4b + 4c$ is equal to:
(A) -2 **(B)** -2
(C) 0 **(D)** 1

6. If $x = \cfrac{1}{1 + \cfrac{1}{x + \cfrac{1}{x + ----}}}$, then the value of x will be
(A) $\frac{2}{3}$ **(B)** $\frac{1}{2}$
(C) $\frac{\sqrt{3}}{2}$ **(D)** $\frac{3}{2}$

7. The solution of the equation $4^x + 2^x = 6$ is/are:
(A) $0, 1$ **(B)** $2, 1$
(C) $1, 0$ **(D)** 1

8. The equation $x + 1 + x - 1 = 0$ has:
(A) one real solution **(B)** two real solutions
(C) No solution **(D)** None of these

9. The roots of the equation $3x + 1 - 1 = x$ are:
(A) 0 **(B)** 1
(C) $0, 1$ **(D)** None

10. The roots of the equation $x^2 + ax - 4 = 0$ are, where $a \in R$:
(A) real and distinct **(B)** equal
(C) imaginary **(D)** real

11. A shopkeeper buys a number of books for Rs. 80. If he had bought 4 more for the same amount, each book would have cost Rs. 1 less. How many books did he buy?
(A) 8 **(B)** 16
(C) 24 **(D)** 28

12. Solve the system of equations: $\begin{cases} 1/x + 1/y = 3/2, \\ 1/x^2 + 1/y^2 = 5/4 \end{cases}$
(A) $(1, 2)$ **(B)** $(1, 3)$
(C) $(2, 3)$ **(D)** $(2, 2)$

13. If one root of the quadratic equation $ax^2 + bx + c = 0$ is double the other then which one of the following is correct:
(A) $b^2 = 3ac$ **(B)** $2b^2 = 5ac$
(C) $2b^2 = 9ac$ **(D)** $2b^2 > 9ac$

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Comprehension (For Q.No. 14. to 16)

The coordinates of the vertex of the parabola $f(x) = 2x^2 + px + q$ are $(-3, 1)$, then

14. The value of p is
(A) 12 **(B)** -12
(C) 19 **(D)** -19

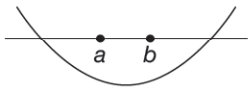
15. The value of q is
(A) -19 **(B)** 19
(C) -12 **(D)** none of these

16. The parabola
(A) touches the x-axis
(B) intersect the x-axis in two real and distinct points
(C) lies completely above the x-axis
(D) lies completely below the x-axis

17. Solution set of the equation $3^{2x^2} - 2.3^{x^2+x+6} + 3^{2(x+6)} = 0$ is:
(A) $\{-3, 2\}$ **(B)** $\{6, -1\}$
(C) $\{-2, 3\}$ **(D)** $\{1, -6\}$

18. If graph of $y = (x - a)(x - b)$ is

, then graph of $y = (x - a)(x - b) - 2$ is

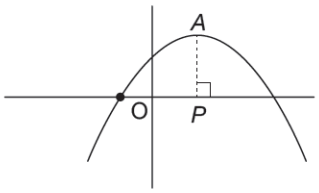
(A) 

(B)

(C)

(D) None of these

19. If $y = ax^2 + bx + c$ represents the curve given in the figure and $b^2 = 2(b + 2ac)$, where $a \neq 0$ and $AP = 3$ units, then $OP =$



(A) 3/2 **(B)** 3/4
(C) 3 **(D)** 6

20. If $b > 0$, $b \neq 1$, then the equation $2\log_x b + \log_{bx} b + 3\log_{b^2x} b = 0$ has:

(A) Exactly one real root
(B) Two real roots
(C) No real roots
(D) Infinite number of real roots

21. If one root of the quadratic equation $px^2 + qx + r = 0$ $(p \neq 0)$ is a surd $\frac{\sqrt{a}}{\sqrt{a} + \sqrt{a-b}}$, where p, q, r, a, b are all rationales then the other root is -

(A) $\frac{\sqrt{b}}{\sqrt{a} - \sqrt{a-b}}$ **(B)** $a + \frac{\sqrt{a(a-b)}}{b}$

(C) $\frac{a + \sqrt{a(a-b)}}{b}$ **(D)** $\frac{\sqrt{a} - \sqrt{a(a-b)}}{\sqrt{b}}$

22. $ax^2 + bx + c = 0$ has real and distinct roots a and $\beta(\beta > \alpha)$. Further $a > 0$, $b < 0$ and $c < 0$, then:

(A) $0 < \beta < |\alpha|$ **(B)** $0 < |\alpha| < \beta$

(C) $\alpha + \beta < 0$ **(D)** $|\alpha| + |\beta| = \left|\frac{b}{a}\right|$

23. Let $p, q \in \{1, 2, 3, 4\}$. Then number of equation of the form $px^2 + qx + 1 = 0$, having real roots, is

(A) 15 **(B)** 9
(C) 7 **(D)** 8

24. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are imaginary, then for all values of a, b, c and $x \in R$, the expression $a^2x^2 + abx + ac$ is

(A) positive
(B) non-negative
(C) negative
(D) may be positive, zero or negative

25. If x, y are rational number such that $x + y + (x - 2y)\sqrt{2} = 2x - y + (x - y - 1)\sqrt{6}$, then

(A) x and y can not be determined
(B) $x = 2, y = 1$
(C) $x = 5$, $y = 1$
(D) none of these

26. The equation whose roots are the squares of the roots of the equation $ax^2 + bx + c = 0$ is -

(A) $a^2x^2 + b^2x + c^2 = 0$
(B) $a^2x^2 - (b^2 - 4ac)x + c^2 = 0$
(C) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$
(D) $a^2x^2 + (b^2 - ac)x + c^2 = 0$

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27. If $\alpha \neq \beta$, $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation whose roots are α/β and β/α, is

(A) $x^2 + 5x - 3 = 0$
(B) $3x^2 + 12x + 3 = 0$
(C) $3x^2 - 19x + 3 = 0$
(D) none of these

28. If $x^2 - 11x + a$ and $x^2 - 14x + 2a$ have a common factor then 'a' is equal to:

(A) 24 **(B)** 1
(C) 2 **(D)** 12

29. If the roots of the equation, $x^3 + Px^2 + Qx - 19 = 0$ are each one more than the roots of the equation, $x^3 - Ax^2 + Bx - C = 0$, where A, B, C, P, and Q are constants then the value of $A + B + C =$

(A) 18 **(B)** 19
(C) 20 **(D)** none

30. If α, β, γ, δ are roots of $x^4 - 100x^3 + 2x^2 + 4x + 10 = 0$, then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ is equal to -

(A) $\frac{2}{5}$ **(B)** $\frac{1}{10}$

(C) 4 **(D)** $-\frac{2}{5}$

31. Number of real solutions of the equation $x^4 + 8x^2 + 16 = 4x^2 - 12x + 9$ is equal to:
(A) 1 **(B)** 2
(C) 3 **(D)** 4

32. For every $x \in R$, the polynomial $x^8 - x^5 + x^2 - x + 1$ is:
(A) positive
(B) never positive
(C) positive as well as negative
(D) negative

33. The roots of the equation $(p-2)x^2 + 2(p-2)x + 2 = 0$ are not real when:
(A) $p \in [1, 2]$ **(B)** $p \in [2, 3]$
(C) $p \in (2, 4)$ **(D)** $p \in [3, 4]$

34. The real roots of the equation $x^2 + 5|x| + 4 = 0$ are:
(A) $(-1, -4)$ **(B)** $(1, 4)$
(C) $(-4, 4)$ **(D)** No real root

35. A quadratic equation whose roots are $\left(\frac{\gamma}{\alpha}\right)^2$ and $\left(\frac{\beta}{\alpha}\right)^2$, where α, β, γ are the roots of $x^3 + 27 = 0$, is:
(A) $x^2 - x + 1 = 0$ **(B)** $x^2 + 3x + 9 = 0$
(C) $x^2 + x + 1 = 0$ **(D)** $x^2 - 3x + 9 = 0$

36. The number of real solutions of the equation $\left(5 + 2\sqrt{6}\right)^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$ is:
(A) 2 **(B)** 4
(C) 6 **(D)** none of these.

37. If roots of the equation $12x^2 + mx + 5 = 0$ are in 3 : 2 them $m =$
(A) $5\sqrt{10}$ **(B)** $3\sqrt{10}$
(C) $2\sqrt{10}$ **(D)** None of these

38. If $a > 0$, $b > 0$ then the roots of the equation $a - bx - x^2 = 0$ are
(A) both positive
(B) both negative
(C) of opposite sign and numerically greater root is positive
(D) of opposite sign and numerically greater root is negative

39. If the quadratic equations $3x^2 + ax + 1 = 0$ and $2x^2 + bx + 1 = 0$ have a common root, then the value of the expression $5ab - 2a^2 - 3b^2$ is:
(A) 0 **(B)** 1
(C) -1 **(D)** none of these

40. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are negatives of each other, then $r =$
(A) $p + q$ **(B)** $p - q$
(C) $\frac{p+q}{2}$ **(D)** $\frac{p-q}{2}$

PART – II KVPY Level Problems

1. Given that $f(x) = x^2 + ax + b$ is a polynomial with integral coefficients. If f is a common factor of polynomials $g(x) = x^4 - 3x^3 + 2x^2 - 3x + 1$ and $h(x) = 3x^4 - 9x^3 + 2x^2 + 3x - 1$, find $f(x)$.Ans.
(A) $f(x) = x^2 \times 3x + 1$ **(B)** $f(x) = x^2 - 3x + 1$
(C) $f(x) = x - 3x + 1$ **(D)** $f(x) = x^2 - 3x - 1$

2. Suppose f is a function satisfying $f(x + x^{-1}) = x^6 + x^{-6}$, for all $x \neq 0$. Determine $400 - f(3)$.
(A) 75 **(B)** 76
(C) 77 **(D)** 78

3. Let x_1, x_2, x_3, x_4 denote the four roots of the equation $x^4 + kx^2 + 90x - 2009 = 0$.
If $x_1x_2 = 49$, find the value of k.
(A) 5 **(B)** 6
(C) 7 **(D)** 8

4. If x, y and z are real numbers such that $x + y + z = 9$ and $xy + yz + zx = 24$, find the largest possible value of z.
(A) 2 **(B)** 3
(C) 4 **(D)** 5

5. For how many integers is the number $x^4 - 51x^2 + 50$ negative?
(A) 10 **(B)** 11
(C) 12 **(D)** 13

6. For what values of b do the equations: $1988x^2 + bx + 8891 = 0$ and $8891x^2 + bx + 1988 = 0$ have a common root?
(A) $b = \mp 10879$ **(B)** $a = -10879$
(C) $b = \times 10879$ **(D)** $c = \mp 10879$

7. When x is real, the greatest possible value of $10^x - 100^x$ is:
(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$
(C) $\frac{1}{2}$ **(D)** $\frac{2}{1}$

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8. Find the real points (x, y) satisfying $3x^2 + 3y^2 - 4xy + 10x - 10y + 10 = 0$.
(A) $(x, y) = (-1, 1)$ **(B)** $(x, y) = (-1, 2)$
(C) $(x, y) = (-1, 3)$ **(D)** $(x, y) = (-1, 4)$

9. The number of roots of the equation $\sqrt{x-2}\,(x^2 - 4x + 3) = 0$ is
(A) three **(B)** four
(C) one **(D)** two

10. The sum of values of x satisfying the equation $(31 + 8\sqrt{15})^{x^2-3} + 1 = (32 + 8\sqrt{15})^{x^2-3}$ is
(A) 3 **(B)** 0
(C) 2 **(D)** none of these

11. A quadratic equation whose product of roots x_1 and x_2 is equal to 4 and satisfying the relation $X_1/(X_1 - 1) + x_2/(x_2 - 1) = 2$ is
(A) $x^2 - 2x + 4 = 0$ **(B)** $x^2 + 2x + 4 = 0$
(C) $x^2 + 4x + 4 = 0$ **(D)** $x^2 - 4x + 4 = 0$

12. The integral values of m for which the roots of the equation $mx^2 + (2m - 1)x + (m - 2) = 0$ are rational are given by the expression [where n is integer]
(A) n^2 **(B)** $n(n + 2)$
(C) $n(n + 1)$ **(D)** none of these

13. If α and β are the roots of the equation $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$, then which of the following is true?
(A) $A_{n+1} = aA_n + bA_{n-1}$
(B) $A_{n+1} = bA_n + aA_{n-1}$
(C) $A_{n+1} = aA_n + bA_{n-1}$
(D) $A_{n+1} = bA_n + aA_{n-1}$

14. Number of values of a for which equations $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root
(A) 0 **(B)** 1
(C) 2 **(D)** infinite

15. Let r, s, and t be the roots of the equation, $8x^3 + 1001x + 2008 = 0$. The value of $(r + s)^3 + (s + t)^3 + (t + r)^3$ is:
(A) 251 **(B)** 751
(C) 735 **(D)** 753

16. Total number of values of a so that $x^2 - x - a = 0$ has integral roots, where $a \in N$ and $6 \le a \le 100$, is equal to:
(A) 2 **(B)** 4
(C) 6 **(D)** 8

17. If α, β are the roots of $x^2 + px + q = 0$ and $x^{2n} + p^n x^n + q^n = 0$ and if (α/β), $\beta/\alpha)$ are the roots of $x^n + 1 + (x+1)^n = 0$, then n $(\in N)$
(A) must be an odd integer
(B) may be any integer
(C) must be an even integer
(D) can not say anything

18. If α, β, γ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, then $\alpha^4 + \beta^4 + \gamma^4$ is
(A) 18 **(B)** 10
(C) 15 **(D)** 36

19. If the quadratic equation $4x^2 - 2(a + c - 1)\,x + ac - b = 0$ $(a > b > c)$
(A) both roots are greater than a
(B) both roots are less than c
(C) both roots lie between $c/2$ and $a/2$
(D) exactly one of the root lies between $c/2$ and $a/2$

20. If the equation $|x^2 + bx + c| = k$ has four real roots, then
(A) $b^2 - 4c > 0$ and $0 < k < \dfrac{4c - b^2}{4}$
(B) $b^2 - 4c < 0$ and $0 < k < \dfrac{4c - b^2}{4}$
(C) $b^2 - 4c > 0$ and $k > \dfrac{4c - b^2}{4}$
(D) none of these

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PART – III PRMO Level Problems

1. Find an integer x that satisfies the equation $x^5 - 101x^3 - 999x^2 + 100900 = 0$.

2. Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If total k polynomials satisfy $P(-1) = -9$, then find $k - 200$?

3. For certain real numbers a, b, and c, the polynomial $g(x) = x^3 + ax^2 + x + 10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial $f(x) = x^4 + x^3 + bx^2 + 100x + c$. What is $\dfrac{-f(1)}{91}$?

4. Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that $P(1) = P(3) = P(5) = P(7) = a$, and $P(2) = P(4) = P(6) = P(8) = -a$. The smallest possible value of $a = 5 \times Q$. What is Q?

5. Monic quadratic polynomials $P(x)$ and $Q(x)$ have the property that $P(Q(x))$ has zeroes at $x = -23, -21, -17$, and -15 and $Q(P(x))$ has zeroes at $x = -59, -57, -51$, and -49. What is the last two digits of the sum of the minimum values of $P(x)$ and $Q(x)$?

6. Let $p(x)$ be a polynomial with real coefficients such that for all real x, $2(1 + p(x)) = p(x - 1) + p(x + 1)$ and $p(0) = 8$, $p(2) = 32$. Determine the value of $p(4)$.

7. Find the number of pairs (m, n) of integers which satisfy the equation $m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1$.

8. If the equation is x has real roots, then find the value of a and b. $x^2 + 2(1 + a)x + (3a^2 + 4ab + 4b^2 + 2) = 0$ find value of a.

9. If x, y are positive real numbers satisfying the system of equations $x^2 + y\sqrt{xy} = 336$, $y^2 + x\sqrt{xy} = 112$, then $x + y$ equals

10. a, b, c are positive integers such that $a^2 + 2b^2 - 2bc = 100$ and $2ab - c^2 = 100$. Then $\frac{a+b}{c}$ is

11. Solve for x, y, and z; if $xy + x + y = 23$, $yz + y + z = 31$, $zx + z + x = 47$. Find positive value of x.

12. Given that the real numbers s, t satisfy $19s^2 + 99s + 1 = 0$, $t^2 + 99t + 19 = 0$, and $st \neq 1$. Find the value of $\frac{st + 4s + 1}{t}$.

13. Let a, b and c be the lengths of the three sides of a triangle. Suppose a and b are the roots of the equation $x^2 + 4(c + 2) = (c + 4)x$, and the largest angle of the triangle is $x°$. Find the value of x.

14. Let p be an integer such that both roots of the equation $5x^2 - 5px + (66p - 1) = 0$ are positive integers. Find the value of p.

15. Find the value of the smallest positive integer m such that the equation $x^2 + 2(m + 5)x + (100m + 9) = 0$ has only integer solutions.

16. If $x + y = 5xy$, $y + z = 6yz$, $z + x = 7zx$. The value of $x + y + z = T$. Find lowest integer greater than T.

17. Let f be a function with the following properties:
(i) $f(1) = 1$, and
(ii) $f(2n) = n \times f(n)$. for any positive integer n.
If the value of $f(2^{100}) = 2^{49xy}$, then the digits x and y, respectively?

18. Find the largest real number x such that $\frac{x^2 + x - 1 + |x^2 - (x-1)|}{2} = 35x - 250$.

19. Define a function on the positive integers recursively by $f(1) = 2$, $f(n) = f(n - 1) + 1$ if n is even, and $f(n) = f(n - 2) + 2$ if n is odd and greater than 1. What is $f(2017) - 2000$?

20. Suppose that $f(x + 3) = 3x^2 + 7x + 4$ and $f(x) = ax^2 + bx + c$. What is $(a + b + c)^4$?

21. Let m, n be integers such that $1 < m \leq n$. Define $f(m, n) = \left(1-\frac{1}{m}\right)\times\left(1-\frac{1}{m+1}\right)\times\left(1-\frac{1}{m+2}\right)\times\ldots\times\left(1-\frac{1}{n}\right)$. If $S = f(2, 2049) + f(3, 2049) + f(4, 2049) + \ldots + f(2049, 2049)$, find the value of $\frac{S}{64}$.

22. If $x + \sqrt{x^2-1} + \frac{1}{x-\sqrt{x^2-1}} = 20$ then if $x^2 + \sqrt{x^4-1} + \frac{1}{x^2-\sqrt{x^4-1}} = k$, find $[k]$

23. How many distinct ordered triples (x, y, z) satisfy the equations:
$x + 2y + 4z = 12$
$xy + 4yz + 2xz = 22$
$xyz = 6$

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EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. The graph of the equation $y = 2x^2 + 4x + 3$ has its lowest point at: **[NSEJS-2010]**
(A) (–1, 9) **(B)** (1, 9)
(C) (–1, 1) **(D)** (0, 3)

2. The product of the roots of the equation $\sqrt[3]{8+x} + \sqrt[3]{8-x} = 1$ is: **[NSEJS-2010]**
(A) –21 **(B)** –189
(C) 9 **(D)** –5

3. If α, β, γ are the roots of the equation $(x-2)(x^2+6x-11) = 0$, therefore, $(\alpha+\beta+\gamma)$ equals: **[NSEJS-2010]**
(A) –4 **(B)** 23/6
(C) 13 **(D)** –8

4. If $2^{2x+1} + 2^{1-2x} = 2$, then the value of x is: **[NSEJS-2011]**
(A) 0.5 **(B)** – 0.5
(C) 1 **(D)** 0

5. Given that a $(a+b)=36$ and $b(a+b)=64$, where a and b are positive, $(a-b)$ equals:
[NSEJS-2011]

(A) 2.8 **(B)** 3.2
(C) −2.8 **(D)** −2.5

6. If one of the roots of the equation $x^2-px+q=0$ is m times the other root then $m/(1+m^2)$ is equal to
[NSEJS-2012]

(A) $\frac{q}{p^2-2q}$ **(B)** $\frac{p}{q^2-2p}$

(C) $\frac{q}{q^2-2p}$ **(D)** $\frac{p}{p^2-2q}$

7. If $a+b+c=1, a^2+b^2+c^2=21$ and $abc=8$, then find the value of $(1-a)(1-b)(1-c)$ [NSEJS-2012]

(A) −10 **(B)** −18
(C) −24 **(D)** −30

8. If $3x+3y-1, 4x^2+y-5, 5x+2y$ are the sides of an equilateral triangle, its area is closest to the integer
[NSEJS-2013]

(A) 84 **(B)** 85
(C) 86 **(D)** 87

9. Let a, b be two positive real number such that $a<b<\frac{1}{a}$ and let $x=\left(a+\frac{1}{a}\right)-\left(b+\frac{1}{b}\right)$. Then: [NSEJS-2013]

(A) x is always greater than zero
(B) x is always less than zero
(C) $x=0$
(D) No such definite conclusion can be drawn about x

10. If a and b are two positive real numbers such that $\frac{a^2+b^2}{ab}=6$, then a positive value of $\frac{a}{b}$ lines between
[NSEJS-2013]

(A) 2 and 3 **(B)** 3 and 4
(C) 4 and 6 **(D)** 5 and 6

11. The number of real values of a for which the cubic equation $x^3-3ax^2+3ax-a=0$ has all real roots, one of which is a itself, is [NSEJS-2013]

(A) 0 **(B)** 1
(C) 2 **(D)** 3

12. The solution set of the inequality $0<\frac{x}{x+1}<1, x\in R$ is: [NSEJS-2014]

(A) Set of all positive real numbers
(B) set of all real numbers except − 1
(C) Set of all non-negative real numbers
(D) set of all numbers satisfying $0\le x\le 1, x\in R$

13. The statement "a is not less than 4" is correct represented by: [NSEJS-2016]

(A) $a<4$ **(B)** $a>4$
(C) $a\ge 4$ **(D)** $a\le 4$

14. The number of natural number $n\ge 30$ for which $\sqrt{n+\sqrt{n+\sqrt{n+\ldots}}}$ is a natural number is
[NSEJS-2016]

(A) 30 **(B)** zero
(C) 6 **(D)** 5

15. The number of natural number $n\le 30$ for which $\sqrt{n+\sqrt{n+\sqrt{n+\ldots}}}$ is a prime numbers is
[NSEJS-2016]

(A) Three **(B)** Zero
(C) Nine **(D)** Two

16. The sum of two numbers is 13 and the sum of their cubes is 1066. Find the product of those two numbers:
[NSEJS-2017]

(A) 26 **(B)** 27
(C) 28 **(D)** 29

17. If $p+q+r2, p^2+q^2+r^2=30$ and $pqr=10$, the value of $(1-p)(1-q)(1-r)$ will be [NSEJS-2017]

(A) − 18 **(B)** − 24
(C) − 27 **(D)** − 35

18. If the roots of the equation $\frac{x^2-bx}{ax-c}=\frac{m-1}{m+1}$ are equal and of opposite signs, then the value of 'm' is _____.
[NSEJS-2017]

(A) $\frac{a-b}{a+b}$ **(B)** $\frac{a+b}{a-b}$

(C) $\frac{ab}{a-b}$ **(D)** $\frac{a+b}{ab}$

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19. Let α and β be the roots of $x^2 - 5x + 3 = 0$ with $\alpha > \beta$. Iϕ $a_n = \alpha^n - \beta^n$ for $n \geq 1$ then the value of $\frac{3a_6 + a_8}{a_7}$ is **[NSEJS-2019]**

(A) 2 **(B)** 3
(C) 4 **(D)** 5

20. The number of triples (x, y, z) such that any one of these numbers is added to the product of the other two, the result is 2, is: **[NSEJS-2019]**

(A) 1 **(B)** 2
(C) 4 **(D)** infinitely many

21. What is the product of all the roots of the equation $\sqrt{5|x|+8} = \sqrt{x^2-16}$? **[NSEJS-2019]**

(A) −64 **(B)** −24
(C) 576 **(D)** 24

22. If the equation $(\alpha^2 - 5\alpha + 6)x^2 + (\alpha^2 - 3\alpha 2)x + (\alpha^2 - 4) = 0$ has more than two roots, then the value of α is: **[NSEJS-2019]**

(A) 2 **(B)** 3
(C) 1 **(D)** None of these

23. The sum of the roots of $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ is zero. The product of roots is: **[NSEJS-2019]**

(A) 0 **(B)** $\frac{a+b}{2}$

(C) $-\frac{1}{2}(a^2 - b^2)$ **(D)** $2(a^2 + b^2)$

PART – II Previous Years' KVPY Problems

1. The real numbers x satisfying $\frac{\sqrt{x+5}}{1-x} > 1$ are precisely those which satisfy **[KVPY-2009]**

(A) $x < 1$ **(B)** $0 < x < 1$
(C) $-5 < x < 1$ **(D)** $-1 < x < 1$

2. The number of pairs of reals (x, y) such that $x^2 = x^2 + y^2$ and $y = 2xy$ is **[KVPY-2009]**

(A) 4 **(B)** 3
(C) 2 **(D)** 1

3. How many positive real numbers x satisfy the equation $x^3 - 3|x| + 2 = 0$? **[KVPY-2009]**

(A) 4 **(B)** 3
(C) 2 **(D)** 1

4. A student notices that the roots of the equation $x^2 + bx + a = 0$ are each 1 less than the roots of the equation $x^2 + ax + b = 0$. Then, $a + b$ is **[KVPY-2010]**

(A) possibly any real number
(B) −2
(C) − 4
(D) −5

5. Suppose a, b, c are three distinct real numbers, let **[KVPY-2011]**

$$P(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)}$$

When simplified, $P(x)$ becomes

(A) 1
(B) x
(C) $\frac{x^2 + (a+b+c)(ab+bc+ca)}{(a-b)(b-c)(c-a)}$
(D) 0

6. Let $a, b, c\ d$ be real numbers such that $|a-b| = 2$, $|b-c| = 3, |c-d| = 4$. Then, the sum of all possible values of $|a-d|$ is: **[KVPY-2011]**

(A) 9 **(B)** 18
(C) 24 **(D)** 30

7. Let $f(x) = ax^2 + bx + bx + c$, where a, b, c are integers, Suppose $f(a) = 0$, $40 < f(6) < 50$, $60 < f(7) < 70$ and $1000t < f()50 < 1000(t+1)$ for some integer t. Then, the value of t is: **[KVPY-2011]**

(A) 2 **(B)** 3
(C) 4 **(D)** 5 or more

8. Let $f(x)$ be a quadratic polynomial with $f(2) = 10$ and $f(-2) = -2$ .Then, the coefficient of x in $f(x)$ is: **[KVPY-2012]**

(A) 1 **(B)** 2
(C) 3 **(D)** 4

9. The number of solutions (x, y, z) to the system of equations $x + 2y + 4z = 9$, $4yz + 2xz + xy = 13$, $xyz = 3$, such that at least two of x, y, z are integers is **[KVPY-2012]**

(A) 3 **(B)** 5
(C) 6 **(D)** 4

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10. Let a, b be non-zero real numbers. Which of the following statements about the quadratic equation $(a+b)x+b=0$ is necessarily true? **[KVPY-2013]**

I. It has at least one negative root.
II. It has at least one positive root.
III. Both its roots are real.

(A) I and II only **(B)** I and III only
(C) II and III only **(D)** All of them

11. The least positive integer n for which $\sqrt{n+1}-\sqrt{n-1}<02$ is **[KVPY-2013]**

(A) 24 **(B)** 25
(C) 26 **(D)** 27

12. Let r be a root of the equation $x^2+2x+6=0$. The value of $(r+2)(r+3)(r+4)(r+5)$ is equal to **[KVPY-2014]**

(A) 51 **(B)** -51
(C) -126 **(D)** 126

13. Let R be the set of all real numbers and let f be a function from R to R such that $f(x)+\left(x+\frac{1}{2}\right)f(1-x)=1$, for all $x\in R$. Then, $2f(0)+3f(1)$ is equal to: **[KVPY-2014]**

(A) 2 **(B)** 0
(C) -2 **(D)** -4

14. The number of integers a in the interval [1, 2014] for which the system of equations $x+y=a, \frac{x^2}{x-1}+\frac{y^2}{y-1}=4$ has finitely many solutions is **[KVPY-2014]**

(A) 0 **(B)** 1007
(C) 2013 **(D)** 2014

15. Two distinct polynomials $f(x)$ and $g(x)$ are defined as follows:
$f(x)=x^2+ax+2$; $g(x)=x^2+2x+a$. If the equations $f(x)=0$ and $g(x)=0$ have a common root, then the sum of the roots of the equation $f(x)+g(x)=0$ is **[KVPY-2015]**

(A) $-\frac{1}{2}$ **(B)** 0
(C) $\frac{1}{2}$ **(D)** 1

16. Let f be a function defined on the set of all positive integers such that $f(xy)=f(x)+f(y)$ for all positive integers x, y. If $f(12)=24$ and $f(8)=15$. The value of $f(48)$ is **[KVPY-2016]**

(A) 31 **(B)** 32
(C) 33 **(D)** 34

17. Suppose a is a positive real number such that $a^5-a^3+a=2$. Then, **[KVPY-2016]**

(A) $a^2<2$ **(B)** $2<a^6<3$
(C) $3<a^6<4$ **(D)** $4\le a^6$

18. Consider the quadratic equation $nx^2+7\sqrt{nx}+n=0$, where n is a positive integer. Which of the following statements are necessarily correct? **[KVPY-2016]**

I. For any n, the roots are distinct.
II. There are infinitely many values of n for which both roots are real.
III. The product of the roots is necessarily an integer.

(A) III only **(B)** I and III
(C) II and III **(D)** I, II, and III

19. The sum of all non-integer roots of the equation $x^5-6x^4+11x^3-5x^2-3x+2=0$ is **[KVPY-2017]**

(A) 6 **(B)** -11
(C) -5 **(D)** 3

20. Let a, b, c, d be real numbers between -5 and 5 such that **[KVPY-2017]**

$$|a|=\sqrt{4-\sqrt{5-a}},\ |b|=\sqrt{4+\sqrt{5-b}},$$
$$|c|=\sqrt{4-\sqrt{5+c}},\ |d|=\sqrt{4+\sqrt{5+d}}.$$

Then the product, a, b, c, d is

(A) 11 **(B)** -11
(C) 121 **(D)** -121

21. The number of pairs (a, b) of positive real numbers satisfying $a^4+b^4<1$ and $a^2+b^2>1$ is: **[KVPY-2018]**

(A) 0 **(B)** 1
(C) 2 **(D)** More than 2

22. The number of real roots of the polynomial equation $x^4-x^2+2x-1=0$ is: **[KVPY-2018]**

(A) 0 **(B)** 2
(C) 3 **(D)** 4

23. The number of polynomials $p(x)$ with integer coefficients such that curve $y=p(x)$ passes through (2, 2) and (4, 5) is: **[KVPY-2018]**

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(A) 0
(B) 1
(C) more than 1 but finite
(D) infinite

24. Let b be a non-zero real number. Suppose the quadratic equation $2x^2 + bx + \frac{1}{b} = 0$ has two distinct real roots. Then **[KVPY-2019]**

(A) $b + \frac{1}{b} > \frac{5}{2}$ **(B)** $b + \frac{1}{b} < \frac{5}{2}$

(C) $b^2 - 3b > -2$ **(D)** $b^2 + \frac{1}{b^2} < 4$

25. Let $p(x) = x^2 + ax + b$ have two distinct real roots, where a, b are numbers, Define $g(x) = p(x^3)$ for all real numbers x. Then which of the following statements are true.? **[KVPY-2019]**

I. g has exactly two distinct real roots $a_2 b$
II. g can have more than two distinct real roots
III. There exists a real number α such that $g(x) \geq \alpha$ for all real x

(A) Only I **(B)** Only I and III
(C) Only II **(D)** Only II and III

26. Let a, b, c, d be distinct real numbers such that a, b are roots of $x^2 - 5cx - 6d = 0$, and c, d are roots of $x^2 - 5ax - 6b = 0$. Then $b + d$ is **[KVPY-2019]**

(A) 180 **(B)** 162
(C) 144 **(D)** 126

PART – III Previous Years' PRMO Problems

1. Let N be the set of natural number. Suppose $f: N \to N$ is a function satisfying the following conditions.

(A) $f(mn) = f(m) f(n)$;
(B) $f(m) < f(n)$ if $m < n$;
(C) $f(2) = 2$.

What is the value of $\sum_{k-1}^{20} f(k)$? **[PRE-RMO – 2012]**

2. Let $x_1 \cdot x_2 \cdot x_3$ be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression $\left(x_1 - \frac{1}{x_1}\right)\left(x_2 - \frac{1}{x_2}\right)\left(x_3 - \frac{1}{x_3}\right)$ **[PRE-RMO – 2012]**

3. It is given that the equation $x^2 + ax + 20 = 0$ has integer roots. What is the sum of all possible value of a? **[PRE-RMO – 2013]**

4. Let $f(x) = x^3 - 3x + b$ and $g(x) = x^2 + bx - 3$, where b is a real number. What is the sum of all possible value of b for which the equation $f(x) = 0$ and $g(x) = 0$ have a common root? **[PRE-RMO – 2013]**

5. Natural numbers k, l, p, and q are such that if a and b are roots of $x^2 - kx + 1 = 0$, then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots $x^2 - px + q = 0$. What is the sum of all possible value of q? **[PRE-RMO – 2014]**

6. For a natural number b, let $N(b)$ denote the number of natural numbers a for which the equation $x^2 + ax + b = 0$ has integer roots. What is the smallest value of b for which $N(b) = 20$? **[PRE-RMO – 2014]**

7. Let f be a ont-to-one function from the set of natural numbers to itself such that $f(mn) = f(m) f(n)$ for all natural numbers m and n. What is the least possible value of $f(999)$?

8. The equations $x^2 - 4x + k = 0$ and $x^2 + kx - 4 = 0$, where k is a real number, have exactly one common root. What is the value of k? **[PRE-RMO – 2015]**

9. Suppose, a, b are positive real number such that $a\sqrt{a} + b\sqrt{b} = 183$, $a\sqrt{b} + b\sqrt{a} = 182$. Find $\frac{9}{5}(a + b)$. **[PRE-RMO – 2017]**

10. Let a, b be integers such that all the roots of the equation $(x^2 + ax + 20)(a^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of $a + b$? **[PRE-RMO – 2017]**

11. Suppose 1, 2, 3 are the roots of the equations $x^4 + ax^2 + bx = c$. Find the value of c.

12. Integers a, b, c satisfy $a + b - c = 1$ and $a^2 + b^2 - c^3 - 1$. What is the sum of all possible values of $a^2 + b^2 + c^3$? **[PRE-RMO – 2018]**

13. Suppose a, b are integers and $a + b$ is a root of $x^2 + ax + b = 0$. What is the maximum possible values of b^2? **[PRE-RMO – 2018]**

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14. Let $f(x) = x^2 + ax + b$. If for all non-zero real x $f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and the roots of $f(x) = 0$ are integers, what is the value of $a^2 + b^2$?

[PRE-RMO – 2019]

15. If $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$ is a root of $x^4 + ax^3 + bx^3 + bx^2 + cx + d = 0$, where a, b, c, d are integers, what is the value of $|a\ b\ c\ d|$?

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ANSWER KEYS

Know Your Progress - 1

1. $(-3, 1)$ **2.** $(-\infty, -2) \cup (-2, -1) \cup (1, +\infty)$ **3.** $[-3, 3]$ **4.** $(-\infty, 0) \cup (3, +\infty)$ **5.** $(-\infty, 3) \cup (4, +\infty)$

6. $(-\infty, +\infty)$ **7.** $(-1, 5)$ **8.** $[1, 3] \cup (5, +\infty)$ **9.** $(-9/2, -2) \cup (3, +\infty)$ **10.** $(-1, 1) \cup (4, 6)$

11. $(1/2, 3)$ **12.** $(-1, 0) \cup (0, 1)$ **13.** $[-\sqrt{2}, -1) \cup (-1, \sqrt{2}\,] \cup [3, 4)$

14. $(-\infty, -\sqrt{7}/2) \cup (-1, \sqrt{7}/2) \cup (4/3, +\infty)$ **15.** $(-\infty, -2] \cup (-1, 4)$ **16.** $(-\infty, -5) \cup (1, 2) \cup (6, +\infty)$

17. $(-\infty, -3) \cup (-2, 3)$ **18.** $(-\sqrt{2}, 0) \cup (1, \sqrt{2}) \cup (2, +\infty)$ **19.** $(-\infty, -2) \cup (-1, 3) \cup (4, +\infty)$

20. $(-\infty, -7) \cup (-4, -2)$ **21.** $(-\infty, -1] \cup (0, 1] \cup (2, 3]$

Know Your Progress - 2

1. $x = -(a-b)$ or $x = -(a+b)$ **2.** $x = \frac{5}{3}, \frac{5}{3}$ **3.** $x = -\frac{-9 \pm \sqrt{77}}{2}$ **4.** $x = -1$

5. $\frac{2a+b}{3}, \frac{a+2b}{3}$. **6.** $3, -\frac{1}{5}$ **7.** A **8.** B **9.** A **10.** B **11.** D **12.** B

13. C **14.** A **15.** D **16.** B **17.** C

Know Your Progress - 3

1. B **2.** A **3.** B **4.** D **5.** D **6.** C **7.** A **8.** B **9.** D **10.** B

11. B **12.** B **13.** B **14.** A

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Know Your Progress - 4

1. $(1, 3)$ **3.** $1, \frac{1}{2}, \frac{1}{4}$ **4.** $-2 < a < 8$ **5.** $k = 0$ or $k = 3$.

6. (a) imaginary; (b) real and distinct ; (c) real and coincident **8.** (a) $c = 0$; (b) $c = 1$; **9.** $b = \frac{9}{2}, c = 5$;

10. $c = 0, 6$ **11.** A **12.** C **13.** C **14.** A **15.** B **16.** A **17.** C **18.** A

19. A **20.** C

Know Your Progress - 5

1. (a) 1, $x = -1$; (b) -1, $x = 2$

2. (1) (i) $a < 0$ (ii) $b < 0$ (iii) $c < 0$ (iv) $D > 0$ (v) $\alpha + \beta < 0$ (vi) $\alpha\beta > 0$

(2) (i) $a < 0$ (ii) $b > 0$ (iii) $c = 0$ (iv) $D > 0$ (v) $\alpha + \beta > 0$ (vi) $\alpha\beta = 0$

(3) (i) $a < 0$ (ii) $b = 0$ (iii) $c = 0$ (iv) $D = 0$ (v) $\alpha + \beta = 0$ (vi) $\alpha\beta = 0$

3. Third quadrant **5.** least value = 0, greatest value = 1

Know Your Progress - 6

1. 1 **2.** -16 **3.** $\frac{13}{12}$ **4.** $x = 0, y = 6, z = 7, u = 3, v = -1$ **5.** 50 **6.** 81 **7.** 25 **8.** 18

Know Your Progress - 7

1. $f(x) = 0$ 2. $f(x) = \dfrac{x^3 - x + 1}{2x(x-1)}$ 3. 6044 4. $f(x) = 1$ 5. $f(x) = a\cos x + b\sin x$

6. $f(x) = \dfrac{x+1}{x-1}$ 7. $f(x) = \dfrac{2x^3 + x^2 + 5x - 2}{24x(x-1)}$

Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. B	2. B	3. D	4. D	5. C	6. D	7. D	8. C	9. C	10. A
11. B	12. A	13. C	14. A	15. B	16. C	17. C	18. A	19. C	20. B
21. C	22. B	23. C	24. A	25. B	26. C	27. C	28. A	29. A	30. D
31. A	32. A	33. A	34. D	35. C	36. B	37. A	38. D	39. B	40. C

PART – II KVPY Level Problems

1. B	2. D	3. C	4. D	5. C	6. A	7. A	8. A	9. D	10. B
11. A	12. C	13. C	14. B	15. D	16. D	17. A	18. A	19. A	20. A

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PART – III PRMO Level Problems

1. 10	2. 20	3. 77	4. 63	5. 0	6. 64	7. 0	8. 1	9. 20	10. 22
11. 5	12. –5	13. 90	14. 76	15. 90	16. 2	17. 50	18. 25	19. 18	20. 16
21. 16	22. 51	23. 6							

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. C	2. B	3. A	4. A	5. C	6. A	7. B	8. B	9. A	10. D
11. C	12. A	13. C	14. D	15. A	16. D	17. B	18. A	19. D	20. B
21. A	22. A	23. C							

PART – II Previous Years' KVPY Problems

1. D	2. A	3. A	4. C	5. A	6. B	7. C	8. C	9. B	10. B
11. C	12. C	13. C	14. D	15. C	16. D	17. C	18. B	19. D	20. A
21. D	22. B	23. A	24. C	25. B	26. C				

PART – III Previous Years' PRMO Problems

1. 201	2. –29/5	3. 0	4. 0	5. 4	6. 60	7. 24	8. 3	9. 73	10. 25
11. 36	12. 18	13. 81	14. 13	15. 93					

Chapter

3 Sequence and Series

Learning Outcomes

After reading this chapter, you will be able to understand

- Sequence
- Series
- Arithmetic Progression (A.P.)
- Geometric Progression (G.P.)
- Harmonic Progression (H.P.)
- Relation Between A.M., G.M., H.M.
- Arithmetic-Geometric Series

Introduction

Sequence

A succession of terms $a_1, a_2, a_3, a_4, \ldots$ formed according to some rule or law.

Examples are: 1, 4, 9, 16, 25

$$-1, 1, -1, 1, \ldots$$

$$\frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, \ldots$$

A finite sequence has a finite (i.e., limited) number of terms, as in the first example above. An infinite sequence has an unlimited number of terms, i.e., there is no last term, as in the second and third examples.

Series

The indicated sum of the terms of a sequence. In the case of a finite sequence $a_1, a_2, a_3, \ldots, a_n$ the corresponding series is:

$$a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k .$$

This series has a finite or limited number of terms and is called a finite series.

Arithmetic Progression (A.P.)

A.P. is a sequence whose terms differ by a fixed number. This fixed number is called the common difference. If a is the first term and d the common difference, then A.P. can be written as

$$a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$$

(a) n^{th} term of AP $T_n = a + (n-1)d$, where $d = t_n - t_{n-1}$

(b) The sum of the first terms:

$$S_n = \frac{n}{2}[a + \ell] = \frac{n}{2}[2a + (n-1)d]$$

where l is n^{th} term.

Notes

1. n^{th} term of an A.P. is of the form $An + B$ i.e., a linear expression in 'n', in such a case the coefficient of n is the common difference of the A.P., i.e., A.
2. Sum of first 'n' terms of an A.P. is of the form $An^2 + Bn$, i.e., a quadratic expression in 'n', in such case the common difference is twice the coefficient of n^2. i.e., $2A$.
3. Also n^{th} term $T_n = S_n - S_{n-1}$.

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Properties of A.P.

(a) If each term of an A.P. is increased, decreased, multiplied, or divided by the some non-zero number, then the resulting sequence is also an A.P.

(b) Three numbers in A.P.: $a-d, a, a+d$
Four numbers in A.P.: $a-3d, a-d, a+d, a+3d$
Five numbers in A.P.: $a-2d, a-d, a, a+d, a+2d$
Six numbers in A.P.:

$$a-5d,\ a-3d,\ a-d,\ a+d,\ a+3d,\ a+5d$$

(c) The common difference can be zero, positive, or negative.

(d) k^{th} term from the last $=(n-k+1)^{\text{th}}$ term from the beginning (If total number of terms $= n$).

(e) The sum of the two terms of an A.P. equidistant from the beginning and end is constant and equal to the sum of first and last terms.

$$\Rightarrow \quad T_k + T_{n-k+1} = \text{constant} = a + l.$$

(f) Any term of an A.P. (except the first) is equal to half the sum of terms that are equidistant from it.

$$a_n = (1/2)(a_{n-k} + a_{n+k}),\ k < n$$

For $k=1,\ a_n = (1/2)(a_{n-k} + a_{n+k})$;

For $k=2,\ a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(g) If a, b, c are in A.P., then $2b = a + c$.

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Example 1

If $(x+1)$, $3x$ and $(4x+2)$ are first three terms of an A.P. then its 5^{th} term is

(A) 14 **(B)** 19 **(C)** 24 **(D)** 28

Solution

$(x+1), 3x, (4x+2)$ are in A.P.

$$\Rightarrow \quad 3x-(x+1) = (4x+2)-3x \Rightarrow x = 3$$

$$\therefore \quad a = 4,\ d = 9-4 = 5$$

$$\Rightarrow \quad T_5 = 4+(4)5 = 24$$ **Ans. (C)**

Example 2

The sum of first four terms of an A.P. is 56 and the sum of it's last four terms is 112. If its first term is 11 then find the number of terms in the A.P.

Solution

$$a+a+d+a+2d+a+3d = 56$$

$$4a+6d = 56$$

$$44+6d = 56 \quad (as\ a = 11)$$

$$6d = 12$$

Hence $d = 2$

Let total number of terms $= n$

Now sum of last four terms.

$$a+(n-1)d+a+(n-2)d+a+(n-3)d+a+(n-4)d = 112$$

$$\Rightarrow \quad 4a+(4n-10)d = 112 \ \Rightarrow 44+(4n-10)2 = 112$$

$$\Rightarrow \quad 4n-10 = 34$$

$$\Rightarrow \quad n = 11$$ **Ans.**

Example 3

The sum of first n terms of two A.Ps. are in ratio $\dfrac{7n+1}{4n+27}$. Find the ratio of their 11th terms.

Solution

Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively then

$$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11th terms

$$\frac{n-1}{2} = 10 \Rightarrow n = 21$$

So ratio of 11th terms is

$$\frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$$ **Ans.**

Example 4

Four numbers are in A.P. If their sum is 20 and the sum of their squares is 120, then the middle terms are:

(A) 2, 4 **(B)** 4, 6 **(C)** 6, 8 **(D)** 8, 10

Solution

Let the numbers are $a-3d, a-d, a+d, a+3d$

given, $\quad a-3d+a-d+a+d+a+3d = 20$

$$\Rightarrow \quad 4a = 20 \Rightarrow a = 5$$

and

$$(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$$

$\Rightarrow \quad 4a^2 + 20d^2 = 120$

$\Rightarrow \quad 4 \times 5^2 + 20d^2 = 120 \Rightarrow d^2 = 1 \Rightarrow d = \pm 1$

Hence, numbers are 2, 4, 6, 8 or 8, 6, 4, 2 **Ans. (B)**

Example 5

If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. where $a_i > 0$ for all i, show that:

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution

L.H.S.

$$= \frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{1}{\sqrt{a_2}+\sqrt{a_1}} + \frac{1}{\sqrt{a_3}+\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{(a_2-a_1)} + \frac{\sqrt{a_3}-\sqrt{a_2}}{(a_3-a_2)} + \ldots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n - a_{n-1}}$$

Let 'd' is the common difference of this A.P. then

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$$

Now **L.H.S.**

$$= \frac{1}{d}\{\sqrt{a_2}-\sqrt{a_1} + \sqrt{a_3}-\sqrt{a_2} + \ldots + \sqrt{a_{n-1}}-\sqrt{a_{n-2}} + \sqrt{a_n}-\sqrt{a_{n-1}}\}$$

$$= \frac{1}{d}\{\sqrt{a_n}-\sqrt{a_1}\}$$

$$= \frac{a_n - a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{a_1 + (n-1)d - a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)}$$

$$= \frac{1}{d}\frac{(n-1)d}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}} = \text{R.H.S.}$$

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Know Your Progress - 1

1. If t_{54} of an A.P. is –61 and $t_4 = 64$, find t_{10}.
   (A) 59 **(B)** 49
   (C) 36 **(D)** 64

2. Find the number of terms in the sequence 4, 12, 20, … 108.
   (A) 12 **(B)** 19
   (C) 13 **(D)** 14

3. Find the sum of all natural numbers divisible by 5, but less than 100.
   (A) 950 **(B)** 925
   (C) 880 **(D)** 1050

4. Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.
   (A) 70441 **(B)** 70821
   (C) 50821 **(D)** 40821

5. If sum of n terms of a sequence is given by $S_n = 2n^2 + 3n$, find its 50^{th} term.
   (A) 250 **(B)** 225
   (C) 201 **(D)** 205

6. If 9^{th} term of an A.P. be zero, then the ratio of its 29^{th} and 19^{th} term is:
   (A) 2 : 1 **(B)** 1 : 2
   (C) 1 : 3 **(D)** 3 : 1

7. If four numbers in A.P. are such that their sum is 50 and the greatest number is 4 times the least, then the numbers are:
   (A) 5, 10, 15, 20 **(B)** 4, 10, 16, 22
   (C) 3, 7, 11, 15 **(D)** None of these

8. If S_1 is the sum of an arithmetic progression of 'n' odd number of terms and S_2 is the sum of the terms of the series in odd places, then $\frac{S_1}{S_2} =$
   (A) $\frac{2n}{n+1}$ **(B)** $\frac{n}{n+1}$
   (C) $\frac{n+1}{2n}$ **(D)** $\frac{n+1}{n}$

9. The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers.
   (A) 3, 4, 5 **(B)** 2, 4, 6
   (C) 2, 5, 8 **(D)** 3, 6, 9

10. The angles of a quadrilateral are in A.P. whose common difference is 10°. Find the angles.

(A) 70°, 80°, 90°, 100° **(B)** 60°, 80°, 100°, 120°
(C) 72°, 85°, 98°, 111° **(D)** 75°, 85°, 95°, 105°

11. Write down the sequence whose nth terms is:

(A) $\frac{2^n}{n}$ **(B)** $\frac{3+(-1)^n}{3^n}$

12. For an A.P, show that $t_m + t_{2n+m} = 2t_{m+n}$

13. If the sum of p terms of an A.P. is q and the sum of its q terms is p, then find the sum of its $(p+q)$ term.

14. Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$, if it is know that

$$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225.$$

15. Find the number of terms common to the two A.P.'s 3, 7, 11, …, 407 and 2, 9, 16, …, 709

Geometric Progression (G.P.)

G.P. is a sequence of non-zero numbers each of the succeeding term is equal to the preceding term multiplied by a constant. Thus, in a G.P. the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the sequence and is obtained by dividing any term by the immediately previous term. Therefore, $a, ar, ar^2, ar^3, ar^4, \ldots$ is a G.P. with 'a' as the first term and 'r' as common ratio.

(a) n^{th} term; $T_n = ar^{n-1}$

(b) Sum of the first n terms; $S_n = \frac{a(r^n-1)}{r-1}$, if $r \neq 1$

(c) Sum of infinite G.P., $S_\infty = \frac{a}{1-r}; 0 < |r| < 1$

Properties of G.P.

(a) If each term of a G.P. be multiplied or divided by the some non-zero quantity, then the resulting sequence is also a G.P.

(b) Three consecutive terms of a G.P.: $a/r, a, ar$;
Four consecutive terms of a G.P.: $a/r^3, a/r, ar, ar^3$ and so on.

(c) If a, b, c are in G.P. then $b^2 = ac$.

(d) If in a G.P., the product of two terms, which are equidistant from the first and the last term, is constant and is equal to the product of first and last term.

$$\Rightarrow \quad T_k \cdot T_{n-k+1} = \text{constant} = a \times l$$

(e) If each term of a G.P. be raised to the same power, then resulting sequence is also a G.P.

(f) In a G.P., $T_r^2 = T_{r-k} \cdot T_{r+k}, k < r, r \neq 1$

(g) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(h) If $a_1, a_2, a_3, \ldots a_n$ is a G.P. of positive terms, then $\log a_1, \log a_2, \ldots \log a_n$ is an A.P. and vice-versa.

(i) If $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3 \ldots$ are two G.P.'s then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} \ldots$ is also in G.P.

Example 6

If a, b, c, d, and p are distinct real numbers such that

$$(a^2+b^2+c^2)p^2 - 2p(ab+bc+cd) + (b^2+c^2+d^2) \leq 0$$

then a, b, c, d are in

(A) A.P. **(B)** G.P.
(C) H.P. **(D)** none of these

Solution

Here, the given condition

$$(a^2+b^2+c^2)p^2 - 2p(ab+bc+ca) + b^2+c^2+d^2 \leq 0$$

$$\Rightarrow \quad (ab-b)^2 + (bp+c)^2 + (cp-d)^2 \leq 0$$

Θ a square can not be negative

$$\therefore \quad ap - b = 0, bp - c =, cp - d = 0$$

$$\Rightarrow \quad p = \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \Rightarrow a, b, c, d$$

are in G.P. **Ans. (B)**

Example 7

If a, b, c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(A) A.P. **(B)** G.P.
(C) H.P. **(D)** none of these

Solution

a, b, c are in G.P. $\Rightarrow b^2 = ac$

Now the equation $ax^2 + 2bx + c = 0$ can be rewritten as $ax^2 + 2\sqrt{acx} + c = 0$

$$\Rightarrow \quad \left(\sqrt{ax} + \sqrt{c}\right)^2 = 0 \Rightarrow -\sqrt{\frac{c}{a}}, -\sqrt{\frac{c}{a}}$$

If the two given equations have a common root, then this root must be $-\sqrt{\frac{c}{a}}$.

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Thus

$$d\frac{c}{a} - 2e\sqrt{\frac{c}{a}} + f0$$

$$\Rightarrow \quad \frac{d}{a} + \frac{f}{c} = \frac{2e}{c}\sqrt{\frac{c}{a}} = \frac{2e}{\sqrt{ac}} = \frac{2e}{b} \Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$$

are in A.P. **Ans. (A)**

Example 8

A number consists of three digits which are in G.P. the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right-hand digits. Find the numbers.

Solution

Let the three digits be a, ar and ar^2 then number is

$$100a + 10ar + ar^2 \qquad \text{(i)}$$

Given, $\quad a + ar^2 = 2ar + 1$

or $\quad a\left(r^2 - 2r + 1\right) = 1$

or $\quad a(r-1)2 = 1 \qquad \text{(ii)}$

Also given $\quad a + ar = \frac{2}{3}\left(ar + ar^2\right)$

$$\Rightarrow \quad 3 + 3r = 2r + 2r^2$$

$$\Rightarrow \quad 2r^2 - r - 3 = 0 \Rightarrow (r+1)(2r-3) = 0$$

$$\therefore \quad r = -1, 3/2$$

for $r = -1$, $\quad a = \frac{1}{(r=1)^2} = \frac{1}{4} \notin I$

$$\therefore \quad r \neq -1$$

for $\quad r = 3/2, a = \frac{1}{\left(\frac{3}{2} - 1\right)^2} = 4 \qquad \{\text{from (ii)}\}$

From (i), number is $400 + 10.4 \cdot \frac{3}{2} + 4 \cdot \frac{9}{4} = 469$ **Ans.**

Example 9

Find the value of $0.32\,\overline{58}$

Solution

Let $\quad R = 0.32\,\overline{58} \Rightarrow R = 0.32585858 \qquad \text{(i)}$

Here number of figures that are not recurring is 2 and number of figures that are recurring is also 2.

then $\quad 100\,R = 32.585858\ldots \qquad \text{(ii)}$

and $\quad 10000\,R = 3258.5858\ldots \qquad \text{(iii)}$

Subtracting (ii) from (iii), we get

$$9900\,R = 3226 \Rightarrow R = \frac{1613}{4950}$$

Aliter Method:

$$R = .32 + .0058 + .0058 + .000058 + \ldots$$

$$= .32 + \frac{58}{10^4}\left(1 + \frac{1}{10} + \frac{1}{10^4} + \ldots\infty\right)$$

$$= .32 + \frac{58}{10^4}\left(\frac{1}{1 - \frac{1}{100}}\right)$$

$$= \frac{32}{100} + \frac{58}{9900} = \frac{3168 + 58}{9900} = \frac{3226}{9900} = \frac{1613}{4950}$$

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Know Your Progress - 2

1. The value of $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{5^n}\right)$ equals

   (A) $\frac{5}{12}$ **(B)** $\frac{5}{24}$

   (C) $\frac{5}{36}$ **(D)** $\frac{5}{16}$

2. 3 geometric means between 1/9 and 9 is:

   (A) $\frac{2}{3}, 1, 3$ **(B)** $\frac{1}{3}, 2, 3$

   (C) $\frac{1}{3}, 1, 6$ **(D)** $\frac{1}{3}, 1, 3$

3. The first term of a G.P. is 7, the last term is 567 and sum of terms is 847. Find the common ratio of the G.P.

   (A) 3 **(B)** 2

   (C) 4 **(D)** 6

4. Find the sum of the infinite G.P.: $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

   (A) $\frac{3}{5}$ **(B)** $\frac{3}{2}$

   (C) $\frac{49}{27}$ **(D)** $\frac{8}{5}$

5. The eighth term of G.P. is 128 and common ratio is 2. The product of its first five terms is

(A) 4^6 **(B)** 4^3
(C) 4^5 **(D)** 4^4

6. Find a three digit number whose consecutive digits form a G.P. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now, if we increase the second digit of the required number by 2, then the resulting digits will form an A.P.

7. If the third term of G.P. is 4, then find the product of first five terms.

8. If a, b, c, are, respectively, the p^{th}, q^{th} and r^{th} terms of the given G.P., then show that $(q-r)\log a+(r-p)\log b+(p-q)\log c=0$, where $a, b, c>0$.

9. Find three numbers in G.P., whose sum is 52 and the sum of whose products in pairs is 624.

10. The rational number that equals the number $2.\overline{357}$ with recurring decimal is

(A) $\frac{2357}{999}$ **(B)** $\frac{2379}{997}$
(C) $\frac{785}{333}$ **(D)** $\frac{2355}{1001}$

H.P. and A.G.P

Harmonic Progression (H.P.)

A sequence is said to be in H.P. if the reciprocal of its terms are in A.P..

If the sequence $a_1, a_2, a_3, \ldots, a_n$ is an H.P. then $1/a_1, 1/a_2, \ldots, 1/a_n$ is an A.P. . Here we do not have the formula for the sum of the n terms of an H.P. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d} \ldots \frac{1}{a+(n-1)d}$

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Note

No term of any H.P. can be zero.

If a, b, c are in H.P., then $b=\frac{2ac}{a+c}$ or $\frac{a}{c}=\frac{a-b}{b-c}$

Example 10

The sum of three numbers are in H.P. is 37 and the sum of their reciprocals is 1/4. Find the numbers.

Solution

Three numbers are in H.P. can be taken as

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

then $$\frac{1}{a-d}+\frac{1}{a}+\frac{1}{a+d}=37 \quad \text{(i)}$$

and $$a-d+a+a+d=\frac{1}{4} \Rightarrow a=\frac{1}{12}$$

from (i), $$\frac{12}{1-12d}+12+\frac{12}{1+12d}=37$$

$$\Rightarrow \quad \frac{12}{1-12d}+\frac{12}{1+12d}=25$$

$$\Rightarrow \quad \frac{24}{1-144d^2}=25 \Rightarrow 1-144d^2=\frac{24}{25} \Rightarrow d^2=\frac{1}{25\times 144}$$

$$\therefore \quad d \pm \frac{1}{60}$$

$\therefore$ $a-d, a, a+d$ are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$

Hence, three numbers in H.P. are 15, 12, 10 or 10, 12, 15 **Ans.**

Example 11

Suppose a is a fixed real number such that

$$\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-y}{rz}$$

If p, q, r are in A.P., then prove that x, y, z are in H.P.

Solution

$\because$ p, q, r are in A.P.

$$\therefore \quad -p=r-q \quad \text{(i)}$$

$$\Rightarrow \quad p-q=q-r=k \quad \text{(let)}$$

Given $$\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz} \Rightarrow \frac{\frac{a}{x}-1}{p}=\frac{\frac{a}{y}-1}{q}=\frac{\frac{a}{z}-1}{r}$$

$$\Rightarrow \quad \frac{\left(\frac{a}{x}-1\right)-\left(\frac{a}{y}-1\right)}{p-q}=\frac{\left(\frac{a}{y}-1\right)-\left(\frac{a}{z}-1\right)}{q-r}$$

(by law of proportion)

$$\Rightarrow \quad \frac{\frac{a}{x}-\frac{a}{y}}{k}=\frac{\frac{a}{y}-\frac{a}{z}}{k} \qquad \{\text{from (i)}\}$$

$$\Rightarrow \quad a\left(\frac{1}{x}-\frac{1}{y}\right)=a\left(\frac{1}{y}-\frac{1}{z}\right)$$

$$\Rightarrow \quad \frac{1}{x}-\frac{1}{y}=\frac{1}{y}-\frac{1}{z}$$

$$\therefore \quad \frac{2}{y}=\frac{1}{x}=\frac{1}{z}$$

$$\therefore \quad \frac{1}{x},\frac{1}{y},\frac{1}{z} \text{ are in A.P.}$$

Hence x, y, z are in H.P.

Arithmetico – Geometric Series

A series, each term of which is formed by multiplying the corresponding term of an A.P. and G.P. is called the Arithmetico-Geometric Series, e.g. $1+3x+5x^2+7x^3+\ldots$

Here 1, 3, 5, … are in A.P. and 1, x, x^2, x^3, … are in G.P.

(a) Sum of n Terms of An Arithmetico-Geometric Series:

Let

$$S_n=a+(a+d)r+(a+2d)r^2+\ldots +\left[a+(n-1)d\right]r^{n-1}$$

then

$$S_n=\frac{a}{1-r}+\frac{dr\left(1-r^{n-1}\right)}{(1-r)^2}-\frac{\left[a+(n-1)d\right]r^n}{1-r};\ r\neq 1$$

(b) Sum To Infinity:

If $0<|r|<1$ and $n\to\infty$, then

$$\lim_{n\to\infty} r^n=0,\ S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

Example 15

Find the sum of series

$$4-9x+16x^2-25x^3+36x^4-49x^5+\ldots\infty.$$

Solution

Let $S=4-9x+16x^2-25x^3+36x^4-49x^5+\ldots\infty$

$$-Sx=-4x+9x^2-16x^3+25x^4-36x^5+\ldots\infty$$

On subtraction, we get

$$S(1+x)=4-5x+7x^2-9x^3+11x^4-13x^5+\ldots\infty$$

$$-S(1+x)x=-4x+5x^2-7x^3+9x^4-11x^5+\ldots\infty$$

On subtraction, we get

$$S(1+x)^2=4-x+2x^2-2x^3+2x^4-2x^5+\ldots\infty$$

$$=4-x+2x^2(1-x+x^2-\ldots\infty)$$

$$=4-x+\frac{2x^2}{1+x}=\frac{4+3x+x^2}{1+x}$$

$$S=\frac{4+3x+x^2}{(1+x)^3} \qquad \textbf{Ans.}$$

Example 16

Find the sum of series up to n terms

$$\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\ldots$$

Solution

For $x\neq 1$, let

$$S=x+3x^2+5x^3+\ldots+(2n-3)x^{n-1}+(2n-1)x^n \quad \text{(i)}$$

$$\Rightarrow xS=x^2+3x^3+\ldots+(2n-5)x^{n-1}+(2n-3)x^n+(2n-1)x^{n+1} \quad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$(1-x)S=x+2x^2+2x^3+\ldots+2x^{n-1}+2x^n-(2n-1)x^{n+1}$$

$$=x+\frac{2x^2\left(1-x^{n-1}\right)}{1-x}-(2n-1)x^{n+1}$$

$$=\frac{x}{1-x}\left[1-x+2x-2x^n-(2n-1)x^n+(2n-1)x^{n+1}\right]$$

$$\Rightarrow \quad S=\frac{x}{(1-x)^2}\left[(2n-1)x^{n+1}-(2n+1)x^n+1+x\right]$$

Thus

$$\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+\ldots+(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$$

$$\left(\frac{2n+1}{2n-1}\right)\left(\frac{2n-1}{2}\right)^2$$

$$\left[(2n-1)\left(\frac{2n+1}{2n-1}\right)^{n+1}-(2n+1)\left(\frac{2n+1}{2n-1}\right)^n+1+\frac{2n+1}{2n-1}\right]$$

$$=\frac{4n^2-1}{4}\cdot\frac{4n}{2n-1}=n(2n+1) \qquad \textbf{Ans.}$$

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Know Your Progress - 3

1. If the 7^{th} term of a H.P. is 8 and the 8^{th} term is 7, then find the 28^{th} term.
2. In a H.P., if 5^{th} term is 6 and 3^{rd} term is 10, find the 2^{nd} term.
3. If the p^{th}, q^{th}, and r^{th} terms of a H.P. are a, b, c, respectively, then prove that $\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0$
4. Find sum to n terms of the series $3+5\times\frac{1}{4}+7\times\frac{1}{4^2}+\ldots$
5. If the sum to the infinity of the series $3+5r+7r^2+\ldots$ is $\frac{44}{9}$, then find the value of r.
6. If the sum to infinity of the series $3+(3+d)\cdot\frac{1}{4}+(3+2d)\cdot\frac{1}{4^2}+\ldots$ is $\frac{44}{9}$, then find d.

Sum of Series

Sigma Notations (Σ)

Theorems

(a) $\sum_{r=1}^{n}(a_r \pm b_r)=\sum_{r=1}^{n}a_r \pm \sum_{r=1}^{n}b_r$

(b) $\sum_{r=1}^{n}ka_r = k\sum_{r=1}^{n}a_r$

(c) $\sum_{r=1}^{n}k = nk$; where k is a constant.

Results

(a) $\sum_{r=1}^{n}r=\frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$

(sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n}r^4=\frac{n^2(n+1)^2}{4}=\left[\sum_{r=1}^{n}r\right]^2$

(sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n}r^4=\frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

(e) $\sum_{r=1}^{n}(2r-1)=n^2$

(sum of first n odd natural numbers)

(f) $\sum_{r=1}^{n}2r=n(n+1)$

(sum of first n even natural numbers)

Note

If n^{th} term of a sequence is given by $T_n=an^3+bn^2+cn+d$ where a, b, c, d are constants, then sum of n terms

$$S_n=\Sigma T_n=a\Sigma n^3+b\Sigma n^2+c\Sigma n+\Sigma d$$

This can be evaluated using the above results.

Example 17

Sum up to 16 terms of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ is

(A) 450 **(B)** 456

(C) 446 **(D)** none of these

Solution

$$t_n=\frac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots(2n-1)}=\frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}}$$

$$=\frac{\frac{n^2(n+1)^2}{4}}{n^2}=\frac{\frac{(n+1)^2}{4}}{n^2}=\frac{n^2}{4}+\frac{n}{2}+\frac{1}{4}$$

$$\therefore\quad S_n\Sigma t_n=\frac{1}{4}\Sigma v^2+\frac{1}{2}\Sigma v+\frac{1}{4}\Sigma 1$$

$$=\frac{1}{4}\cdot\frac{(n-1)(2n-1)}{6}+\frac{1}{2}\cdot\frac{n(n+1)}{2}+\frac{1}{4}\cdot n$$

$$\therefore\quad S_{16}=\frac{16.17.33}{24}+\frac{16.17}{4}+\frac{16}{4}=446$$ **Ans. (C)**

Method of Difference

Some times the n^{th} term of a sequence or a series can not be determined by the method, we have discussed earlier. So

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we compute the difference between the successive terms of given sequence for obtained the n^{th} terms.

If $T_1, T_2, T_3, \ldots, T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2, \ldots$ constitute an A.P./G.P. n^{th} term of the series is determined and the sum to n terms of the sequence can easily be obtained.

Case 1

(a) If difference series are in A.P., then
Let $T_n = an^2 + bn + c$, where a, b, c are constant

(b) If difference of difference series are in A.P.
Let $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constant

Case 2

(a) If difference are in G.P., then
Let $T_n = ar^n + b$, where r is common ratio and a, b are constant

(b) If difference of difference are in G.P., then
Let $T_n = ar^n + bn + c$, where r is common ratio and a, b, c are constant.
Determine constant by putting $n = 1, 2, 3, \ldots, n$ and putting the value of $T_1, T_2, T_3, \ldots$ and sum of series $(S_n) = \Sigma T_n$

Know Your Progress - 4

1. Find the sum of the series upto n terms

$$1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \frac{1+2+3+4}{4} + \ldots$$

2. Find the sum of 'n' terms of the series whose n^{th} term is $t_n = 3n^2 + 2n$.

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Miscellaneous Example

Example 18

If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

Solution

$$T_n = S_n - S_{n-1}$$

$$\sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r = \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)(n+1)(n+2)}{8}$$

$$= \frac{n(n+1)(n+2)}{8}\left[(n+3)-(n-1)\right]$$

$$T_n = \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \quad \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} = \frac{(n+2)-n}{n(n+1)(n+2)} \qquad \text{(i)}$$

$$= \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$$

Let $V_n = \frac{1}{n(n+1)}$

$$\therefore \quad \frac{1}{T_n} = V_n - V_{n+1}$$

Putting $n = 1, 2, 3, \ldots, n$

$$\Rightarrow \quad \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} + \ldots + \frac{1}{T_n} = (V_1 - V_{n+1})\sum_{r=1}^{n} \frac{1}{T_r}$$

$$= \frac{n^2 + 3n}{2(n+1)(n+2)} \quad \textbf{Ans.}$$

Example 19

Find the sum of n terms of the series $1.3.5 + 3.5.7 + 5.7.9 + \ldots$

Solution

The n^{th} term is $(2n-1)(2n+1)(2n+3)$

$$T_n = (2n-1)(2n+1)(2n+3)$$

$$T_n = \frac{1}{8}(2n-1)(2n+1)(2n+3)\{(2n+5)-(2n-3)\}$$

$$= \frac{1}{8}(V_n - V_{n-1})\left[\text{Let } V_n = (2n-1)(2n+1)(2n+3)(2n+5)\right]$$

$$S_n = \Sigma T_n = \frac{1}{8}[V_n - V_0]$$

$$\therefore \quad S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \frac{15}{8}$$

$$= n\left(2n^3 + 8n^2 + 7n - 2\right) \quad \textbf{Ans.}$$

Example 20

Find the sum of n terms of the series $3 + 7 + 14 + 24 + 37 + \ldots$

Solution

Clearly, here the differences between the successive terms are

$7 - 3, 14 - 7, 24 - 14, \ldots$ i.e. 4, 7, 10 ,13, …, which are in A.P.

Let $S = 3 + 7 + 14 + 24 + \ldots + T_n$

$S = 3 + 7 + 14 + \ldots + T_{n-1} + T_n$

Subtracting, we get

$0 = 3 + [4 + 7 + 10 + 13 + \ldots (n-1) \text{ terms}] - T_n$

$\therefore \quad T_n = 3 + S_{n-1}$ of an A.P. whose $a = 4$ and $d = 3$.

$$\therefore \quad T_n = 3 + \left(\frac{n-1}{2}\right)(2.4(n-2)3) = \frac{6 + (n-1)(3n+2)}{4} \text{ or,}$$

$$T_n = \frac{1}{2}\left(3n^2 - n + 4\right)$$

Now putting $n = 1, 2, 3, \ldots, n$ and adding

$$\therefore \quad S_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n\ ] \quad \textbf{Ans.}$$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$

$$= \frac{n}{2}\left(n^2 + n + 4\right)$$

Aliter Method:

Let $T_n = an^2 + bn + c$

Now, $T_1 = 3 = a + b + c$ (i)

$T_2 = 7 = 4a + 2b + c$ (ii)

$T_3 = 14 = 8a + 3b + c$ (iii)

Solving (i), (ii) and (iii) we get

$$a = \frac{3}{2}, b = -\frac{1}{2} \text{ and } c = 2$$

$$\therefore \quad T_n = \frac{1}{2}\left(3n^2 - n + 4\right)$$

$$\Rightarrow \quad s_n = \Sigma T_n = [3\Sigma n^2 - \Sigma n + 4n]$$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$

$$= \frac{n}{2}\left(n^2 + n + 4\right) \quad \textbf{Ans.}$$

Example 21

Find the sum of n-terms of the series $1 + 4 + 10 + 22 + \ldots$

Solution

Let $S = 1 + 4 + 10 + 22 + \ldots + T_n$ (i)

$S = 1 + 4 + 10 + \ldots + T_{n-1} + T_n$ (ii)

$(i) - (ii) \Rightarrow T_n = 1 + \left(3 + 6 + 12 + \ldots + T_n - T_{n-1}\right)$

$$T_n = 1 + 3\left(\frac{2^{n-1} - 1}{2 - 1}\right)$$

$$T_n = 3.2^{n-1} - 2$$

So $S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$$= 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3 \quad \textbf{Ans.}$$

Aliter Method:

Let $T_n = ar^n + b$, where $r = 2$

Now $T_1 = 1 = ar + b$ (i)

$T_2 = 4 = ar^2 + b$ (ii)

Solving (i) and (ii), we get

$$a = \frac{3}{2}, b = -2$$

$\therefore \quad T_n = 3.2^{n-1} - 2$

$\Rightarrow \quad S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$$= 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3 \quad \textbf{Ans.}$$

Example 22

The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9) … and so on. Show that the sum of the numbers in n^{th} group is $n^3 + (n-1)^3$.

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Solution

The groups are $(1), (2, 3, 4), (5, 6, 7, 8, 9) \ldots$

The number of terms in the groups are 1, 3, 5, …

$\therefore$ The number of terms in the nth group $= (2n-1)$

the last term of the n^{th} group is n^2

If we count from last term common difference should be -1

So the sum of numbers in the nth group

$$= \left(\frac{2n-1}{2}\right)\left\{2n^2 + (2n-2)(-1)\right\}$$

$$= (2n-1)\left(n^2 - n + 1\right) = 2n^3 - 3n^2 + 3n - 1 = n^3 + (n-1)^3$$

Example 23

Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16\left(2^n - 1\right)$, where the function f satisfied $f(x+y) = f(x) \cdot f(y)$ for all natural number x, y and further $f(1) = 2$.

Solution

It is given that

$$f(x+y) = f(x)\, f(y) \text{ and } f(1) = 2$$

$$f(1+1) = f(1)\, f(1)$$

$\Rightarrow$ $f(2) = 2^2,\ f(1+2) = f(1)\, f(2)$

$\Rightarrow$ $f(3) = 2^3,\ f(2+2) = f(2)\, f(2)$

$\Rightarrow$ $f(4) = 2^4$

Similarly $f(k) = 2^k$ and $f(a) = 2^a$

Hence, $\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a) f(k) = f(a) \sum_{k=1}^{n} f(k)$

$$= 20^a \left\{2^1 + 2^2 + \ldots + 2^n\right\}$$

$$2^a \left\{\frac{2\left(2^n - 1\right)}{2-1}\right\} = 2^{a+1}(2n-1)$$

But $\sum_{k=1}^{n} f(a+k) = 16\left(2^n - 1\right)$

$$2^{a+1}\left(2^n - 1\right) = 16\left(2^n - 1\right)$$

$\therefore$ $2^{a+1} = 2^4$

$\therefore$ $a + 1 = 4$

$\therefore$ $a = 3$ **Ans.**

Different Means

Means

Arithmetic Mean

If three terms are in A.P., then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a and c. So A.M. of a and $c = \frac{a+c}{2} = b$.

n-Arithmetic Means Between Two Numbers

If a, b be any two given numbers and $a, A_1, A_2, \ldots, A_n, b$ are in A.P., then $A_1, A_2, \ldots, A_n$ are the 'n' A.M's between a and b then. $A_1 = a + d$, $A_2 = a + 2d, \ldots, A_n = a + nd$ or $b - d$, where

$$d = \frac{b-a}{n+1}$$

$$\Rightarrow \qquad A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \left(\frac{2b-a}{n+1}\right), \ldots$$

Note

Sum of n A.M's inserted between a and b is equal to n times the single A.M. between a and b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a and b.

Geometric Mean

If a, b, c are in G.P., then b is the G.M. between a and c, $b^2 = ac$. So G.M. of a and $c = \sqrt{ac} = b$

n-Geometric Means Between Two Numbers

If a, b are two given positive numbers and $a, G_1, G_2, \ldots, G_n, b$ are in G.P. Then $G_1, G_2, G_3, \ldots, G_n$ are 'n' G.Ms between a and b, where $b = ar^{n+1} \Rightarrow r = (b/a)^{1/n+1}$

$$G_1 = a(b/a)^{1/n+1},$$

$$G_2 = a(b/a)^{2/n+1} \ldots,$$

$$G_n = a(b/a)^{1/n+1},$$

$$= ar, \ = \ ar^2, \ldots = ar^n = b/r$$

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Note

The product of n G.Ms between a and b is equal to n^{th} power of the single G.M. between a and b, i.e.,

$$\prod_{r=1}^{n} G_r = (G)^n$$

where G is the single G.M. between a and b

Harmonic Mean

If a, b, c are in H.P., then b is H.M. between a and c. So H.M. of a and $c = \dfrac{2ac}{a+c} = b$.

Insertion of 'n' HM's between a and b:

$$a, H_1, H_2, H_3, \ldots, H_n, b \rightarrow \text{H.P}$$

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots, \frac{1}{H_n}, \frac{1}{b} \rightarrow \text{A.P.}$$

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \Rightarrow D = \frac{b\ a}{n+1}$$

$$\frac{1}{H_n} = \frac{1}{a} + n\left(\frac{\frac{1}{b} - \frac{1}{a}}{n+1}\right)$$

Important Note

(i) If A, G, H, are, respectively, A.M., G.M., H.M. between two positive number a and b then (a) $G^2 = AH$ (A, G, H constitute a G.P.) (b) $A \geq G \geq H(c)$ $A = G = H \Leftrightarrow a = b$

(ii) Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as

$$A = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

$$G = (a_1 a_2 \ldots a_n)^{1/n}$$

and

$$H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots = a_n$

Example 12

If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that $a + b \geq 6\left(2^{1/3} + 4^{1/3}\right)$.

Solution

Let α, β, γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $a_1 = -a, a_2 = -b, a_3 = -g \Rightarrow a_1 + a_2 + a_3 = \dfrac{a}{2}$

$$a_1a_2 + a_2a_3 + a_3a_1 = \frac{b}{2}\ a_1a_2a_3 = 2$$

Applying A.M. ≥ G.M., we have

$$\frac{\alpha_1\alpha_2\alpha_3}{3} \geq (a_1a_2a_3)^{1/3} \Rightarrow a \geq 6 \times 2^{1/3}$$

Also $\dfrac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \Rightarrow b \geq 6 \times 4^{1/3}$

Therefore, $a + b^3 6\left(2^{1/3} + 4^{1/3}\right)$.

Example 13

If $a_i > 0\ \forall\ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$, then prove that $(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n) \geq 2^n$

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Solution

Using A.M. ≥ G.M.

$$1 + a_1 \geq 2\sqrt{a_1}$$

$$1 + a_2 \geq 2\sqrt{a_2}$$

$$1 + a_n\ ^3 2\sqrt{a_n}$$

$$\Rightarrow \quad (1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n (a_1\ a_2\ a_3\ \ldots\ a_n)^{1/2}$$

As $a_1a_2a_3\ldots a_n = 1$

Hence $(1+a_1)(1+a_2)\ldots(1+a_n)^3 \geq 2^n$.

Example 14

If a, b, x, y are positive natural numbers such that $\dfrac{1}{x} + \dfrac{1}{y} = 1$ then prove that $\dfrac{a^x}{x} + \dfrac{b^y}{y} \geq ab$.

Solution

Consider the positive numbers a^x, a^x,..., y times and b^y, b^y,..., x times For all these numbers,

$$\text{A.M.} = \frac{\{a^x \cdot a^x \ldots y \text{ time}\} + \{b^y + b^y + \ldots x \text{ times}\}}{x + y}$$

$$= \frac{ya^x + xa^y}{(x + y)}$$

$$\text{G.M.} = \{(a^x, a^x, \ldots, y \text{ times})(b^y \cdot b^y, \ldots, x \text{ times})\}^{\frac{1}{(x+y)}}$$

$$= \left[\left(a^{xy}\right) \cdot \left(b^{xy}\right)\right]^{\frac{1}{(x+y)}} = (ab)^{\frac{xy}{(x+y)}}$$

As $\frac{1}{x} + \frac{1}{y} = 1, \frac{x+y}{xy} = 1$, i.e, $x + y = xy$

So using AM ≥ GM $\frac{ya^x + xa^y}{x + y} \geq (ab)^{\frac{xy}{x+y}}$

$\therefore$ $\frac{ya^x + xa^y}{xy} \geq ab$ or $\frac{a^x}{x} + \frac{a^y}{y} \geq ab.$

Know Your Progress - 5

1. If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the G.M. between a and b then find the value of 'n'.

2. If b is the harmonic mean between a and c, then prove that $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$.

3. **(a)** The harmonic mean of two numbers is 4. The arithmetic mean A and the geometric mean G satisfy the relation $2A + G^2 = 27$. Find the two numbers.

(b) The AM of two numbers exceeds their GM by 15 and HM by 27. Find the numbers.

4. If $A_1, A_2, A_3, \ldots, A_{51}$ are arithmetic means inserted between the numbers a and b, then find the value of $\left(\frac{b + A_{51}}{b - A_{51}}\right) - \left(\frac{A_1 + a}{A_1 - a}\right)$.

5. An H.M. is inserted between the number 1/3 and an unknown number. If we diminish the reciprocal of the inserted number by 6, it is the G.M. of the reciprocal of 1/3 and that of the unknown number. If all the terms of the respective H.P. are distinct, then

(A) the unknown number is 27

(B) the unknown number is 1/27

(C) the H.M. is 15

(D) the G.M. is 21

6. If one A.M. 'a' and two G.M.'s, p and q be inserted between any two given numbers, then show that $p^3 + q^3 = 2apq$.

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in A.P. such that $a_4 - a_7 + a_{10} = m$, then the sum of first 13 terms of this A.P., is:
(A) 15 m **(B)** 10 m
(C) 12 m **(D)** 13 m

2. Let $a_1, a_2, a_3, \ldots$ be an A.P., such that $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + a_3 + \ldots + a_q} = \frac{p^3}{q^3}$; $p \neq q$. Then $\frac{a_6}{a_{21}}$ is equal to:
(A) $\frac{121}{1861}$ **(B)** $\frac{11}{41}$
(C) $\frac{121}{1681}$ **(D)** $\frac{41}{11}$

3. Given sum of the first n terms of an A.P. is $2n + 3n^2$. Another A.P. is formed with the same first term and double of the common difference, the sum of n terms of the new A.P. is:
(A) $n + 4n^2$ **(B)** $n^2 + 4n$
(C) $3n + 2n^2$ **(D)** $6n^2 - n$

4. If a, b, c are in A.P., then $(a - c)^2$ equals
(A) $4(b^2 - ac)$ **(B)** $4(b^2 + ac)$
(C) $4b^2 - ac$ **(D)** $b^2 - 4ac$

5. If the sum of n terms of an A.P. is $Pn + Qn^2$, where P, Q are constants, then its common difference is
(A) $2Q$ **(B)** $P + Q$
(C) $2P$ **(D)** $P - Q$

6. Given a sequence of 4 numbers, first three of which are in G.P. and the last three are in A.P. with common difference six. If first and last terms of this sequence are equal, then the last term is:
(A) 8 **(B)** 16
(C) 2 **(D)** 4

7. The first term of an infinite G.P. is 1 and every term is equal to the sum of the successive terms, then its fourth term will be:
(A) $\frac{1}{2}$ **(B)** $\frac{1}{8}$
(C) $\frac{1}{4}$ **(D)** $\frac{1}{16}$

8. If G be the GM between x and y, then the value of $\frac{1}{G^2 - x^2} + \frac{1}{G^2 - y^2}$ is equal to:
(A) G^2 **(B)** $\frac{2}{G^2}$
(C) $\frac{1}{G^2}$ **(D)** $3G^2$

9. $2 + 4 + 7 + 11 + 16 + \ldots$ to n terms =
(A) $\frac{1}{6}(n^2 + 3n + 8)$ **(B)** $\frac{n}{6}(n^2 + 3n + 8)$
(C) $\frac{1}{6}(n^2 - 3n + 8)$ **(D)** $\frac{n}{6}(n^2 - 3n + 8)$

10. If a, b, c are in H.P., then $\frac{a - b}{b - c}$ is equal to
(A) $\frac{a}{b}$ **(B)** $\frac{b}{a}$
(C) $\frac{a}{c}$ **(D)** $\frac{c}{b}$

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11. If a, b, and c are positive real numbers, then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is greater than or equal to
(A) 3 **(B)** 6
(C) 27 **(D)** 5

12. If $a_1, a_2, a_3, \ldots, a_n \in R^+$ and $a_1 \cdot a_2 \cdot a_3 \ldots a_n = 1$, then minimum value of $(1 + a_1 + a_1^2)$ $(1 + a_2 + a_2^2)$ $(1 + a_3 + a_3^2) \ldots (1 + a_n + a_n^2)$ is equal to:
(A) 3^{n+1} **(B)** 3^n
(C) 3^{n-1} **(D)** none of these

13. The value of $1^2 + 3^2 + 5^2 + \ldots + 25^2$ is:
(A) 1728 **(B)** 1456
(C) 2925 **(D)** 1469

14. The sum of the series: $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots$ up to 10 terms, is:
(A) $\frac{22}{13}$ **(B)** $\frac{18}{11}$
(C) $\frac{20}{11}$ **(D)** $\frac{16}{9}$

15. If $a = \sum_{n=0}^{\infty} x^n, b = \sum_{n=0}^{\infty} y^n, c = \sum_{n=0}^{\infty} (xy)^n$ where $|x|$, $|y| < 1$; then:

(A) $abc = a + b + c$ **(B)** $ab + bc = ac + b$
(C) $ac + bc = ab + c$ **(D)** $ab + ac = bc + a$

16. If $r > 1$ and $x = a + \frac{a}{r} + \frac{a}{r^2} + \ldots$ to ∞, $y = b - \frac{b}{r} + \frac{b}{r^2} - \ldots$ to ∞ and $z = c + \frac{c}{r^2} + \frac{c}{r^4} + \ldots$ to ∞, then $\frac{xy}{z}$

(A) $\frac{ab}{c}$ **(B)** $\frac{ac}{b}$
(C) $\frac{bc}{a}$ **(D)** None of these

17. If a, b, c are positive real numbers such that $ab^2c^3 = 64$, then minimum value of $\left(\frac{1}{a} + \frac{2}{b} + \frac{3}{c}\right)$ is equal to:

(A) 6 **(B)** 2
(C) 3 **(D)** None of these

18. In a G.P., first term is 1. If $4T_2 + 5T_3$ is minimum, then its common ratio is:

(A) $\frac{2}{5}$ **(B)** $-\frac{2}{5}$
(C) $\frac{3}{5}$ **(D)** $-\frac{3}{5}$

19. The sum $\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \ldots$ upto 11-terms is:

(A) $\frac{11}{4}$ **(B)** $\frac{60}{11}$
(C) $\frac{7}{2}$ **(D)** $\frac{11}{2}$

20. The sum of the series: $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$ upto 10 terms is:

(A) 11300 **(B)** 12100
(C) 12300 **(D)** 11200

21. Let $a_1, a_2, a_3, \ldots$ be terms of an A.P. If $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + \ldots + a_q} = \frac{p^2}{q^2}$, $p \neq q$ then $\frac{a_6}{a_{21}}$ equals:

(A) $\frac{2}{7}$ **(B)** $\frac{11}{41}$
(C) $\frac{41}{11}$ **(D)** $\frac{7}{2}$

22. If $a_1, a_2, \ldots, a_n$ are in H.P., then the expression $a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n$ is equal to:

(A) na_1a_n **(B)** $(n-1)a_1a_n$
(C) $n(a_1 - a_n)$ **(D)** $(n-1)(a_1 - a_n)$

23. In a G.P. consisting of positive terms, each term equals the sum of the next two terms. Then, the common ratio of this progression equals:

(A) $\frac{1}{2}\sqrt{5}$ **(B)** $\sqrt{5}$
(C) $\frac{1}{2}\left(\sqrt{5} - 1\right)$ **(D)** $\frac{1}{2}\left(1 - \sqrt{5}\right)$

24. The first two terms of a G.P. add up to 12. The sum of the third and the fourth terms is 48. If the terms of the G.P. are alternately positive and negative, then the first term is:

(A) –4 **(B)** –12
(C) 12 **(D)** 4

25. The sum to infinity of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots$ is:

(A) 4 **(B)** 6
(C) 2 **(D)** 3

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26. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in an A.P. with common difference –2, then the time taken by him to count all notes is:

(A) 24 minutes **(B)** 34 minutes
(C) 125 minutes **(D)** 135 minutes

27. A man saves ₹200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹40 more than the saving of immediately previous month. His total saving from the start of service will be ₹11040 after:

(A) 20 months **(B)** 21 months
(C) 18 months **(D)** 19 months

28. Let a_n be the n^{th} term of an A.P. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the A.P. is:

(A) $\frac{\alpha - \beta}{200}$ **(B)** $\alpha - \beta$
(C) $\frac{\alpha - \beta}{100}$ **(D)** $\beta - \alpha$

29. **Statement-1:** The sum of the series $1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \ldots + (361 + 380 + 400)$ is 8000.

Statement-2: $\sum_{k=1}^{n}\left(k^3 - (k-1)^3\right) = n^3$, for any natural number n.

(A) Statement-1 is true, Statement-2 is false.
(B) Statement-1 is false, Statement-2 is true.
(C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(D) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

30. If 100 times the 100^{th} term of an A.P. with non-zero common difference equals the 50 times its 50^{th} term, then the 150^{th} term of this A.P. is:
(A) zero
(B) –150
(C) 150 times its 50^{th} term
(D) 150

31. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, …, is:
(A) $\frac{7}{81}(179 - 10^{-20})$ **(B)** $\frac{7}{9}(99 - 10^{-20})$
(C) $\frac{7}{81}(179 + 10^{-20})$ **(D)** $\frac{7}{9}(99 - 10^{-20})$

32. Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is:
(A) $\frac{\sqrt{61}}{9}$ **(B)** $\frac{2\sqrt{17}}{9}$
(C) $\frac{\sqrt{34}}{9}$ **(D)** $\frac{2\sqrt{13}}{9}$

33. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is:
(A) $\sqrt{2} + \sqrt{3}$ **(B)** $3 + \sqrt{2}$
(C) $2 - \sqrt{3}$ **(D)** $2 + \sqrt{3}$

34. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to:
(A) $\frac{121}{10}$ **(B)** $\frac{441}{100}$
(C) 100 **(D)** 110

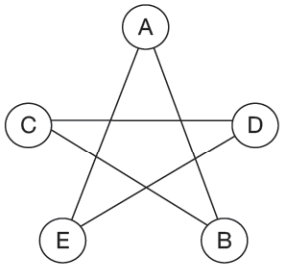
35. If m is the A.M. of two distinct real numbers λ and $n(\lambda, n > 1)$ and G_1, G_2, and G_3 are three geometric means between λ and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals
(A) $4\lambda mn^2$ **(B)** $4\lambda^2 m^2 n^2$
(C) $4\lambda^2 mn$ **(D)** $4\lambda m^2 n$

36. If the 2^{nd}, 5^{th}, and 9^{th} terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is:
(A) $\frac{7}{4}$ **(B)** $\frac{8}{5}$
(C) $\frac{4}{3}$ **(D)** 1

37. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$, is $\frac{16}{5}m$, then m is equal to:
(A) 99 **(B)** 102
(C) 101 **(D)** 100

PART – II KVPY Level Problems

1. In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9 although not necessarily in this order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$ form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?



(A) 8 **(B)** 9
(C) 10 **(D)** 11

2. The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?
(A) $-\sqrt{3}$ **(B)** $-\frac{2\sqrt{3}}{3}$
(C) $-\frac{\sqrt{3}}{3}$ **(D)** $\sqrt{3}$

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3. Figures 0, 1, 2, and 3 consists of 1, 5, 13, and 25 non-overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would there be in Figure 100?

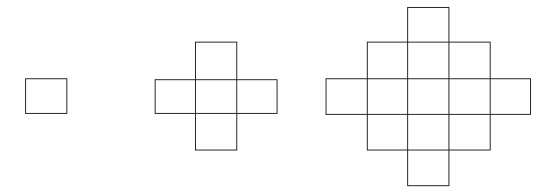

Figure 0 Figure 1 Figure 2

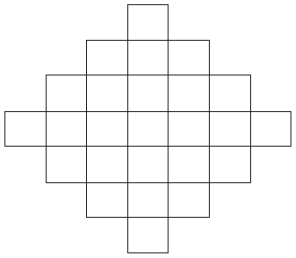

Figure 3

(A) 10, 401 **(B)** 19, 801
(C) 20, 201 **(D)** 39, 801

4. Let 1, 4, … and 9, 16, … be two arithmetic sequences. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

(A) 3722 **(B)** 3732
(C) 3914 **(D)** 3924

5. A sequence of three real numbers forms an arithmetic sequence whose first term is 9. If the first term is unchanged, 2 is added to the second term, and 20 is added to the third term, then the three resulting numbers form a geometric sequence. What is the smallest possible value for the third term of the geometric progression?

(A) 1 **(B)** 4
(C) 36 **(D)** 49

6. Let x and y be positive real numbers. What is the smallest possible value of $\frac{16}{x}+\frac{108}{y}+xy$?

(A) 34 **(B)** 35
(C) 36 **(D)** 37

7. James calculates the sum of the first n positive integers and finds that the sum is 5053. If he has counted one integer twice, which one is it?

(A) 02 **(B)** 03
(C) 04 **(D)** 05

8. Let $a_1, a_2, \ldots$ be a sequence for which $a_1 = 2$ $a_2 = 3$ and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for each positive integer $n \geq 3$. What is a_{2006}?

(A) 02 **(B)** 03
(C) 04 **(D)** 05

9. If S_p denotes the sum of the series $1 + r^P + r^{2P} + \ldots$ to ∞ and s_p denotes the sum of the series $1 - r^P + r^{2P} - r^{3P} + \ldots$ to ∞, $|r| < 1$, then $S_p + s_p$ is

(A) $2S_{2p}$ **(B)** 0
(C) $\frac{1}{2}S_{2p}$ **(D)** $-\frac{1}{2}S_{2p}$

10. Let T_r and S_r be the r^{th} term and sum up to r^{th} term of a series, respectively. If for an odd number n, $S_n = n$ and $T_n = \frac{T_{n-1}}{n^2}$, then T_m (m being even) is:

(A) $\frac{2}{1+m^2}$ **(B)** $\frac{2m^2}{1+m^2}$
(C) $\frac{(m+1)^2}{2+(m+1)^2}$ **(D)** $\frac{2(m+1)^2}{1+(m+1)^2}$

11. The sum of first n terms of an A.P. is given by $S_n = (1 + T_n)(n + 2)$. Then T_2 is:

(A) $-\frac{11}{6}$ **(B)** $\frac{-5}{3}$
(C) $\frac{5}{3}$ **(D)** 2

12. The first and last terms of an A.P. are a and ℓ, respectively. If s be the sum of all the terms of the A.P., then common difference is:

(A) $\frac{\ell^2 - a^2}{2s-(\ell+a)}$ **(B)** $\frac{\ell^2 - a^2}{2s-(\ell-a)}$
(C) $\frac{\ell^2 - a^2}{2s+(\ell+a)}$ **(D)** $\frac{\ell^2 - a^2}{2s-(\ell+a)}$

13. In a sequence of 21 terms first 11 terms are in A.P. with common difference 2 and last 11 terms are in G.P. with common ratio 2. If middle term of A.P. is equal to middle term of G.P. then, middle term in the complete sequence is:

(A) $\frac{10}{1-2^5}$ **(B)** $\frac{10(1-2^{11})}{(2^{10}-1)}$
(C) $\frac{1-2^{11}}{2^{10}-1}+10$ **(D)** $\frac{20}{2^{10}-1}$

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14. $x + y + z = 15$ when a, x, y, z, bare in A.P. and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$ when a, x, y, z, bare in H.P., then the quadratic equation whose roots are $\frac{1}{a}$ and $\frac{1}{b}$ is

(A) $x^2 - 10x + 9 = 0$ **(B)** $x^2 + 10x - 9 = 0$
(C) $9x^2 - 10x + 1 = 0$ **(D)** $9x^2 + 10x + 1 = 0$

15. Harmonic mean of the reciprocal of even numbers from 12 to 190 is

(A) $\frac{1}{101}$ **(B)** $\frac{2}{111}$
(C) $\frac{1}{111}$ **(D)** $\frac{2}{101}$

16. If $\log_{10}(x^3 + y^3) - \log_{10}(x^2 + y^2 - xy) \le 2$. Where x, y are positive real numbers. Then the maximum value of 'xy' is

(A) 2500 **(B)** 3000
(C) 1200 **(D)** 3500

17. If $a + b + c + d + e + f = 12$ then the maximum value of $ab + be + cd + de + ef + fa$.
(a, b, c, d, e, f are non negative real numbers)

(A) 36 **(B)** 24
(C) 30 **(D)** none of these

18. The maximum value of the expression, $f(x, y, z) = xyz(4d - ax - by - cz)$ where a, b, c, d are positive constants, x, y, z are positive variables and $4d - ax - by - cz > 0$ is:

(A) $\frac{abc}{d}$ **(B)** $\frac{d^2}{abc}$
(C) $\frac{abc}{d^3}$ **(D)** $\frac{d^4}{abc}$

19. The minimum value of $\frac{(a^2+3a+1)(b^2+3b+1)(c^2+3c+1)}{abc}$, where $a, b, c \in R^+$ is:

(A) $\frac{11^3}{2^3}$ **(B)** 125
(C) 25 **(D)** 27

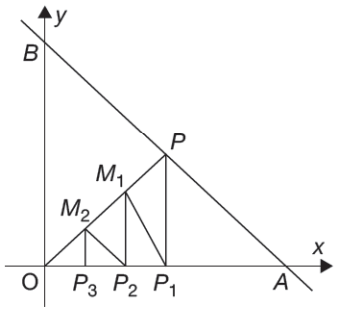
20. If p, q are two numbers such that $3 < p < 4$, $q > 4$ or $q < 3$ then $p^2(7q - 12)$ is:

(A) less than $q^2(7p - 12)$
(B) greater than $q^2(7p - 12)$
(C) less than $q^2(12p - 7)$
(D) greater than $q^2(12p - 7)$

21. The 288th term of the series $a, b, b, c, c, c, d, d, d, d, e, e, e, e, e, f, f, f, f, f, f, \ldots$ is:

(A) u **(B)** v
(C) w **(D)** x

22. The line $x + y = 1$ meets x-axis at A and y-axis at B, P is the mid-point of AB; P_1 is the foot of the perpendicular from P to OA; M_1 is that of P_1 from OP; P_2 is that of M_1 from OA; M_2 is that of P_2 from OP; P_3 is that of M_2 from OA; and so on.
If P_n denotes the n^{th} foot of the perpendicular on OA, then $OP\,n$ is equal to



(A) $\left(\frac{1}{2}\right)^{n-1}$ **(B)** $\left(\frac{1}{2}\right)^{n}$
(C) $\left(\frac{1}{2}\right)^{n+1}$ **(D)** None of these

23. ABC is a right-angled triangle in which $\angle B = 90°$ and $BC = a$. If n points $L_1, L_2, \ldots, L_n$ on AB is divided in $n + 1$ equal parts and $L_1M_1, L_2M_2, \ldots L_nM_n$ are line segments parallel to BC and $M_1, M_2, \ldots M_n$ are on AC, then the sum of the lengths of $L_1M_1, L_2M_2, \ldots, L_nM_n$ is

(A) $\frac{a(n+1)}{2}$ **(B)** $\frac{a(n-1)}{2}$
(C) $\frac{an}{2}$ **(D)** None of these

24. Suppose that $F(n + 1) = \frac{2F(n)+1}{2}$ for $n = 1, 2, 3, \ldots$ and $F(1) = 2$. Then $F(101)$ equals

(A) 50 **(B)** 52
(C) 54 **(D)** none of these

25. In a geometric progression with common ratio 'q' the sum of the first 109 terms exceeds the sum of the first 100 terms by 12. If the sum of the first nine terms of the progression is $\frac{\lambda}{q^{100}}$, then the value of λ equals to:

(A) 10 **(B)** 14
(C) 12 **(D)** 22

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26. If $a_1, a_2, \ldots, a_6$ $(a_i > 0)$ are the roots of the equation $x^6 - 12x^5 + ax^4 - bx^3 + cx^2 - dx + 64 = 0$ then b is equal to

(A) 20 **(B)** 60
(C) 160 **(D)** 240

27. $\sum_{k=1}^{n} \frac{k}{k^4 + \frac{1}{4}} =$

(A) $\frac{4n^2 - 4n + 4}{2n^2 + 2n + 1}$ **(B)** $\frac{3n^2 - n + 2}{2n^2 + 2n + 1}$

(C) $\frac{2n^2 - 2n}{2n^2 + 2n + 1}$ **(D)** $\frac{n^2 + 5n - 2}{2n^2 + 2n + 1}$

28. If $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are in A.P. and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are in G.P., then $\frac{\cos^2\alpha + \cos^2\gamma - 4\cos\alpha\cos\gamma}{1 - \sin\alpha\sin\gamma} =$

(A) -2 **(B)** -1
(C) 0 **(D)** 2

PART – III PRMO Level Problems

1. If $f(x) = \frac{1}{\sqrt[3]{x^2 + 2x + 1} + \sqrt[3]{x^2 - 1} + \sqrt[3]{x^2 - 2x + 1}}$ for all positive integers x, find the value of $f(1) + f(3) + f(5) + \ldots + f(997) + f(999)$.

2. There are n balls in a box, and the balls are numbered 1, 2, 3, …, n respectively. One of the balls is removed from the box, and it turns out that the sum of the numbers on the remaining balls in the box is 5048. If the number on the ball removed from the box is m, find the value of m.

3. Consider the sequence of numbers: 4, 7, 1, 8, 9, 7, 6, … For $n > 2$, the nth term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. What is the last two digits of smallest value of n for which $S_n > 10{,}000$?

4. Suppose that the sequence $\{a_n\}$ is defined by $a_1 = 2$, and $a_n + 1 = a_n = 2n$, when $n \geq 1$. What are last two digits of a_{100}?

5. The increasing sequence of positive integers a_1, a_2, a_3, … has the property that $a_{n+2} = a_n + a_{n+1}$, for all $n \geq 1$. Suppose that $a_7 = 120$. What is $a_8/2$?

6. Consider the sequence of numbers: 4, 7, 1, 8, 9, 7, 6, … For $n > 2$, the nth term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. If smallest value of n for $S_n > 10{,}000$ is $abcd$ then find $a + b + c + d$:

7. Consider a sequence of real number $\{a_n\}$ defined by $a_1 = 1$ and $a_{n+1} = \frac{a_n}{1 + na_n}$ for $n \geq 1$. Find the value of $\frac{1}{a_{2005}} - 2009000$.

8. A sequence $\{a_n\}$ is defined by $a_1 = 2$, $a_n = \frac{1 + a_{n-1}}{1 - a_{n-1}}$, $n \geq 2$. Find the value of $(1100 + 2008\ a_{2007})$.

9. Find the minimum value of $\sqrt{\sum_{k=1}^{100} |n - k|}$, where n ranges over all positive integers.

10. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of $S2$?

11. The sequence $\log_{12}162$, $\log_{12}x$, $\log_{12}y$, $\log_{12}z$, $\log_{12}1250$, is an arithmetic progression. What is $\frac{x}{10}$?

12. If $S = \sum_{k=1}^{99} \frac{(-1)^{k+1}}{\sqrt{k(k+1)}\left(\sqrt{k+1} - \sqrt{k}\right)}$, find the value of $10S$.

13. Given that $x + (1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where each a_r is an integer, $r = 0, 1, 2, \ldots, n$. Find the value of n such that $a_0 + a_2 + a_3 + a_4 + \ldots + a_{n-2} + a_{n-1} = 60 - \frac{n(n+1)}{2}$.

14. A finite sequence of three-digit integers has the property that the tens and units digits of the each terms are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

15. The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous terms. Find $\frac{T_{2005}}{10}$

16. Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. Then a_{12} is

17. Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If x, $2y$, $3z$ is an arithemetic sequence, then, $27r$ is?

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18. Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that $a_1 + a_2 + \ldots + a_{2018} = 2018^{2018}$. What is the remainder when $a_1^3 + a_2^3 + \ldots + a_{2018}^3$ is divided by 6?

19. A function f is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n-1) - f(n-2) + n$ for all integers $n \geq 3$. What is the value of $[f(2018) - 2000]$?

20. How many four digit integers $abcd$, with $a \neq 0$, have the property that the three two-digit integers $ab < bc < cd$ form an increasing arithmetic sequence? One such number is 4692, where $a = 4$, $b = 6$, $c = 9$, and $d = 2$.

21. The product (8)(888….8), where the second factor has k digits, is an integer whose digits have a sum of 1000. Find the value of $(1000 - k)$?

22. Let $[x]$ denote the greatest integer part of a real number x. If $M = \sum_{n=1}^{40} \left[\frac{n^2}{2}\right]$ then M equals (last two digits)

23. The sum of 7 consecutive positive integers is equal to the sum of the next five consecutive integers. What is the largest among the 12 numbers?

24. Let a sequence have 200 zeroes to start with. In step 1, to every position in the sequence we add 1. In step 2, to every even position in the sequence we add 1. In step 3, to every position which is a multiple of 3, we add 1. This is continued up to 200-th step. After the 200-th step, what will be the value in the 128th position?

25. In the sequence $a_1, a_2, a_3, \ldots, a_n$, the sum of any three consecutive terms is 40; if the third term is 10 and the eighth term is 8; find the 2013th term.

26. A sequence has first term 2007, after which every term is the sum of the squares of the digits of the preceding term. Find last two digits of this sequence upto 2013 terms

27. Find fifth term of finite sequence of 16 numbers, such that
(a) it reads same form left to right as form right to left
(b) the sum of any 7 consecutive terms is –1
(c) the sum of any 11 consecutive terms is +1.

28. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the largest possible value for the third term in the G.P.?

29. In the sequence 2001, 2002, 2003, …, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is 2001 + 2002 – 2003 = 2000. What is the 2004th term in this sequence?

30. Consider the sequence of numbers: 4, 7, 1, 8, 9, 7, 6, …. For $n > 2$, the nth term of the sequence is the units digit of the sum of the two previous terms. Let Sn denote the sum of the first n terms of this sequence. If smallest value of n for $Sn > 10{,}000$ is $abcd$ then find $a + b + c + d$:

31. Let x and y be positive real numbers. What is the smallest possible value of $\frac{16}{x} + \frac{108}{y} + xy$?

32. Find the value of the positive integer n if
$$\frac{1}{\sqrt{4}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{7}} + \ldots + \frac{1}{\sqrt{n}+\sqrt{n+1}} = 5.$$

33. Consider a sequence of real number $\{a_n\}$ defined by
$a_1 = 1$ and $a_{n+1} = \frac{a_n}{1 + na_n}$ for $n \geq 1$.
Find the value of $\frac{1}{a_{2005}} - 2009000$.

34. James calculates the sum of the first n positive integers and finds that the sum is 5053. If he has counted one integer twice, which one is it?

35. What is the value of
$2100 - (x+1)(x+2006)$
$$\left[\frac{1}{(x-1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots + \frac{1}{(x+2005)(x+2006)}\right]?$$

36. Suppose an denotes the last two digits of 7^n. For example, $a_2 = 49$, $a_3 = 43$. The value of $a_1 + a_2 + a_3 + \ldots + a_{2007}$ is given by x. Find sum of all the digits of x.

37. A sequence $\{a_n\}$ is defined by $a_1 = 2$, $a_n = \frac{1 + a_{n-1}}{1 - a_{n-1}}$, $n \geq 2$. Find the value of $(1100 + 2008\, a_{2007})$.

38. Find the minimum value of $\sqrt{\sum_{k=1}^{100} |n-k|}$, where n ranges over all positive integers.

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39. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

40. Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a G.P. What is the smallest possible value of $10c$?

41. The sequence $S_1, S_2, S_3, \ldots, S_{10}$ has the property that every term beginning with the third is the sum of the previous two. That is, $S_n = S_{n-2} + S_{n-1}$ for $n \geq 3$. Suppose that $S_9 = 110$ and $S_7 = 42$. What is S_4?

42. The sequence $\log_{12}162$, $\log_{12}x$, $\log_{12}y$, $\log_{12}z$, $\log_{12}1250$, is an arithmetic progression. What is $\frac{x}{10}$?

43. The internal angles of quadrilateral $ABCD$ form an arithmetic progression. Triangles ABD and DCB are similar with $\angle DBA = \angle DCB$ and $\angle ADB = \angle CBD$. Moreover, the angles in each of these two triangles also form an arithmetic progression. In degrees, the largest possible sum of two largest angles of $ABCD$ is R. What is $\frac{R}{3}$?

44. Two non-decreasing sequences of nonnegative integers have different first terms, Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. The smallest possible value of N is n. What is half of n?

45. The first four terms of an arithmetic sequence are p, 9, $3p - q$, and $3p + q$. What is the sum of digits of the 2010^{th} term of the sequence?

46. Let $a + ar_1 + ar_1 2 + ar_1 3 \ldots$ and $a + ar_2 + ar_2 2 + ar_2 3 + \ldots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $31(r_1 + r_2)$?

47. For each positive integer n, the mean of the first n terms of a sequence is n. What will be the answer when 2008^{th} term of the sequence is divide by 55?

48. The geometric series $a + ar + ar^2 + \ldots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is $10(a + r)$?

49. Find the maximum value of $\sqrt{x-144} + \sqrt{722-x}$.

50. If $S = \sum_{k=1}^{99} \frac{(-1)^{k+1}}{\sqrt{k(k+1)}\left(\sqrt{k+1} - \sqrt{k}\right)}$, find the value of $10S$.

51. Let $a_1, a_2, a_3, \ldots$ be the sequence of all positive integers that are relatively prime to 75, where $a_1 < a_2 < a_3 < \ldots$ (The first five terms of the sequence are: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 8$.)
Then the value of a_{2008} is defined as $p10^3 + q10^2 + r10 + s$. Find $(p + q + r + s)$.

52. Given that
$x + (1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^n$
$= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$,
where each a_r is an integer, $r = 0, 1, 2, \ldots, n$.
Find the value of n such that $a_0 + a_2 + a_3 + a_4 + \ldots + a_{n-2} + a_{n-1} = 60 - \frac{n(n+1)}{2}$.

53. Given that $a_{n+1} = \frac{a_{n-1}}{1 + na_{n-1}a_n}$, where $n = 1, 2, 3, \ldots$ and $a_0 = a_1 = 1$, find the value of $\left(2 \times 10^4 - \frac{1}{a_{199}a_{200}}\right)$.

54. Suppose that (u_n) is a sequence of real numbers satisfying $u_{n+2} = 2u_{n+1} + u_n$, and that $u_3 = 9$ and $u_6 = 128$. what is u_5?

55. For each positive integer n, the mean of the first n terms of a sequence is n. What is the square root of 2008^{th} term of the sequence to the nearest integer?

56. A finite sequence of three-digit integers has the property that the tens and units digits of the each terms are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

57. How many non-similar triangle have angles whose degree measures are distinct positive integers in arithmetic progression?

58. Let $a_1, a_2, \ldots$ be a sequence for which $a_1 = 2$ $a_2 = 3$ and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for each positive integer $n \geq 3$. What is a_{2006}?

59. The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous terms. Find $\frac{T_{2005}}{10}$

60. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the

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three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

61. Let a_1, a_2, …, be a sequence with the following properties.
I. $a_1 = 1$, and
II. $a_{2n} = n \cdot a_n$ for any positive integer n.
Find the sum of digits of 'n' if $a_2100 = 2^n$.

62. In the sequence 2001, 2002, 2003, …, each term after the third is found by sub-tracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is 2001 + 2002 – 2003 = 2000. What is the 2004^{th} term in this sequence?

63. Let 1, 4, … and 9, 16, … be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. Suppose there are 'n' distinct numbers in set 'S', then find the sum of the digits of 'n'.

64. Suppose that $\{a_n\}$ is an arithmetic sequence with $a_1 + a_2 + \ldots + a_{100} = 100$ and $a_{101} + a_{102} + \ldots + a_{200} = 200$. What is the value of $10^3(a_2 - a_1)$?

65. Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. Then a_{12} is

66. Let a_1, a_2, …, a_k be a finite arithemetic sequence with $a_4 + a_7 + a_{10} = 17$ and $a_4 + a_5 + a_6 + \ldots + a_{12} + a_{13} + a_{14} = 77$. If $a_k = 13$, the $k =$

67. Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If x, $2y$, $3z$ is an arithemetic sequence, then, $27r$ is?

68. The sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 1, 2, … consists of 1's separated by blocks of 2's with n 2's in the n^{th} block. The sum of digits of the sum of the first 1234 terms of this sequence is

69. Define a sequence of real numbers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and $a_{n+1}^3 = 99a_n^3$ for all $n \geq 1$. Then $(a_{100})^{1/33}$ equals

70. The number of terms in an A.P. is even. The sums of the odd and even numbered terms are 24 and 30, respectively. If the last term exceeds the first by 10.5, the number of terms in the A.P. is

71. In a geometric series of positive terms the difference between the fifth and fourth terms is 576, and the difference between the second and first terms is 9. What is the last two digits of sum of the first five terms of this series?

72. Let a_1, a_2, …, a_{2018} be a strictly increasing sequence of positive integers such that $a_1 + a_2 + \ldots + a_{2018} = 2018^{2018}$. What is the remainder when $a_1^3 + a_2^3 + \ldots + a_{2018}^3$ is divided by 6?

73. A function f is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n-1) - f(n-2) + n$ for all integers $n \geq 3$. What is the value of $[f(2018) - 2000]$?

74. Define a sequence recursively by $F_0 = 0$, $F_1 = 1$, and F_n = the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus, the sequence starts 0, 1, 1, 2, 0, 2 … What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2022} + F_{2023} + F_{2024}$?

75. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Find the value of $[S(n+1) - 1200]$?

76. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

77. How many four digit integers $abcd$, with $a \neq 0$, have the property that the three two-digit integers $ab < bc < cd$ form an increasing arithmetic sequence? One such number is 4692, where $a = 4$, $b = 6$, $c = 9$, and $d = 2$.

78. The product (8)(888….8), where the second factor has k digits, is an integer whose digits have a sum of 1000. Find the value of $(1000 - k)$?

79. Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequences is N. If the smallest possible value of N is k, find $\frac{k}{2}$.

80. Consider the set of numbers $\{1, 10, 10^2, 10^3, \ldots, 10^{10}\}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. The sum of all natural numbers less than 400 that are NOT divisible by 6, is: **[NSEJS-2009]**

(A) 13266 **(B)** 66534
(C) 79800 **(D)** 93066

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2. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}$ and $\frac{a+b-c}{c}$ are in A.P. and $a+b+c \neq 0$, then **[NSEJS-2011]**

(A) $b = \frac{2ac}{a-c}$ **(B)** $b = \frac{2ac}{a+c}$

(C) $b = \frac{a+c}{2}$ **(D)** $b = \sqrt{ac}$

3. If $a^2 + b^2 + c^2 + d^2 = 25$, out of the following statement the correct one is **[NSEJS-2011]**

(A) $ab + bc + cd + da \leq 25$

(B) $ab + bc + cd + da \geq 25$

(C) $ab + bc + cd + da \leq \frac{25}{2}$

(D) $ab + bc + cd + da \geq \frac{25}{2}$

4. In a school, the average score of all 1400 students at an examination was found to be 69.5 the average score of boys in the school was 68 and that of girls was 72. Hence, the number of boys and the number of girls respectively are **[NSEJS-2011]**

(A) 680 and 720 **(B)** 720 and 680

(C) 875 and 525 **(D)** 800 and 600

5. Find the value $\frac{2}{15} + \frac{2}{35} + \frac{2}{63} + \frac{2}{99} + \ldots + \frac{2}{9999}$ **[NSEJS-2012]**

(A) 8/33 **(B)** 2/11

(C) 98/303 **(D)** 222/909

6. $x + y + z = 1$; $x \neq z$; x, y, z are real number and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = m$. How many of he following values of m are always possible. **[NSEJS-2012]**

(i) $m = 6$ (ii) $m = 8$ (iii) $m = 10$ (iv) $m = 12$

(A) 1 **(B)** 2

(C) 3 **(D)** all

7. The sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{2012} + \frac{1}{2013}$ equals **[NSEJS-2013]**

(A) $\frac{1}{1006} + \frac{1}{1007} + \frac{1}{1008} + \ldots + \frac{1}{2013}$

(B) $\frac{1}{1007} + \frac{1}{1008} + \frac{1}{1009} + \ldots + \frac{1}{2013}$

(C) $\frac{1}{1006} + \frac{1}{1007} + \frac{1}{1008} + \ldots + \frac{1}{2012}$

(D) $\frac{1}{1007} + \frac{1}{1008} + \frac{1}{1009} + \ldots + \frac{1}{2012}$

8. The houses of a row are numbered consecutively from 1 to 49. Find the value of x such that the sum of the numbers of houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. **[NSEJS-2014]**

(A) 25 **(B)** 35

(C) 37 **(D)** No such value exists

9. Along a road lie an odd number of stones placed at intervals of 10 metre. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. If a man starts from one of the end stones, and by carrying them in succession he covers 3 km to pile all stones at the centre. The number of stones is then: **[NSEJS-2014]**

(A) 12 **(B)** 15

(C) 30 **(D)** 25

10. A number is said to be triangular number if it is the sum of consecutive numbers beginning with 1. Which one of the following is not a triangular number: **[NSEJS-2014]**

(A) 1431 **(B)** 190

(C) 28 **(D)** 506

11. There are ten numbers in a certain A.P. The sum of first three terms is 321. The sum of last three numbers is 405. Find the sum of all the tern numbers. **[NSEJS-2015]**

(A) 1165 **(B)** 1210

(C) 1221 **(D)** 1252

12. The sum of first four terms of an A.P. is 56. The sum of last four terms of same A.P. is 112. The first term of the A.P. is 11. Find the number of terms in that A.P. **[NSEJS-2015]**

(A) 7 **(B)** 8

(C) 11 **(D)** 13

13. $\frac{3}{4} + \frac{3}{28} + \frac{3}{70} + \frac{3}{130} + \ldots + \frac{3}{9700} = ?$ **[NSEJS-2015]**

(A) 0.97 **(B)** 0.99

(C) 1 **(D)** 1.03

14. What is the sum of all three digit even numbers divisible by seventeen? **[NSEJS-2015]**

(A) 18846 **(B)** 18684

(C) 14688 **(D)** 16848

15. If $a, b > 0$ then **[NSEJS-2016]**

(A) $a + b \leq \sqrt{ab}$

(B) $a + b > \sqrt{ab}$

(C) $a + b \geq \sqrt{2ab}$

(D) None of the above inequalities will hold

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16. If for $x, y > 0$ we have $\frac{1}{x}+\frac{1}{y}=2$, then the minimum value of xy is: **[NSEJS-2016]**

(A) 2 **(B)** 1
(C) 4 **(D)** $\sqrt{2}$

17. What is the sum of all odd numbers between 500 and 600? **[NSEJS-2017]**

(A) 26000 **(B)** 27000
(C) 27500 **(D)** 29500

18. $1\frac{1}{2}+1\frac{1}{6}+1\frac{1}{12}+1\frac{1}{20}+1\frac{1}{30}+\ldots+1\frac{1}{380}=$ _____. **[NSEJS-2017]**

(A) 20.25 **(B)** 20.04
(C) 19.95 **(D)** 19.85

19. The mean of the following frequency distribution is _____. **[NSEJS-2017]**

Class interval					
Frequency					

(A) 25 **(B)** 28
(C) 30 **(D)** 32

20. Let T_k denote the k-th term of an arithmetic progression. Suppose there are positive integers $m \neq n$ such that $T_m = 1/n$ and $T_n = 1/m$. Then T_{mn} equals: **[NSEJS-2018]**

(A) $\frac{1}{mn}$ **(B)** $\frac{1}{m}+\frac{1}{n}$
(C) 1 **(D)** 0

21. For a regular k-sided polygon, let $a(k)$ denotes its interior angle. Suppose $n > 4$ is such that $a(n-2)$, $a(n)$, $a(n+3)$ forms an arithmetic progression. The sum of digits of n is: **[NSEJS-2018]**

(A) 2 **(B)** 3
(C) 4 **(D)** 5

22. The sum of 5 numbers in geometric progression is 24. The sum of their reciprocals is 6. The product of the terms of the geometric progression is a **[NSEJS-2018]**

(A) 36 **(B)** 32
(C) 24 **(D)** 18

23. The integer closest to $\sqrt{111\ldots1-222\ldots2}$, where there are 2018 ones and 1009 two's, is **[NSEJS-2018]**

(A) $\frac{10^{1009}-1}{3}$ **(B)** $\frac{10^{1009}-1}{9}$
(C) $\frac{10^{2018}-1}{3}$ **(D)** $\frac{10^{2018}-1}{9}$

24. The solution of the equation $1+4+7+\ldots+x=925$ is **[NSEJS-2019]**

(A) 73 **(B)** 76
(C) 70 **(D)** 74

PART – II Previous Years' KVPY Problems

1. The number of positive integers n in the set $\{1, 2, 3, \ldots, 100\}$ for which the number $\frac{1^2+2^2+3^2+\ldots+n^2}{1+2+3+\ldots+n}$ is an integer is: **[KVPY-2010]**

(A) 33 **(B)** 34
(C) 50 **(D)** 100

2. The expression $\frac{2^2+1}{2^2-1}+\frac{3^2+1}{3^2-1}+\frac{4^2+1}{4^2-1}+\ldots+\frac{(2011)^2+1}{(2011)^2-1}$ lies in the interval **[KVPY-2011]**

(A) $\left(2010, 2010\frac{1}{2}\right)$

(B) $\left(2011-\frac{1}{2011}, 2011-\frac{1}{2012}\right)$

(C) $\left(2011, 2011\frac{1}{2}\right)$

(D) $\left(2012, 2012\frac{1}{2}\right)$

3. For $\frac{2^2+4^2+6^2+\ldots+(2n)^2}{1^2+3^2+5^2+\ldots+(2n-1)^2}$ to exceed 1.01, the maximum value of n is **[KVPY-2012]**

(A) 99 **(B)** 100
(C) 101 **(D)** 150

4. Let x, y, z be three non-negative integers such that $x+y+z=10$. The maximum possible value of $xyz+xy+yz+zx$ is **[KVPY-2013]**

(A) 52 **(B)** 64
(C) 69 **(D)** 73

5. Let a, b, c, d, e be natural numbers in an arithmetic progression such that $a+b+c+d+e$ is the cube of an integer and $b+c+d$ is square of an integer. The least possible value of the number of digits of c is: **[KVPY-2013]**

(A) 2 **(B)** 3
(C) 4 **(D)** 5

6. The value of $\sum_{n=0}^{1947}\frac{1}{2^n+\sqrt{2^{1947}}}$ is equal to **[KVPY-2014]**

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(A) $\frac{487}{\sqrt{2^{1945}}}$ **(B)** $\frac{1946}{\sqrt{2^{1947}}}$

(C) $\frac{1947}{\sqrt{2^{1947}}}$ **(D)** $\frac{1948}{\sqrt{2^{1947}}}$

7. If n is the smallest natural number such that $n + 2n + 3n + \ldots + 99n$ is a perfect square, then the number of digits of n^2 is: **[KVPY-2015]**

(A) 1 **(B)** 2
(C) 3 **(D)** more than 3

8. The average incomes of the people in two villages are P and Q, respectively. Assume that $P \neq Q$. A person moves from the first village to the second village. The new average incomes are P' and Q', respectively. Which of the following is not possible? **[KVPY-2015]**

(A) $P' > P$ and $Q' > Q$
(B) $P' > P$ and $Q' < Q$
(C) $P' = P$ and $Q' = Q$
(D) $P' < P$ and $Q' < Q$

9. Suppose the quadratic polynomial $P(x) = ax^2 + bx + c$ has positive coefficients a, b, c in arithmetic progression in that order. If $P(x) = 0$ has integer roots α and β. Then, $\alpha + \beta + \alpha\beta$ is equal to: **[KVPY-2016]**

(A) 3 **(B)** 5
(C) 7 **(D)** 14

10. Let A, G, and H be the arithmetic mean, geometric mean and harmonic mean, respectively, of two distinct positive real numbers. If a is the smallest of the two roots of the equation $A(G - H)x^2 + G(H - A)x + H(A - G) = 0$ then, **[KVPY-2017]**

(A) $-2 < a < -1$ **(B)** $0 < a < 1$
(C) $-1 < a < 0$ **(D)** $1 < a < 2$

11. Let $n \geq 4$ be a positive integer and let $l_1, l_2, \ldots, l_n$ be the lengths of the sides of arbitrary n sided non-degenerate polygon P. Suppose $\frac{l_1}{l_2} + \frac{l_2}{l_3} + \ldots + \frac{l_{n-1}}{l_n} + \frac{l_n}{l_1} = n$ **[KVPY-2017]**

Consider the following statements:
I. The lengths of the sides of P are equal.
II. The angles of P are equal.
III. P is a regular polygon if it is cyclic.
Then,

(A) I is true and I implies II
(B) II is true
(C) III is false
(D) I and III are true

12. In a Mathematics test, the average marks of boys is $x\%$ and the average marks of girls is $y\%$ with $x \neq y$. If the average marks of all students is $z\%$, the ratio of the number of girls to the total number of students is: **[KVPY-2017]**

(A) $\frac{z - x}{y - x}$ **(B)** $\frac{z - y}{y - x}$

(C) $\frac{z + y}{y - x}$ **(D)** $\frac{z + x}{y - x}$

13. Suppose the sum of the first m terms of an arithmetic progression is n and the sum of its first n terms is m, where $m \neq n$. Then, the sum of the first $(m + n)$ terms of the arithmetic progression is: **[KVPY-2018]**

(A) $1 - mn$ **(B)** $mn - 5$
(C) $-(m + n)$ **(D)** $m + n$

14. The median of all 4-digit numbers that are divisible by 7 is: **[KVPY-2018]**

(A) 5797 **(B)** 5498.5
(C) 5499.5 **(D)** 5490

15. A 100 mark examination was administered to a class of 50 students. Despite only integer marks being given, the average score of the class was 47.5. Then, the maximum number of students who could get marks more than the class average is **[KVPY-2018]**

(A) 25 **(B)** 35
(C) 45 **(D)** 49

16. Let $a_n, n \geq 1$, be an arithmetic progression with first term 2 and common difference 4. Let M_n be the average of the first n terms. Then the sum $\sum_{n=1}^{10} M_n$ is **[KVPY-2019]**

(A) 110 **(B)** 335
(C) 770 **(D)** 1100

17. The number of three digit number $\overline{abc}$ such that the arithmetic mean of b and c the square of their geometric mean are equal is **[KVPY-2019]**

(A) 9 **(B)** 18
(C) 36 **(D)** 54

18. Let N be the set of positive integers. For all $n \in N$, let

$$f_n = (n+1)^{1/3} - n^{1/3}$$

and

$$A = \left\{ n \in N : f_{n+1} < \frac{1}{3(n+1)^{2/3}} < f_n \right\}$$

Then **[KVPY-2019]**

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(A) $A = N$
(B) A is a finite set
(C) the complement of A in N is nonempty, but finite
(D) A and its complement in N are both infinite

PART – III Previous Years' PRMO Problems

1. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$, What is the value of $\sum_{n=1}^{99} \frac{1}{S_n + S_{n-1}}$? **[PRE-RMO – 2013]**

2. The First term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014^{th} term of the sequence? **[PRE-RMO – 2014]**

3. Let u, w be real numbers is geometric progression such that $u > v > w$. Suppose $u^{40} = v^n = w^{60}$. Find the value of n. **[PRE-RMO – 2017]**

4. Integers 1, 2, 3, …, n where $n > 2$ are written on a board. Two numbers m, k (such that $1 < m < n$, $1 < k < n$) are removed and the average of the remaining numbers is found to be 17. What is the maximum sun of the two removed numbers? **[PRE-RMO – 2017]**

5. Five distinct 2-digit numbers are in a geometric progression. Find the middle term. **[PRE-RMO – 2017]**

6. What is the value of $\sum_{\substack{1\le i\le j\le 10 \\ i+j=\text{odd}}} (i+j) - \sum_{\substack{1\le i\le j10 \\ i+j=\text{even}}} (i+j)$? **[PRE-RMO – 2018]**

7. Let $N = 6 + 66 + 666 + \ldots + 666\ldots.66$, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N? **[PRE-RMO – 2018]**

8. What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}} + \frac{b^2}{\frac{c}{29}+\frac{a}{31}} + \frac{c^2}{\frac{a}{29}+\frac{b}{31}} \ge n(a+b+c)$$

holds for all positive real numbers a, b, c. **[PRE-RMO – 2018]**

9. What is the greatest integer not exceeding the sum $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$ **[PRE-RMO – 2019]**

10. Let and form the sequence a_n, $n \ge 2$ by $a_n = 100a_{n-1} + 134$. The first few terms are 24, 2534, 253534, 25353534, …. What is the least value of n for which a_n is divisible by 99? **[PRE-RMO – 2019]**

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ANSWER KEYS

Know Your Progress - 1

1. B **2.** D **3.** A **4.** B **5.** C **6.** A **7.** A **8.** A **9.** B **10.** D

11. **(a)** $\frac{2}{1}, \frac{4}{2}, \frac{8}{3}, \frac{16}{4}, \ldots,$ **(b)** $\frac{2}{3}, \frac{4}{9}, \frac{2}{27}, \frac{4}{81}, \ldots$ **13.** $-(p+q)$ **14.** 900 **15.** 14

Know Your Progress - 2

1. C **2.** D **3.** A **4.** B **5.** C **6.** 931 **7.** 4^5 **9.** 4, 12, 36 **10.** C

Know Your Progress - 3

1. 2 **2.** 15 **3.** – **4.** $4+\frac{8}{9}\left(1-\frac{1}{4^{n-1}}\right)-\left(\frac{2n+1}{3\times 4^{n-1}}\right)$ **5.** $\frac{1}{4}$ **6.** 2

Know Your Progress - 4

1. $\frac{n(n+3)}{4}$ **2.** $\frac{n(n+1)(2n+3)}{2}$

Know Your Progress - 5

1. $\frac{1}{2}$ **3.** **(a)** 6, 3 **(b)** 120, 30 **4.** 102 **5.** B

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. D	**2.** C	**3.** D	**4.** A	**5.** A	**6.** A	**7.** B	**8.** C	**9.** B	**10.** C
11. A	**12.** B	**13.** C	**14.** C	**15.** C	**16.** A	**17.** C	**18.** B	**19.** D	**20.** B
21. B	**22.** B	**23.** C	**24.** B	**25.** D	**26.** B	**27.** B	**28.** C	**29.** C	**30.** A
31. C	**32.** D	**33.** D	**34.** C	**35.** D	**36.** C	**37.** C			

PART – II KVPY Level Problems

1. B	**2.** B	**3.** C	**4.** A	**5.** A	**6.** A	**7.** C	**8.** B	**9.** A	**10.** D
11. A	**12.** A	**13.** A	**14.** C	**15.** A	**16.** A	**17.** A	**18.** D	**19.** B	**20.** A
21. D	**22.** B	**23.** C	**24.** B	**25.** C	**26.** C	**27.** C	**28.** A		

PART – III PRMO Level Problems

1. 5	**2.** 2	**3.** 99	**4.** 2	**5.** 97	**6.** 28	**7.** 11	**8.** 96	**9.** 50	**10.** 16
11. 27	**12.** 11	**13.** 5	**14.** 37	**15.** 25	**16.** 78	**17.** 9	**18.** 4	**19.** 17	**20.** 17
21. 8	**22.** 60	**23.** 23	**24.** 8	**25.** 8	**26.** 36	**27.** 5	**28.** 49	**29.** 0	**30.** 28
31. 36	**32.** 48	**33.** 11	**34.** 3	**35.** 95	**36.** 24	**37.** 96	**38.** 50	**39.** 16	**40.** 20
41. 10	**42.** 27	**43.** 80	**44.** 52	**45.** 13	**46.** 31	**47.** 73	**48.** 25	**49.** 34	**50.** 11
51. 20	**52.** 5	**53.** 99	**54.** 53	**55.** 63	**56.** 37	**57.** 59	**58.** 3	**59.** 25	**60.** 1
61. 18	**62.** 0	**63.** 14	**64.** 10	**65.** 78	**66.** 18	**67.** 9	**68.** 16	**69.** 99	**70.** 8
71. 23	**72.** 4	**73.** 17	**74.** 9	**75.** 39	**76.** 4	**77.** 17	**78.** 7	**79.** 52	**80.** 9

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. B	**2.** B	**3.** A	**4.** C	**5.** C	**6.** B	**7.** B	**8.** B	**9.** D	**10.** D
11. B	**12.** C	**13.** B	**14.** C	**15.** B	**16.** B	**17.** C	**18.** C	**19.** C	**20.** C
21. B	**22.** B	**23.** A	**24.** A						

PART – II Previous Years' KVPY Problems

1. B	**2.** C	**3.** D	**4.** C	**5.** B	**6.** A	**7.** C	**8.** C	**9.** C	**10.** B
11. D	**12.** A	**13.** C	**14.** B	**15.** D	**16.** A	**17.** B	**18.** A		

PART – III Previous Years' PRMO Problems

1. 9	**2.** 370	**3.** 48	**4.** 51	**5.** 36	**6.** 55	**7.** 33	**8.** 14	**9.** 78	**10.** 88

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Chapter

4 Number System

Learning Outcomes

After reading this chapter, you will be able to understand

- Logarithms
- Factors & Multiples
- Divisibility
- Cyclicity
- Base system
- Anti-Logarithms
- HCF & LCM
- Remainders
- Highest Power Dividing a factorial
- Alpha Numeric Puzzles

Logarithms

Every positive real number N can be expressed in exponential form as $a^x = N$ where 'a' is also a positive real number different than unity and is called the base and 'x' is called an exponent. We can write the relation $a^x = N$ in logarithmic form as $\log_a N = x$. Hence $a^x = N \Leftrightarrow \log_a N = x$.

Hence, logarithm of a number to some base is the exponent by which the base must beraised in order to get that number.

Limitations of logarithm: $\log_a N$ is defined only when

(a) $N > 0$
(b) $a > 0$
(c) $a \neq 1$

Notes

(i) For a given value of N, $\log_a N$ will give us a unique value.
(ii) Logarithm of zero does not exist.
(iii) Logarithm of negative reals are not defined in the system of real numbers.

Fundamental Identities

Using the basic definition of logarithm, we have 3 important deductions:

(a) $\log_a 1 = 0$ i.e., logarithm of unity to any base is zero.
(b) $\log_N N = 1$ i.e., logarithm of a number to the same base is 1.
(c) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e., logarithm of a number to the base as its reciprocal is –1

Note

$N = (a)^{\log_a N}$ e.g. $2^{\log_2 7} = 7$

The Principal Properties of Logarithms

If m, n are arbitrary positive numbers where $a > 0$, $a \neq 1$ and x is any real number, then:

(a) $\log_a mn = \log_a m + \log_a n$
(b) $\log_a \frac{m}{n} = \log_a m - \log_a n$
(c) $\log_a m^x = x \log_a m$

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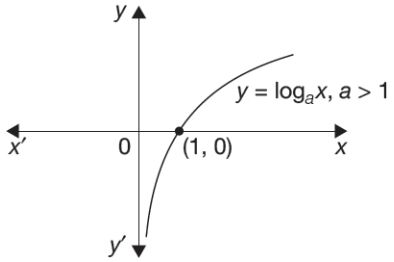
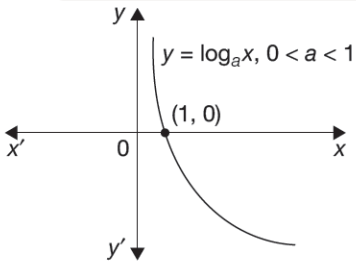
Some Useful Results

1. If $a > 1$, then
   (a) $\log_a x < 0$ [for all x satisfying $0 < x < 1$]
   (b) $\log_a x = 0$ for $x = 1$
   (c) $\log_a x > 0$ for $x > 1$
   (d) $x > y \Rightarrow \log_a x > \log_a y$ i.e., $\log_a x$ is an increasing function.
   $\therefore$ Graph of $y = \log_a x$, $a > 1$



2. If $0 < a < 1$, then
   (a) $\log_a x < 0$ for all $x > 1$
   (b) $\log_a x = 0$ for $x = 1$
   (c) $\log_a x > 0$ for all x satisfying $0 < x < 1$
   (d) $x > y \Rightarrow \log_a x < \log_a y$ i.e., $\log_a x$ is a decreasing function.
   $\therefore$ Graph of $y = \log_a x$, $0 < a < 1$.



Base Changing Theorem

Can be stated as 'quotient of the logarithm of two numbers is independent of their common base'.

Symbolically, $\log_b m = \dfrac{\log_a m}{\log_a b}$, where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

Notes

(i) $\log_b a \cdot \log_a b = \dfrac{\log a}{\log b} \cdot \dfrac{\log b}{\log a} = 1$; hence $\log_b a = \dfrac{1}{\log_a b}$

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) Base power formula: $\log_{a^k} m = \dfrac{1}{k} \log_a m$

Systems of Logarithm

The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and e (= 2.718 approx). Logarithms of numbers to the base 10 are named as **'common logarithm'** and the logarithms of numbers to the base e are called **Natural or Napierian logarithm**. We will consider log x as $\log_e x$ or ln$x$.

Conversion of base e to base 10 and vice versa:

$$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a; \log_{10} a = \frac{\log_e a}{\log_e 10} =$$

$$\log_{10} e \times \log_e a = 0.434 \log_e a$$

Points to Remember

(a) It must be noted that whenever the number and the base are on the same side of unity, then logarithm of that number to that base is positive; however, if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

E.g., $\log_{10} \sqrt[3]{10} = \frac{1}{3}$; $\log_{\sqrt{7}} 49 = 4$; $\log_{\frac{1}{2}} \left(\frac{1}{8}\right) = 3$; $\log_2 \left(\frac{1}{32}\right) = -5$; $\log_{10}(0.001) = -3$

(b) $n \geq 2$, $n \in N$

$\sqrt[n]{a} = a^{1/n} \Rightarrow n^{\text{th}}$ root of 'a' ('a' is a non-negative number)

(c) Some important values: $\log_{10} 2 \approx 0.3010$; $\log_{10} 3 \approx 0.4771$; $\ln 2 \approx 0.693$, $\ln 10 \approx 2.303$

Characteristic and Mantissa

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction. The integer part is called characteristic and the fractional part is called mantissa. When the value of log n is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \geq 1$) or the number of zeros after decimal and before first non–zero digit in the number (if $0 < n < 1$).

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Notes

(i) The mantissa part of logarithm of a number is always non–negative $(0 \le m < 1)$

(ii) If the characteristic of $\log_{10} N$ be n, then the number of digits in N is $(n + 1)$, $N \in$ natural number.

(iii) If the characteristic of $\log_{10} N$ be $(-n)$, then there exist $(n - 1)$ zeros after decimal in N.

Antilogarithm

The positive real number 'n' is called the antilogarithm of a number 'm' if $\log n = m$

Thus, $\log n = m \Leftrightarrow n = \text{antilog } m$

Example 1

If $\log_3 a = 4$, find value of a.

Solution

$$\because \log_3 a = 4$$

$$\therefore a = 3^4 \Rightarrow a = 81.$$

Example 2

Find the value of $\log\frac{9}{8} - \log\frac{27}{32} + \log\frac{3}{4}$.

Solution

Given: $\log\frac{9}{8} - \log\frac{27}{32} + \log\frac{3}{4} = \log\left(\frac{9}{8} \div \frac{27}{32}\right) + \log\frac{3}{4}$

$$= \log\left(\frac{9}{8} \times \frac{32}{27} \times \frac{3}{4}\right)$$

$$= \log 1 = 0. \qquad [\because \log_a 1 = 0]$$

Example 3

If $2\log_4 x = 1 + \log_4(x - 1)$, find the value of x.

Solution

Given $2\log_4 x = 1 + \log_4(x - 1)$

$$\Rightarrow \log_4 x^2 - \log_4(x - 1) = 1$$

$$\Rightarrow \log_4 \frac{x^2}{x-1} = 1 \Rightarrow 4^1 = \frac{x^2}{x-1}$$

$$\Rightarrow x^2 = 4x - 4$$

$$\Rightarrow x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2.$$

Example 4

Evaluate: $3^{2-\log_3 5}$.

Solution

Given $3^{2-\log_3 5} = 3^2 . 3^{-\log_3 5}$ $[\because a^{m+n} = a^m \cdot a^n]$

$$= 9 \cdot 3^{\log_3 5^{-1}} = 9 \times 5^{-1} = \frac{9}{5}.$$

Example 5

If $A = \log_{27} 625 + 7^{\log_{11} 13}$ and $B = \log_9 125 + 13^{\log_{11} 7}$, then find the relation between A and B.

Solution

$$A = \log_{27} 625 + 7^{\log_{11} 13} = \log_{3^3} 5^4 + 7^{\log_{11} 13}$$

or, $$A = \frac{4}{3}\log_3 5 + 7^{\log_{11} 13} \qquad \text{(i)}$$

and

$$B = \log_9 125 + 13^{\log_{11} 7} \quad \text{or} \quad B = \log_{3^2} 5^3 + 7^{\log_{11} 13}$$

or, $$B = \frac{3}{2}\log_3 5 + 7^{\log_{11} 13} \qquad \text{(ii)}$$

By (i) and (ii) we have,

$$A - \frac{4}{3}\log_3 5 = B - \frac{3}{2}\log_3 5$$

$$\because \frac{4}{3}\log_3 5 < \frac{3}{2}\log_3 5$$

$$\therefore A < B.$$

Example 6

Find the value of $\log_{25} 125 - \log_8 4$

Solution

Given, $\log_{25} 125 - \log_8 4 = \log_{5^2} 5^3 - \log_{2^3} 2^2 = \frac{3}{2}\log_5 5 - \frac{2}{3}\log_2 2$

$$= \frac{3}{2} - \frac{2}{3} \qquad [\because \log_a a = 1]$$

$$= \frac{5}{6}.$$

Example 7

The value of N, satisfying $\log_a [1 + \log_b \{1 + \log_c (1 + \log_p N)\}] = 0$ is:

(A) 4 **(B)** 3

(C) 2 **(D)** 1

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Solution

$$1 + \log_b\{1 + \log_c (1 + \log_p N)\} = a^0 = 1$$
$$\Rightarrow \log_b\{1 + \log_c(1 + \log_p N)\} = 0$$
$$\Rightarrow 1 + \log_c(1 + \log_p N) = 1$$
$$\Rightarrow \log_c (1 + \log_p N) = 0$$
$$\Rightarrow 1 + \log_p N = 1$$
$$\Rightarrow \log_p N = 0$$
$$\Rightarrow N = 1$$ **Ans. (D)**

Example 8

Find the value of $2 \log\frac{2}{5} + 3 \log\frac{25}{8} - \log\frac{625}{128}$

Solution

$$2 \log\frac{2}{5} + 3 \log\frac{25}{8} + \log\frac{128}{625}$$
$$= \log\frac{2^2}{5^2} + \log\left(\frac{5^2}{2^3}\right)^3 + \log\frac{2^7}{5^4}$$
$$= \log\frac{2^2}{5^2}.\frac{5^6}{2^9}.\frac{2^7}{5^4} = \log 1 = 0$$

Example 9

Evaluate: $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution

$$81^{\log_3 5} + 3^{3\log_9 36} + 3^{4\log_9 7}$$
$$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2}$$
$$= 625 + 216 + 49 = 890.$$

Example 10

Find the number of digit in 18^{20}.

Solution

$N = 18^{20}$

Taking log on both sides, $\log N = \log 18^{20}$

$$= 20\ [\log_{10} 9 + \log_{10} 2]$$
$$= 20\ [2 \times 0.4771 + 0.3010]$$
$$= 25.104$$

Number of digit = C + 1 = 25 + 1 = 26

Example 11

If $\log_4 m = 1.5$, then find the value of m.

Solution

$$\log_4 m = 1.5 \Rightarrow m = 4^{3/2} \Rightarrow m = 8$$

Example 12

If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4 q^4}{100} = 100^{2a-1}$

Solution

$$\log_5 p = a \Rightarrow p = 5^a$$
$$\log_2 q = a \Rightarrow q = 2^a$$
$$\Rightarrow \frac{p^4 q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

Example 13

If $\log_e x - \log_e y = a$, $\log_e y - \log_e z = b$ and $\log_e z - \log_e x = c$, then find the value of

$$\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$$

Solution

$$\log_e x - \log_e y = a \Rightarrow \log_e \frac{x}{y} = a \Rightarrow \frac{x}{y} = e^a$$
$$\log_e y - \log_e z = b \Rightarrow \log_e \frac{y}{z} = b \Rightarrow \frac{y}{z} = e^b$$
$$\log_e z - \log_e x = c \Rightarrow \log_e \frac{z}{x} = c \Rightarrow \frac{z}{x} = e^c$$
$$\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$$
$$= \left(e^a\right)^{b-c} \times \left(e^b\right)^{c-a} \times \left(e^c\right)^{a-b}$$
$$= e^{a(b-c)+b(c-a)+c(a-b)} = e^0 = 1$$

Example 14

If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution

$$a^2 + b^2 = (a + b)^2 - 2ab = 23ab$$
$$\Rightarrow (a + b)^2 = 25ab \Rightarrow a + b = 5\sqrt{ab} \quad \text{(i)}$$

Using (i)

$$\text{L.H.S.} = \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab$$
$$= \frac{1}{2}(\log a + \log b) = \text{R.H.S.}$$

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Example 15

If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where $a, b, x \in R^+ - \{1\}$):

(A) $\frac{1}{p}+\frac{1}{q}$ **(B)** $\frac{1}{2p}+\frac{1}{q}$

(C) $\frac{1}{p}+\frac{1}{2q}$ **(D)** $\frac{1}{2p}+\frac{1}{2q}$

Solution (B)

$$\log_a x = p \quad \Rightarrow \quad a^p = x \quad \Rightarrow \quad a = x^{1/p}.$$

Similarly, $b^q = x^2 \quad \Rightarrow \quad b = x^{2/q}$

$$\text{Now, } \log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}}$$

$$= \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\frac{1}{2}} = \frac{1}{2p}+\frac{1}{q}$$

Example 16

If a, b, c are distinct positive real numbers different from 1 such that

$(\log_b a \cdot \log_c a - \log_a a) + (\log_a b \cdot \log_c b - \log_b b) + (\log_a c \cdot \log_b c - \log_c c) = 0$, then abc is equal to:

(A) 0 **(B)** e

(C) 1 **(D)** none of these

Solution

$$(\log_b a \cdot \log_c a - 1) + (\log_a b \cdot \log_c b - 1) + (\log_a c \cdot \log_b c - 1) = 0$$

$$\Rightarrow \frac{\log a}{\log b}.\frac{\log a}{\log c} + \frac{\log b}{\log a}.\frac{\log b}{\log c} + \frac{\log c}{\log a}.\frac{\log c}{\log b} = 3$$

$$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3 \log a \cdot \log b \cdot \log c$$

$$\Rightarrow (\log a + \log b + \log c) = 0$$

[$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$$\Rightarrow \log abc = \log 1 \quad \Rightarrow \quad abc = 1$$

Example 17

Find value of x in $\log_a b = x$, where $a = 7^7$ and $b = 7^{7^7}$

Solution

$$\Rightarrow a^x = b$$

$$\Rightarrow 7^{7x} = 7^{7^7} \Rightarrow 7x = 7^7$$

$$\Rightarrow x = 7^6$$

Example 18

Show that $\log_4 18$ is an irrational number.

Solution

$$\log_4 18 = \log_4 (3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\frac{\log_2 3}{\log_2 4} + \frac{1}{\log_2 4}$$

$= \log_2 3 + \frac{1}{2}$ assume the contrary, that this number $\log_2 3$ is rational number.

$$\Rightarrow \log_2 3 = \frac{p}{q}$$

Since $\log_2 3 > 0$ both numbers p and q may be regarded as natural number

$$\Rightarrow 3 = 2^{p/q} \Rightarrow 2^p = 3^q$$

But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

Example 19

If in a right-angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that $\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a \cdot \log_{c-b} a$.

Solution

We know that in a right-angled triangle

$$c^2 = a^2 + b^2$$

$$c^2 - b^2 = a^2 \qquad \text{(i)}$$

$$\text{LHS} = \frac{1}{\log_a (c+b)} + \frac{1}{\log_a (c-b)}$$

$$= \frac{\log_a (c-b) + \log_a (c+b)}{\log_a (c+b).\log_a (c-b)}$$

$$= \frac{\log_a (c^2 - b^2)}{\log_a (c+b).\log_a (c-b)} = \frac{\log_a a^2}{\log_a (c+b).\log_a (c-b)}$$

(using (i))

$$= \frac{2}{\log_a (c+b).\log_a (c-b)} = 2\log_{(c+b)} a \cdot \log_{(c-b)} a = \text{RHS}$$

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Know Your Progress - 1

1. Evaluate: $\log_{10}(0.06)^6$
2. Determine number of cyphers (zeros) between decimal and first significant digit in $\left(\frac{1}{6}\right)^{200}$.
3. Find antilog of $\frac{5}{6}$ to the base 64.

Find value of following

4. $\log_{\sqrt{3}-\sqrt{2}}\left(\sqrt{5-2\sqrt{6}}\right)$
5. $\log_{\sqrt{5}+1}(6+2\sqrt{5})$
6. $\log_7 3 \cdot \log_5 2 \cdot \log_3 7 \cdot \log_2(125)$
7. $25^{\log_5 3}$
8. $6^{\log_6 5} + 3^{\log_9 16}$
9. $\log_6 4 + \frac{1}{\log_9 6}$
10. Solve for 'x' if $2\log_x 7 + \log_{7x} 7 + 3\log_{49x} 7 = 0$.
11. If $\log_{10}[\log_{10}(\log_{10} x)] = 0$, then find the value of x.
12. Find the value of x, if $\log_3(\log_2 x) + 2\log_9(\log_7 8) = 2$.
13. Find x if $\log_{10} 1250 + \log_{10} 80 = x$.
14. If $\log 2 = a$ and $\log 3 = b$ then $[\log(1) + \log(1+3) + \log(1+3+5) + \ldots + \ldots + \log(1+3+5+7+\ldots+19)] - 2[\log 1 + \log 2 + \log 3 + \ldots + \log 7] = p + qa + rb$ where p, q, r are constants. Find the value of $p + 2q + 3r$ if all logs are in base 10.
15. Let 'L' denote the antilog of 0.4 to the base 1024 and 'M' denote the number of digits in 6^{10} (Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$)and 'N' denote the number of positive integers that have the characteristic 2, when base of the logarithm is 6.

Find the value of LMN.

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Factors and Multiples

Factors

'a' is a factor of 'b' if there exists a relation such that $a \times n = b$, where 'n' is any natural number.

1 is a factor of all numbers as $1 \times b = b$.

Factor of a number cannot be greater than the number (in fact the largest factor will be the number itself). Thus factors of any number will lie between 1 and the number itself (both inclusive) and they are limited.

Multiples

'a' is a multiple of 'b' if there exists a relation of the type $b \times n = a$. Thus, the multiples of 6 are $6 \times 1 = 6$, $6 \times 2 = 12$, $6 \times 3 = 18$, $6 \times 4 = 24$, and so on.

The smallest multiple will be the number itself and the number of multiples would be infinite.

Note

To understand what multiples are, let's just take an example of multiples of 3. The multiples are 3, 6, 9, 12, … so on. We find that every successive multiple appears as the third number after the previous.

So if one wishes to find the number of multiples of 6 less than 255, we could arrive at the number through $\frac{255}{6} = 42$ (and the remainder 3). The remainder is of no consequence to us. So in all there are 42 multiples.

If one wishes to find the multiples of 36, find $\frac{255}{36} = 7$ (and the remainder is 3).

Hence, there are 7 multiples of 36.

Factorization

It is the process of splitting any number into a form where it is expressed only in terms of the most basic prime factors.

For example, $36 = 2^2 \times 3^2$. It is expressed in the factorized form in terms of its basic prime factors.

Number of Factors

For any composite number C, which can be expressed as

$$C = a^p \times b^q \times c^r \times \ldots,$$

where $a, b, c \ldots$ are all prime factors and p, q, r are positive integers, the number of factors is equal to $(p + 1) \times (q + 1) \times (r + 1) \ldots$

E.g., $36 = 2^2 \times 3^2$. So the factors of $36 = (2 + 1) \times (2 + 1) = 3 \times 3 = 9$.

HCF and LCM

(i) **LCM (Least Common Multiple):** The LCM of given numbers, as the name suggests is the smallest positive number that is a multiple of each of the given numbers

(ii) **HCF (Highest Common factor):** The HCF of given numbers, as the name suggests is the largest factor of the given set of numbers.

Consider the numbers 12, 20, and 30. The factors and the multiples are:

Factors	Given Numbers	Multiples
1, 2, 3, 4, 6, 12	12	12, 24, 36, 48, 60, 72, 84, 96, 108, 120 …
1, 2, 4, 5, 10, 20	20	20, 40, 60, 80, 100, 120 …
1, 2, 3, 5, 6, 10, 15, 30	30	30, 60, 90, 120 …

The common factors are 1 and 2 and the common multiples are 60, 120...

Thus, the HCF is 2 and the LCM is 60. The meaning of HCF is the largest number that divides all the given numbers.

Also, since a number divides its multiple, the meaning of LCM is that it is the smallest number that can be divided by the given numbers.

HCF will be lesser than or equal to the least of the numbers and LCM will be greater than or equal to the greatest of the numbers.

For any two numbers x and y,

$$x \times y = \text{HCF}(x, y) \times \text{LCM}(x, y).$$

HCF and LCM of fractions:

$$\text{LCM of fractions} = \frac{\text{LCM of numerators}}{\text{HCF of denominators}}$$

$$\text{HCF of fractions} = \frac{\text{HCF of numerators}}{\text{LCM of denominators}}$$

Make sure the fractions are in the most reducible form.

Example 20

How many numbers from 200 to 600 are divisible by 4, 5, 6?

Solution

Every such number must be divisible by LCM of (4, 5, 6) = 60.

$$\left[\frac{600}{60}\right] - \left[\frac{200}{60}\right] = 10 - 3 = 7.$$

Such numbers are 240, 300, 360, 420, 480, 540, and 600.

Clearly, there are 7 such numbers.

Example 21

If $N = 12^3 \times 3^4 \times 5^2$, find the total number of even factors of N.

Solution

The factorized form of N is $(2^2 \times 3^1)^3 \times 3^4 \times 5^2 \Rightarrow 2^6 \times 3^7 \times 5^2$.

Hence, the total number of factors of N is $(6 + 1)(7 + 1)(2 + 1) = 7 \times 8 \times 3 = 168$.

Some of these are odd multiples and some are even. The odd multiples are formed only with the combination of $3s$ and $5s$.

So, the total number of odd factors is $(7 + 1)(2 + 1) = 24$.

Therefore, the number of even factors is $168 - 24 = 144$.

Example 22

A number N when factorized can be written $N = a^4 \times b^3 \times c^7$. Find the number of perfect squares which are factors of N (the three prime numbers $a, b, c > 2$).

Solution

In order that the perfect square divides N, the powers of 'a' can be 0, 2 or 4, i.e., 3.

Powers of 'b' can be 0, 2, i.e., 2. Power of 'c' can be 0, 2, 4 or 6, i.e., 4.

Hence, a combination of these powers given $3 \times 2 \times 4$ i.e., 24 numbers.

So, there are 24 perfect squares that divides N.

Example 23

Directions: (i to iv) Answer the questions based on the given information.

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There are one thousand lockers and one thousand students in a school. The principal asks the first student to go to each locker and open it. Then he asks the second student go to every second locker and close it. The third student goes to every third locker, and if it is closed, he opens it, and if it is open, he closes it. The fourth student does it to every fourth locker and so on. The process is completed with all the thousand students.

(i) How many lockers are closed at the end of the process?

(ii) How many students can go to only one locker?

(iii) How many lockers are open after 970 students have done their job?

(iv) How many students go to locker no. 840?

Solution

(i to iv): Whether the locker is open or not depends on the number of times it is accessed. If it is accessed odd number of times, then it is open, while if it is accessed even number of times then it is closed.

How many times a locker will be accessed depends on the locker number. If it contains odd number of factors, then it will be open and if it contains even number of factors, then it will be closed. We know that a perfect square contains odd number of factors while a non–perfect square contains even number of factors. Thus, the lockers with perfect square number will be open and the number of these perfect squares from 1 to 1000 determines the number of open lockers.

(i) Number of closed lockers = No. of non-perfect square numbers from 1 to 1000 = 1000 – 31 = 969.

(ii) Up to 500 students can go to two or more than two lockers, while the rest 500 can go to only one locker.

(iii) The 31 perfect squares (the last being 312 = 961) will be open while the lockers from 971 to 1000 is yet to be accessed last time so they all are open. The total being = 31 + 30 = 61

(iv) The no. of students that have gone to locker no. 840 is same as the no. of factors of 840.
$840 = 2^3 \times 3 \times 5 \times 7$.
So, the no. of factors = (3 + 1) (1 + 1) (1 + 1) (1 + 1) = 32.

Example 24

Find a number greater than 3 that when divided by 4, 5, and 6 always leaves the same remainder 3.

Solution

The smallest number which, when divided by 4, 5, and 6, leaves the remainder 3 in each case is LCM (4, 5 and 6) + 3 = 60 + 3 = 63.

Example 25

In a school, 437 boys and 342 girls have been divided into classes, so that each class has the same number of students and no class has boys and girls mixed. What is the least number of classes needed?

Solution

We should have the maximum number of students in a class. So, we have to find HCF

$$(437, 342) = 19.$$

HCF is also the factor of difference of the number.

$$\therefore \text{ Number of classes } = \frac{437}{19} + \frac{342}{19} = 23 + 18 = 41$$ classes.

Example 26

Find the HCF and LCM of $\frac{8}{9}, \frac{16}{81}, \frac{2}{3}$ and $\frac{10}{27}$.

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Solution

$$\text{HCF of given fractions} = \frac{\text{HCF of } (2,8,16,10)}{\text{LCM of } (3,9,81,27)} = \frac{2}{81}$$

$$\text{LCM of given fractions} = \frac{\text{LCM of } (2,8,16,10)}{\text{HCF of } (3,9,81,27)} = \frac{80}{3}.$$

Example 27

Find the least number that when divided by 6, 7, 8, 9, and 10 leaves remainder 1.

Solution

As the remainder is same

Required number = LCM of divisors + remainder = LCM (6, 7, 8, 9, 10) + 1 = 2520 + 1 = 2521.

Example 28

Six bells start tolling together and they toll at intervals of 2, 4, 6, 8, 10, 12 sec., respectively, find:

(i) after how much time will all six of them toll together?

(ii) how many times will they toll together in 30 min?

Solution

The time after which all six bells will toll together must be multiple of 2, 4, 6, 8, 10, 12.

Therefore, required time = LCM of time intervals = LCM (2, 4, 6, 8, 10, 12) = 120 sec.

Therefore, after 120 sec, all six bells will toll together.

After each 120 sec, i.e., 2 min, all bells are tolling together.

Therefore, in 30 min, they will toll together $\left(\frac{30}{2}+1\right)$ = 16 times

1 is added as all the bells are tolling together at the start also, i.e., 0^{th} second.

Example 29

LCM of two distinct natural numbers is 211. What is their HCF?

Solution

211 is a prime number. So there is only one pair of distinct numbers possible whose LCM is 211,

i.e., 1 and 211. HCF of 1 and 211 is 1.

Example 30

An orchard has 48 apple trees, 60 mango trees, and 96 banana trees. These have to be arranged in rows such that each row has the same number of trees and all are of the same type. Find the minimum number of such rows that can be formed.

Solution

Total number of trees are 204 and each of the trees are exactly divisible by 12. HCF of (48, 60, 96).

$\therefore \frac{204}{12}$ = 17 such rows are possible.

Know Your Progress - 2

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1. If a 3-digit number '*abc*' has 3 factors, then find the total number of factors of the 6-digit number *abcabc*.
2. Find the number of natural numbers less than 100 that cannot be written as multiple of perfect square greater than 1.
3. If $N = 3^{15} \times 7^{43}$, then find the number of factors of N^2 that are less than N but not divide N completely.
4. If both 112 and 33 are factors of number $N = a \times 65804544$, then find the smallest possible value of a.
5. A number N^2 has 15 factors. Find all the possible number of factors of N?
6. There are two numbers a and b such that $a > b$, if HCF of $(a, b) = h$ and LCM of $(a, b) = l$, then find the LCM of $(a - b)$ and b.
7. In a large school auditorium, the students are made to sit to watch the programs. If the teachers make a row of student of 16 each, there will be 12 students left. If they make row of 24 each, there will be 20 students left, if they make rows of 25 each, there will be 21 students left and if they make down of 30 each, there will be 26 students left. Find minimum number of students present in school?
8. Find all pairs of positive integer (x, y) such that HCF (x, y) + LCM $(x, y) = 91$.
9. How many pairs of positive integers (x, y) exist such that HCF of $(x, y) = 35$ and $x + y = 1085$?
10. Manas and his friend met at Nehru Place after a long time. Manas stays at Vivek Vihar and his friend stays in Gurgaon. Both of them commute by bus. They reached the bus stop and got to know that a bus had left just for each of their destinations. Neither wanted to leave the other alone at bus stop. If frequency of buses to Gurgaon was 7 min and that to Vivek Vihar was 11 min then.
    - **(a)** How long would they wait at the bus stop?
    - **(b)** How many buses going to their destinations would each one decide not to board?

Divisibility, Cyclicity, Factorial, Binomial Theorem, and Remainders

Division Algorithm

General representation of result is,

$$\frac{Dividend}{Divisor} = \text{Quotient} + \frac{Remainder}{Divisor}$$

$$\text{Dividend} = (\text{Divisor} \times \text{Quotient}) + \text{Remainder}$$

Basic Knowledge

"Divisible by" means "when you divide one number by another number, the result is a whole number. "Divisible by" and "can be evenly divided by" mean the same thing.

The expressions $\overline{abc}$, abc, and abc are the same $\overline{abc} = abc = 100a + 10b + c$. They represent a three-digit number such as $\overline{234} = 234 = 234$

Divisibility Rules

Divisibility Rule for 2, 4, 8, and 16

A number is divisible by 2 if the last digit of the number is divisible by 2 or 2^1.

A number is divisible by 4 if the last two digit of the number are divisible by 4 or 2^2.

A number is divisible by 8 if the three digits of the number are divisible by 8 or 2^3.

A number is divisible by 16 if the last four last four digits of the number are divisible by 16 or 2^4.

Divisibility Rule for 5, 25, 125, and 625

A number is divisible by 5 if the last of the number is divisible by 5 or 5^1.

A number is divisible by 25 if last two digits of the number form a number that is divisible by 25 or 5^2.

A number is divisible by 125 if the last three digits of the number form a number that is divisible by 125 or 5^3.

Divisibility Rule for 3 and 9

A number is divisible by 3 if sum of the digits of the number is divisible by 3.

A number is divisible by 9 if the sum of the digits of the number is divisible by 9.

Divisibility Rule for 7, 11 and 13

1. If you double the last digit and subtract if from the rest of the number and the answer is divisibly by 7, the number is divisible by 7, you can apply this rule to that answer again if necessary.
2. To find out if a number is divisible by 11, add every other digit, and call that sum 'x'. Add together the remaining digits, and call that sum 'y'. Take the positive difference of x and y. If the difference is zero or a multiple of eleven, then so is the original number. Repeat the rule if necessary.
3. Delete the last digit from the number and then subtract 9 times the deleted digit from the remaining number. If what is left is divisible by 13, then so is the original number. Repeat the rule if necessary.
4. If the positive difference of the last three digit and the rest of the digits is divisible by 7, 10, or 13, then the number is divisibly by 7, 11, or 13, respectively.

Divisibility Rule for 17 and 19

1. Five times the last digit of the number and subtract from previous number the result obtained should be either 0 or divisible by 17.
2. Double the last digit of given number and add to remaining number The result obtained should be divisible by 19.

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Divisibility Rule for 6, 10, 12, 14, 15, 18, 24 and 36

A number is divisible by 6, if the number is divisible by both 2 and 3.

A number is divisible by 10, if the number is divisible by both 2 and 5.

A number is divisible by 12, if the number is divisible by both 3 and 4.

A number is divisible by 14, if the number is divisible by both 2 and 7.

A number is divisible by 15, if the number is divisible by both 3 and 5.

A number is divisible by 18, if the number is divisible by both 2 and 9.

A number is divisible by 24, if the number is divisible by both 3 and 8.

A number is divisible by 36, if the number is divisible by both 4 and 9.

Remainders

The method of finding the remainder without actually performing the process of division is termed as remainder theorem.

Remainder should always be positive. For example, if we divide –22 by 7, generally we get –3 as quotient and –1 as remainder. But this is wrong because remainder is never

be negative hence the quotient should be – 4 and remainder is +6. We can also get remainder 6 by adding –1 to divisor 7 (7 – 1 = 6).

Euclid's Division Lemma

If we have two positive integers a and b, then there exist unique integer q and r, which satisfies the condition

$$a = b \times q + r$$

where $0 \le r \le b$

Euclid's Division Algorithm

To obtain the HCF of two positive integers, say c and d, with $c > d$, follow the steps below:

Step 1: Apply Euclid's division lemma, to c and d. So, we find whole numbers, q and r such that

$$c = dq + r,\ 0 \le r < d.$$

Step 2: If $r = 0$, d is the HCF of c and d. If $r \neq 0$, apply the division lemma to d and r.

Step 3: Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

This algorithm works because HCF(c, d) = HCF(d, r) where the symbol HCF (c, d) denotes the HCF of c and d, etc.

Euclid's division algorithm is a technique to compute the HCF of two given positive integers. Recall that the HCF of two positive integers a and b is the largest positive integer d that divides both a and b.

Let us see how the algorithm works, through an example first. Suppose we need to find the HCF of the integers 455 and 42. We start with the larger integer, that is, 455. Then we use Euclid's lemma to get

$$455 = 42 \times 10 + 35$$

Now consider the divisor 42 and the remainder 35, and apply the division lemma to get

$$42 = 35 \times 1 + 7$$

Now consider the divisor 35 and the remainder 7, and apply the division lemma to get

$$35 = 7 \times 5 + 0$$

Notice that the remainder has become zero, and we cannot proceed any further.

We claim that the HCF of 455 and 42 is the divisor at this stage, i.e., 7.

Congruence of Integers

Definition 1

When an integer n is divided by a non-zero integer m, there must be an integral quotient q and a remainder r, where $0 \le |r| < m$.

$$n = mq + r,$$

and the process for getting this relation is called division with remainder.

Definition 2

Two integers a and b are said to be congruent modulo m, denoted by $a \equiv b \pmod m$, if a and b have the same remainder when they are divided by a non-zero integer m. If the remainders are different, then a and b are said to be **not congruent modulo** m, denoted by $a \not\equiv b \pmod m$.

By the definition of congruence, the following four equivalent relations are obvious:

$a \equiv b \pmod m \Leftrightarrow a - b = km \Leftrightarrow a - b = 0 \Leftrightarrow m \mid (a - b)$.

Basic Properties of Congruence

In what follows n, b, c, d, x, y are integers.

1. $a \equiv a \pmod n$ (Reflexive relation for all $a \in Z$)
2. $a \equiv b \pmod n \Leftrightarrow b \equiv a \pmod n$
   (Symmetric relation for all $a, b \in Z$)
3. $a \equiv b \pmod n,\ b \equiv c \pmod n \Rightarrow a \equiv c \pmod n$
   (Transitive relation for all integers a, b, c)
4. If $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then
   - (i) $a + c \equiv b + d \pmod n$
   - (ii) $a - c \equiv b - d \pmod n$
   - (iii) $ac \equiv bd \pmod n$
   - (iv) $ax + cy \equiv bx + dy \pmod n$
5. If $a \equiv b \pmod n$, then
   - (i) $a + c \equiv b + c \pmod n$
   - (ii) $a - c \equiv b - c \pmod n$
   - (iii) $ac \equiv bc \pmod n$
   - (iv) $a + k_1 n \equiv b + k_2 n \pmod n$; $k_1, k_2 \in Z$
   - (v) $a^m \equiv b^m \pmod n$ $m \in N$
6. $a \equiv b \pmod n \Rightarrow f(a) \equiv f(b) \pmod n$
   Where f is a polynomial over Z. i.e., $f(x)$ is a polynomial with integer coefficients.
7. $a\lambda \equiv b\lambda \pmod n \Rightarrow a \equiv b \left(\mod \frac{n}{(\lambda, n)}\right)$
8. If $n \neq 0$ and $(a, n) = 1$, then there exists an integer a' such that $aa' \equiv 1 \pmod n$
   Which is called the inverse of 'a' modulo n.

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The Units Digit of Power of Positive Integers a

Let P be the units digit of a positive integer a, an n be the positive integer power of a. Then the units digit of a^n is determined by the units digits of P^n, denoted by $U(P^n)$, and the sequence $(U(P^n))$, $n = 1, 2, 3, \ldots)$ follows the following rules:

(i) The sequence takes constant values for $P = 0, 1, 5, 6$, i.e. $U(P^n)$ does not change as n changes.
(ii) The sequence is periodic with a period 2 for $P = 4$ or 9.
(iii) The sequence is periodic with a period 4 for $P = 2, 3, 7, 8$.

The Last Two Digits of Some Positive Integers

(i) The last two digits of 5^n $(n \geq 2)$ is 25.
(ii) The ordered pair of last two digits of 6^n $(n \geq 2)$ changes with the period '36, 96, 76, 56' as n changes.
(iii) The ordered pair of last two digits of 7^n $(n \geq 2)$ changes with the period '09, 49, 43, 01' as n changes.
(iv) The ordered pair of last two digits of 76^n is always 76.

Fermat's little theorem states that if p is a prime number, then for any integer a, the number $a^p - a$ is an integer multiple of p. In the notation of modular arithmetic, this is expressed as $a^p \equiv a \pmod{p}$.

For example, if $a = 2$ and $p = 7$, then $2^7 = 128$, and $128 - 2 = 126 = 7 \times 18$ is an integer multiple of 7.

If a is not divisible by p, Fermat's little theorem is equivalent to the statement that $a^{p-1} - 1$ is an integer multiple of p, or in symbols:

$$a^{p-1} \equiv 1 \pmod{p}.$$

For example, if $a = 2$ and $p = 7$, then $2^6 = 64$, and $64 - 1 = 63 = 7 \times 9$ is thus a multiple of 7.

Factorial Notation

(i) A Useful Notation: $n!$ (factorial n) $= n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$; $n! = n \cdot (n-1)!$ Where $n \in N$
(ii) $0! = 1! = 1$
(iii) Factorials of negative integers are not defined.
(iv) $n!$ is also denoted by $\lfloor \underline{n}$
(v) $(2n)! = 2^n \cdot n!$ $[1.\ 3.\ 5.\ 7 \ldots (2n-1)]$
(vi) Prime factorization of $n!$: Let p be a prime number and n be a positive integer, then exponent of p in $n!$ is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots + \left[\frac{n}{p^k}\right]$$

where, $p^k \leq n < p^{k+1}$ and $[x]$ denotes the integral part of x.

If we isolate the power of each prime contained in any number n, then n can be written as $n = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4} \ldots$, where α_i are whole numbers.

Example 31

Find the exponent of 6 in 50!

Solution

$$E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] + \left[\frac{50}{64}\right]$$

(where [ ] denotes integral part)

$$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$$

$$E_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right]$$

$$E_3(50!) = 16 + 5 + 1 + 0 = 22$$

$\Rightarrow$ 50! can be written as $50! = 2^{47} \cdot 3^{22} \ldots$

Therefore, exponent of 6 in 50! = 22 **Ans.**

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${}^nC_r$ NOTATION

${}^nC_r$ denotes the number of combinations of n different things taken r at a time $(n \in N, r \in W, r \leq n)$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Note

(i) ${}^nC_r$ is also denoted by $\binom{n}{r}$ or $C(n, r)$.
(ii) ${}^nC_r = {}^nC_{n-r}$; ${}^nC_0 = {}^nC_n = 1$
(iii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$
(iv) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
(v) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$
(vi) ${}^nC_r = \frac{n}{r}\left({}^{n-1}C_{r-1}\right)$

Binomial Expression

Any algebraic expression that contains two dissimilar terms is called binomial expression.

For example: $x - y$, $xy + \frac{1}{x}$, $\frac{1}{z} - 1$, $\frac{1}{(x-y)^{1/3}} + 3$ etc.

Binomial Theorem

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as BINOMIAL THEOREM

If $x, y \in R$ and $n \in N$, then:

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 + \ldots + {}^nC_r$$

$$x^{n-r}\, y^r + \ldots + {}^nC_n\, y^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$$

This theorem can be proved by induction.

Observations:

(a) The number of terms in the expansion is $(n + 1)$, i.e., one more than the index.

(b) The sum of the indices of x and y in each term is n.

(c) The binomial coefficients of the terms **($^nC_0$, $^nC_1$ ...)** equidistant from the beginning and the end are equal. i.e. $^nC_r = {}^nC_{n-r}$

(d) Symbol ^nC_r can also be denoted by $\binom{n}{r}$, $C(n, r)$ or A_r^n.

Some important expansions:

(i) $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_n\, x^n$.

(ii) $(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + (-1)^n \cdot {}^nC_n\, x^n$.

Note

The coefficient of x^r in $(1 + x)^n = {}^nC_r$ and that in $(1 - x)^n = (-1)^r \cdot {}^nC_r$

Example 32

Expand: $(y + 2)^6$.

Solution

$${}^6C_0 y^6 + {}^6C_1\, y^5 \cdot 2 + {}^6C_2 y^4 \cdot 2^2 + {}^6C_3\, y^3 \cdot 2^3 + {}^6C_4 y^2 \cdot 2^4 + {}^6C_5 y^1 \cdot 2^5 + {}^6C_6 \cdot 2^6.$$

$$= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64.$$

Example 33

Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution

$${}^7C_0,\ {}^7C_1\left(-\frac{2y^2}{5}\right),\ {}^7C_2\left(-\frac{2y^2}{5}\right)^2,\ {}^7C_3\left(-\frac{2y^2}{5}\right)^3$$

Example 34

If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and –6, respectively, then m is:

(A) 6 **(B)** 9

(C) 12 **(D)** 24

Solution

$$(1+x)^m (1-x)^n = \left[1 + mx + \frac{(m)(m-1).x^2}{2} + \ldots\ldots\right]$$

$$= \left[1 - nx + \frac{n(n-1)}{2} x^2 + \ldots\ldots\right]$$

Coefficient of $x = m - n = 3$ (i)

Coefficient of $x^2 = -mn + \frac{n(n+1)}{2} + \frac{m(m-1)}{2} = -6$ (ii)

Solving (i) and (ii), we get

$m = 12$ and $n = 9$.

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Pascal's Triangle

$(x+y)^0$	1	1
$(x+y)^1$	$x + y$	1 1
$(x+y)^2$	$x^2 + 2xy + y^2$	1 2 1
$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1
$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^3$	1 4 6 4 1

Pascal's triangle

(a) **Pascal's triangle:** A triangular arrangement of numbers as shown. The numbers give the binomial coefficients for the expansion of $(x + y)^n$. The first row is for $n = 0$, the second for $n = 1$, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

(b) Pascal triangle is formed by binomial coefficient.

(c) The number of terms in the expansion of $(x + y)^n$ is $(n + 1)$, i.e., one more than the index.

(d) The sum of the indices of x and y in each term is n.

(e) Power of first variable (x) decreases while of second variable (y) increases.

(f) Binomial coefficients are also called **combinatorial coefficients.**

(g) Binomial coefficients of the terms equidistant from the beginning and end are equal.

(h) r^{th} term from the beginning in the expansion of $(x + y)^n$ is same as r^{th} term from end in the expansion of $(y + x)^n$.

(i) r^{th} term from the end in $(x + y)^n$ is **$(n - r + 1)^{th}$ term from the beginning**.

Important Terms In The Binomial Expansion

General Term

The general term or the $(r + 1)$th term in the expansion of $(x + y)n$ is given by $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$.

Example 35

Find:

(a) The coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

Solution

(a) In the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$, the general term is:

$$T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r \cdot \frac{a^{11-r}}{b^r} \cdot x^{22-3r}$$

putting $22 - 3r = 7$

$\therefore \quad 3r = 15 \Rightarrow r = 5$

$\therefore \quad T_6 = {}^{11}c_5 \frac{a^6}{b^5} \cdot x^7$

Hence, the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5 a^6 b^{-5}$. **Ans.**

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

(b) In the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, general term is:

$$T_{r+1} = {}^{11}C_r\,(ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r$$

$$= (-1)^r\, {}^{11}C_r \frac{a^{11-r}}{b^r} \cdot x^{11-3r}$$

Putting $11 - 3r = -7$

$\therefore \quad 3r = 18 \Rightarrow r = 6$

$\therefore T_7 = (-1)^6 \cdot {}^{11}C_6 \frac{a^5}{b^6} \cdot x^{-7}$

Hence, the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6\, a^5\, b^{-6}$. **Ans.**

Also given:

Coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

$\Rightarrow {}^{11}C_5\, a^6\, b^{-5} = {}^{11}C_6\, a^5\, b^{-6}$

$\Rightarrow ab = 1 \qquad (\because {}^{11}C_5 = {}^{11}C_6)$

which is the required relation between a and b. **Ans.**

Example 36

Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution

The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r \left(9^{\frac{1}{4}}\right)^{1000-r} \left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\, 3^{\frac{1000-r}{2}}\, 2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000 - r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, …, 1000}

Hence, number of rational terms is 501

Example 37

On dividing 15,968 by a certain number, the quotient is 89 and the remainder is 37. Find the divisor.

Solution

$$\text{Divisor} = \frac{\text{Dividend Remainder}}{\text{Quotient}} = \frac{15968 - 37}{89} = 179.$$

Example 38

Check whether 413 is divisible by 7 or not.

Solution

Last digit = 3, remaining number = 41, $41 - (3 \times 2) = 35$ (divisible by 7), i.e., 413 is divisible by7.

This rule can also be used for number having more than 3 digits.

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Example 39

Check whether 6545 is divisible by 7 or not.

Solution

Last digit = 5, remaining number 654, $654 - (5 \times 2) = 644$; $64 - (4 \times 2) = 56$ divisible by 7, i.e., 6545 is divisible by 7.

Example 40

Check whether 234 is divisible by 13 or not .

Solution

234, $(4 \times 9) - 23 = 13$ (divisible by 13), i.e., 234 is divisible by 13.

Example 41

Check whether 357 is divisible by 17 or not.

Solution

357, $(7 \times 5) - 35 = 0$, i.e., 357 is divisible by 17.

Example 42

Check whether 589 is divisible by 19 or not.

Solution

589, $(9 \times 2) + 58 = 76$ (divisible by 19), i.e., the number is divisible by 19.

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Example 43

Find the smallest number of six digits which is exactly divisible by 111.

Solution

Smallest number of 6 digits is 100,000.

On dividing 100,000 by 111, we get 100 as remainder.

$\therefore$ Number to be added = $(111 - 100) = 11$.

Hence, required number = 100,011.

Example 44

Find the largest four digit number that when reduced by 54, is perfectly divisible by all even natural numbers less than 20.

Solution

Even natural numbers less than 20 are 2, 4, 6, 8, 10, 12, 14, 16, 18.

Their LCM = 2 × LCM of first 9 natural numbers = $2 \times 2520 = 5040$.

This happens to be the largest four–digit number divisible by all even natural numbers less than 20.

54 was subtracted from our required number to get this number.

Hence, (required number – 54) = 5040 $\Rightarrow$ Required number = 5094.

Example 45

Ajay multiplied 484 by a certain number to get the result 3823*a*. Find the value of '*a*'.

Solution

3823*a* is divisible by 484, and 484 is a factor of 3823*a*.

4 is a factor of 484 and 11 is also a factor of 484.

Hence, 3823*a* is divisible by both 4 and 11.

To be divisible by 4, the last two digits have to be divisible by 4.

'*a*' can take two values 2 and 6.

38232 is not divisible by 11, but 38236 is divisible by 11.

Hence, 6 is the correct choice.

Example 46

Which digits should come in place of * and $ if the number 62684*$ is divisible by both 8 and 5?

Solution

Since the given number is divisible by 5, so 0 or 5 must come in place of $. But, a number ending with 5 is never divisible by 8. So, 0 will replace $.

Now, the number formed by the last three digits is 4*0, which becomes divisible by 8, if * is replaced by 4 or 8.

Hence, digits in place of * and $ are (4 or 8 or 0) and 0 respectively.

Example 47

For what digit(s) *x* will be 7-digit number 3xx6xx2 be divisible by 4?

Solution

1, 3, 5, 7 and 9

The two-digit number *x*2 needs to be divisible by 4 by the divisible rule.

The following numbers work: 12, 32, 52, 72 and 92

Example 48

The six-digit number $\overline{713EF5}$ is divisible by 125. How many such six-digit numbers are there?

Solution

(713125, 713375, 713625 and 713875)

If the given number is divisible by 125, $\overline{EF5}$ should be divisible by 125.

$125 \times 1 = 125$, $125 \times 3 = 375$, $125 \times 5 = 625$ and $125 \times 7 = 875$.

Example 49

Which number is not divisible by 7? 616, 567, 798, or 878

Solution

616 : 61 – 6 × 2 = 61 – 12 = 49 (divisible by 7)
567 : 56 – 7 × 2 = 56 – 14 = 42 (divisible by 7)
798 : 79 – 8 × 2 = 79 – 16 = 63 (divisible by 7)
878 : 87 – 8 × 2 = 87 – 16 = 71 (not divisible by 7)

Example 50

Which number is not divisible by 7?
630616, 588567, 28105 or 40111

Solution

630616 : 630 – 616 = 14 (divisible by 7)
588567 : 588 – 567 = 21 (divisible by 7)
28105 : 105 – 28 = 77 (divisible by 7)
40111 : 111 – 40 = 71 (not divisible by 7)

Example 51

What is the greatest three-digit number that is divisible by 6?

Solution

The number must be divisible by both 2 and 3. So the number is even and the sum of its digits is divisible by 3. If the number is $99x$, $x = 6$ words.

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Example 52

When a three digit number is divided by 2, 3, 4, 5, and 7, the remainders are all 1. Find the minimum and maximum values of such three digit numbers.

Solution

Let x be a three digit with the remainder 1 when divided by 2, 3, 4, 5 and 7. Then $x - 1$ is divisible by each of 2, 3, 4, 5, 7, so $x - 1 = k$, $[2, 3, 4, 5, 7] = 420k$.

Thus, the minimum value of x is $420 + 1 = 421$, the maximum value of x is $2 \times 420 + 1 = 841$

Example 53

It is known that 2726, 4472, 5054, 6412 have the same remainder. When they are divided by some two digit natural number m. Find the value of m.

Solution

For excluding the effect of the unknown remainder, the three differences by the four given numbers can be used to replace the original four numbers. Then,

$m \mid (4472 - 2726) \Rightarrow m \mid 1746$. $1746 = 2 \cdot 3^2 \cdot 97$;

$m \mid (5054 - 4472) \Rightarrow m \mid 582$. $582 = 2 \cdot 3 \cdot 97$;

$m \mid (6412 - 5054) \Rightarrow m \mid 1358$ $1358 = 2 \cdot 7 \cdot 97$.

Example 54

Use Euclid's algorithm to find the HCF of 4052 and 12576.

Solution

Step 1: Since $12576 > 4052$, we apply the division lemma to 12576 and 4052, to get

$$12576 = 4052 \times 3 + 420$$

Step 2: Since the remainder $420 \neq 0$, we apply the division lemma to 4052 and 420, to get

$$4052 = 420 \times 9 + 272$$

Step 3: We consider the new divisor 420 and the new remainder 272, and apply the division lemma to get

$$420 = 272 \times 1 + 148$$

We consider the new divisor 272 and the new remainder 148, and apply the division lemma to get

$$272 = 148 \times 1 + 124$$

We consider the new divisor 148 and the new remainder 124, and apply the division lemma to get

$$148 = 124 \times 1 + 24$$

We consider the new divisor 124 and the new remainder 24, and apply the division lemma to get

$$124 = 24 \times 5 + 4$$

We consider the new divisor 24 and the new remainder 4, and apply the division lemma to get

$$24 = 4 \times 6 + 0$$

The remainder has now become zero, so our procedure stops. Since the divisor at this stage is 4, the HCF of 12576 and 4052 is 4.

Notice that 4 = HCF (24, 4) = HCF (124, 24) = HCF (148, 124) =

HCF (272, 148) = HCF (420, 272) = HCF (4052, 420) = HCF (12576, 4052).

Example 55

Two numbers, x and y, are such that when divided by 6, they leave remainders 4 and 5, respectively.

Find the remainder when $(x^2 + y^2)$ is divided by 6.

Solution

Suppose $x = 6k_1 + 4$ and $y = 6k_2 + 5$

$$x^2 + y^2 = (6k_1 + 4)^2 + (6k_2 + 5)^2 = 36k_1^2 + 48k_1 + 16 + 36k_2^2 + 60k_2 + 25$$

$$= 36k_1^2 + 48k_1 + 36k_2^2 + 60k_2 + 41$$

Obviously, when this is divided by 6, the remainder will be 5.

Example 56

A number when divided by 259 leaves a remainder 139. What will be the remainder when the same number is divided by 37?

Solution

Let the number be P.

So, $P - 139$ is divisible by 259.

Let Q be the quotient then, $\frac{P-139}{259} = Q$

$$\Rightarrow P = 259Q + 139 \quad \Rightarrow \quad \frac{P}{37} = \frac{259Q+139}{37}$$

$\because$ 259 is divisible by 37,

$\therefore$ When 139 divided by 37, leaves a remainder of 28.

Example 57

A number being successively divided by 3, 5, and 8 leaves remainders 1, 4, and 7, respectively. Find the respective remainders if the order of divisors be reversed.

Solution

3	x	
5	y	1
8	z	4
	1	7

$\therefore z = (8 \times 1 + 7) = 15$; $y = (5z + 4) = (5 \times 15 + 4) = 79$;
$x = (3y + 1) = (3 \times 79 + 1) = 238$.

Now,

8	238	
5	29	6
3	5	4
	1	2

$\therefore$ Respective remainders are 6, 4, 2.

Example 58

A number was divided successively in order by 4, 5, and 6. If the remainders were, respectively, 2, 3, and 4, then find out the number.

Solution

4	x	
5	y	2
6	z	3
	1	4

$\therefore z = (6 \times 1 + 4) = 10$

$\therefore y = (5 \times z + 3) = (5 \times 10 + 3) = 53$

$\therefore x = (4 \times y + 2) = (4 \times 53 + 2) = 214$

Hence, the required number is 214.

Example 59

In dividing a number by 585, a student employed the method of short division. He divided the number successively by 5, 9, and 13 (factors of 585) and got the remainders 4, 8, and 12. If he had divided number by 585, then find out the remainder.

Solution

5	x	
9	y	4
13	z	8
	1	12

Now, 1169 when divided by 585 gives remainder = 584.

To find the remainder of big number

Notes

(i) Binomial Expansion:

$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots. + b^n$, *or*

$(a-b)^n = a^n - \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 - \ldots. + (-1)^n b^n$.

Hence, first term is purely of a, i.e., an and last digit is purely of b, i.e., b^n.

(ii) Total number of terms in the expansion of $(a+b)^n$ is $(n+1)$.

Example 60

What is the remainder when 7^{38} is divided by 48.

Solution

$\frac{7^{38}}{48} = \frac{(7^2)^{19}}{48} = \frac{(49)^{19}}{48} = \frac{(48+1)^{19}}{48}$ so by using binomial expansion, we can say that 18 terms are completely divisible by 48, but the last term, which is $\frac{(+1)^{19}}{48}$, is not divisible.

So, 119 = 1 is the remainder.

Example 61

What is the remainder if 7^{25} is divided by 4?

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Solution

7^{25} can be written $(8-1)^{25}$. There are 26 terms in all and first 25 terms are divisible by 8, hence also by 4. The last term is $(-1)^{25}$. Hence, $(8-1)^{25}$ can be written $8X-1$ or $4Y-1$ (where $Y=2X$). So, $4Y-1$ divided by 4 leaves the remainder 3.

Example 62

What is the remainder if 3^{45} is divided by 8?

Solution

3^{45} can be written as 922×3. 9 can be written as $(8+1)$. Hence, any power of 9 can be written as $8N+1$. In other words, any power of 9 is 1 more than a multiple of 8. Hence, $(8N+1) \times 3$ leaves remainder 3 when divided by 8.

Example 63

What is the remainder when $14^{15^{16}}$ is divided by 5?

Solution

$14^{15^{16}} = (15-1)^{\text{odd}} = 15n + (-1)^{\text{odd}}$, i.e., a (multiple of 5) – 1. Thus when divided by 5 the remainder will be (–1), i.e., 4.

Example 64

What is the remainder when $3^{57} + 27$ is divided by 28?

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Solution

$$3^{57} = (3^3)^{19}$$

$$\therefore\ 3^{57} + 27 = (27)^{19} + 27 = (28-1)^{19} + 27$$

$$= 28M + (-1)^{19} + 27 \quad \text{[Expand by binomial theorem]}$$

$$= 28M - 1 + 27 = 28M + 26$$

When $28M + 26$ divided by 28, the remainder is 26.

Hence, the required remainder is 26.

Example 65

What is the remainder when $82^{361} + 83^{361} + 84^{361} + 85^{361} + 86^{361}$ is divided by 7?

Solution

$$82^{361} + 83^{361} + 84^{361} + 85^{361} + 86^{361} = [(84-2)^{361} + (84-1)^{361} + 84^{361} + (84+1)^{361} + (84+2)^{361}]$$

Since 84 is a multiple of 7, then the remainder will be when, $(-2)^{361} + (-1)^{361} + 1^{361} + 2^{361}$ is divided by 7 is $(-2)^{361} + (-1)^{361} + 1^{361} + 2^{361} = 0$. So the remainder is zero.

Cyclicity

The following terms have the same meanings: last digits, one digit, and units digits.

Patterns of the Last Digit of a^n

The last digits of a^n have patterns shown in the table below.

For example, when $a = 2$.

$2^1 = 2$ $\quad 2^2 = 4$ $\quad 2^3 = 8$ $\quad 2^4 = 16$

$2^5 = 32$ $\quad 2^6 = 64$ $\quad 2^7 = 128$ $\quad 2^8 = 256, \ldots$

The last digits of 2^n demonstrate a pattern: 2, 4, 8, 6, 2, 4, 8, 6, etc. …

N	1	2	3	4	Period
1^n	1				1
2^n	2	4	8	6	4
3^n	3	9	7	1	4
4^n	4	6			2
5^n	5				1
6^n	6				1
7^n	7	9	3	1	4
8^n	8	4	2	6	4
9^n	9	1			2

For example:

(i) In $(32)^{33}$ unit digit is equal to the unit digit of 2^1, i.e., 2.

(ii) In $(23)^{15}$ unit digit is equal to the unit digit of 3^3, i.e., 7.

(iii) In $(57)^9$ unit digit is equal to the unit digit of 7^1, i.e., 7.

(iv) In $(678)^{22}$ unit digit is equal to the unit digit of 8^2, i.e., 4.

Example 66

What is the unit digit in the number named by 7^{22}?

Solution

The pattern for the last digit is: 7, 9, 3, 1, 7, 9, 3, 1, etc.

When 22 is divided by 4, the quotient is 5 and the remainder is 2. The last digit of 7^{22} is the same the last digit of 7^2, so the last digit is 9.

Or $7^{22} = 7^{4\times5+2} = 7^2 = 49$. The last digit is 9.

Example 67

What is the units digit of 1998^{1998}?

Solution

The units digit of 1998^{1998} is the same as the units digit of 8^{1998},

The pattern for the last digit is: 3, 4, 2, 6, 8, 4, 2, 6, etc.

When 1998 is divided by 4, the quotient is 499 and the remainder is 2. The last digit of 8^{1998} is the same as the last digit of 8^2, so the last digit is 4.

OR $8^{1998} = 8^{4 \times 499 + 2} = 8^2 = 64$. The last digit is 4.

For example:

(i) In $(34)^{33}$ unit digit is 4.

(ii) In $(29)^{15}$ unit digit is 9.

(iii) In $(49)^{18}$ unit digit is 1.

(iv) In $(25)^{15}$ unit digit is 5.

(v) In $(46)^{13}$ unit digit is 6.

Example 68

Find the last digit of

(i) 3^{57}

(ii) 13^{59}

Solution

(i) The cyclicity of 3 is 4. Hence, $\frac{57}{4}$ gives the remainder 1. So, the last digit of 3^{57} is same as the last digit of 3^1, i.e., 3.

(ii) The number of digits in the base will not make a difference to the last digit. It is last digit of the base that decides the last digit of the number itself. For 13^{59}, we find $\frac{57}{4}$ that gives a remainder 3. So the last digit of 13^{59} is same as the last digit of 3^3, i.e., 7.

Example 69

Find unit's digit in $y = 7^{17} + 7^{34}$

Solution

$7^{17} + 7^{34} = 7^1 + 7^2 = 56$, Hence, the unit digit is 6

Example 70

What will be the last digit of $(73)^{75^{64^{76}}}$

Solution

Let $(73)^{75^{64^{76}}} = (73)^x$ where $x = 75^{64^{76}} = (75)^{\text{even power}}$

$\because$ Cyclicity of 3 is 4

$\therefore$ To find the last digit, we have to find the remainder when x is divided by 4.

$x = (75)^{\text{even power}} = (76 - 1)^{\text{even power}}$, where n is divided by 4 so remainder will be 1.

Therefore, the last digit of $(73)^{75^{64^{76}}}$ will be $31 = 3$.

Example 71

What will be the unit digit of $(87)^{75^{63^{55}}}$.

Solution

Let $(87)^{75^{63^{55}}} = (87)^x$ where $x = 75^{63^{55}} = (75)^{\text{odd}}$

$\because$ Cyclicity of 7 is 4.

$\therefore$ To find the last digit we have to find the remainder when x is divided by 4.

$$x = (75)^{\text{odd power}} = (76 - 1)^{\text{odd power}}$$

where x is divided by 4 so remainder will be -1 or 3, but remainder should be always positive.

Therefore, the last digit of $(87)^{75^{63^{55}}}$ will be $7^3 = 343$.

Hence, the last digit is of $(87)^{75^{63^{55}}}$ is 3.

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Know Your Progress - 3

1. Find the total numbers of natural number 'n' for which 111 divides $16^n - 1$ where 'n' is less than 1000.
2. How many positive integers are there from 1 to 1000 such that leaves remainder of 3 on division by 7 and leaves remainder of 2 on division by 4?
3. How many numbers between 1 and 1200 both included are not divisible by any of the numbers 2, 3 and 5?
4. How many five digit number multiples of 11 are there, if the five digit be 7, 6, 5, 4, and 3 are in some order?

5. The digits 1, 2, 3, 4, and 5 are each used once to compose a five digit number *ABCDE*, such that the three-digit number *ABC* is divisible by 4, *BCD* is divisible by 5 and *CDE* is divsible by 3.

   Find the value of the smallest five digit number *ABCDE*.

6. Find the remainder when 1234567 … 4647484950 is divided by 16.

7. Find the remainder when the $(1!)^2 + (2!)^2 \ldots + (100!)^2$ is divided by 1152.

8. Find the remainder when $3^{21} + 9^{21} + 27^{21} + 81^{21}$ is divided by $(3^{20} + 1)$

9. Find the remainder when $\underbrace{222.....2}_{300\text{ times}}$ is divided by 999.

10. Find the remainder when 2510 is divided by 576.

11. Find the smallest natural number n that has the following properties:
    (a) Its decimal representation has 6 as the last digit.
    (b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

12. Find all the positive integers n, such that $n \cdot 2^n + 1$ is divisible by 3.

13. Find the unit digit of the number equal to sum of 1! + 2! + 3! + 4! + … 100!.

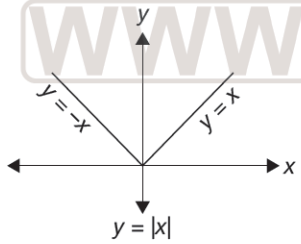
14. Find the unit digit of the number equal to sum of 3! + $(3!)^2 + (3!)^3 \ldots + (3!)^{100}$.

15. Find the unit digit of $(378)^{95^{25^{51}}}$.

16. Find the unit digit of $x = (35)^{87} + (93)^{46} + (44)^{91}$.

17. Find the unit digit of $(2^2 + 4^2 + 6^2 \ldots 100^2)^{64}$.

Modulus, GIF, FPF



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Modulus

The symbol of modulus function is $|x|$ and is defined as:

$$y = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Properties of Modulus

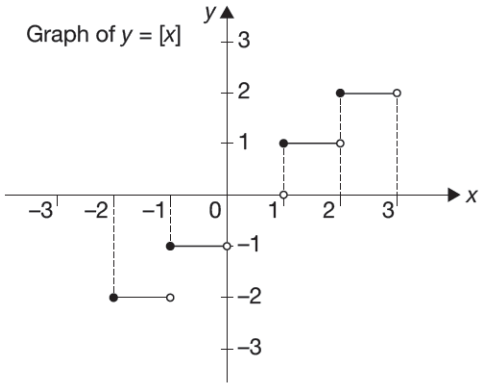
For any $a, b \in R$

(a) $|a| \geq 0$
(b) $|a| = |-a|$
(c) $|ab| = |a|\,|b|$
(d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
(e) $|a + b| \leq |a| + |b|$
(f) $|a| - |b| \leq |a - b|$
(g) $|a + b| = |a| + |b| \Rightarrow ab \geq 0$
(h) $\sqrt{x^2} = |x|$

Greatest Integer Function

The function $y = f(x) = [x]$ is called the Greatest Integer Function (GIF) where $[x]$ denotes the greatest integer less than or equal to x. Note that for:

x	$[x]$
$[-2, -1)$	-2
$[-1, 0)$	-1
$[0, 1)$	0
$[1, 2)$	1



Domain: R
Range: I

Properties of Greatest Integer Function

(i) $[x] \le x < [x]+1$ and $x-1 < [x] \le x$, $0 \le x-[x] < 1$

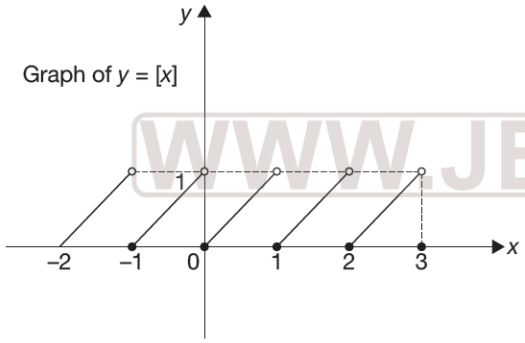
(ii) $[x+m] = [x]+m$, if m is an integer.

(iii) $[x]+[-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

Fractional Part Function

It is defined as: $g(x) = \{x\} = x - [x]$, e.g., the fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.

x	$\{x\}$
$[-2,-1)$	$x+2$
$[-1,0)$	$x+1$
$[0,1)$	x
$[1,2)$	$x-1$



Domain: R
Range: $[0, 1)$

Properties of Fractional Part Function

(i) $0 \le \{x\} < 1$

(ii) $\{[x]\} = [\{x\}] = 0$

(iii) $\{\{x\}\} = \{x\}$

(iv) $\{x+m\} = \{x\}$, $m \in I$

(v) $\{x\}+\{-x\} = \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$

Example 72

If $||x-1|-2| = 5$, then find x.

Solution

$$|x-1|-2 = \pm 5$$
$$|x-1| = 7, -3$$

Case 1: When $|x-1| = 7 \Rightarrow x-1 = \pm 7 \Rightarrow x = 8, -6$

Case 2: When $|x-1| = -3$ (reject)

Example 73

If $|x-1| + |x+1| = 2$, then find x.

Solution

Case 1: If $x \le -1$

$$-(x-1)-(x+1) = 2$$
$$\Rightarrow -x+1-x-1 = 2$$
$$\Rightarrow -2x = 2 \Rightarrow x = -1 \quad \text{(i)}$$

Case 2: If $-1 < x < 1$

$$-(x-1)+(x+1) = 2$$
$$\Rightarrow -x+1+x+1 = 2$$
$$\Rightarrow 2 = 2 \Rightarrow -1 < x < 1 \quad \text{(ii)}$$

Case 3: If $x \ge 1$

$$x-1+x+1 = 2$$
$$\Rightarrow x = 1 \quad \text{(iii)}$$

Thus from (i), (ii), and (iii) $-1 \le x \le 1$

Example 74

If $y = 2[x]+3$ and $y = 3[x-2]+5$, then find $[x+y]$ where $[.]$ denotes greatest integer function.

Solution

$$y = 3[x-2]+5 = 3[x]-1$$

so
$$3[x]-1 = 2[x]+3$$
$$[x] = 4 \Rightarrow 4 \le x < 5$$

then $y = 11$

so $x+y$ will lie in the interval $[15, 16)$

so $[x+y] = 15$

Example 75

Find the value of $\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+.....\left[\frac{1}{2}+\frac{2946}{1000}\right]$ where $[.]$ greatest integer function?

Solution

$$\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+.....\left[\frac{1}{2}+\frac{499}{1000}\right]+\left[\frac{1}{2}+\frac{500}{1000}\right]+....$$
$$\left[\frac{1}{2}+\frac{1499}{1000}\right]+\left[\frac{1}{2}+\frac{1500}{1000}\right]+.....+\left[\frac{1}{2}+\frac{2499}{1000}\right]+$$
$$\left[\frac{1}{2}+\frac{2500}{1000}\right]+.....\left[\frac{1}{2}+\frac{2946}{1000}\right]$$

$$= 0+1\times 1000+2\times 1000+3\times 447$$
$$= 3000+1341 = 4341 \quad \textbf{Ans.}$$

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Example 76

Solve the equation $|2x-1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and $\{.\}$ denotes fractional part function.

Solution

We are given that, $|2x-1| = 3[x] + 2\{x\}$

Let, $2x - 1 \le 0$, i.e., $x \le \frac{1}{2}$. The given equation yields:

$$1 - 2x = 3[x] + 2\{x\}$$

$$\Rightarrow \quad 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \quad \Rightarrow \quad 1 - 5[x] = 4\{x\}$$

$$\Rightarrow \{x\} = \frac{1-5[x]}{4} \quad \Rightarrow \quad 0 \le \frac{1-5[x]}{4} < 1$$

$$\Rightarrow 0 \le 1 - 5[x] < 4 \quad \Rightarrow \quad -\frac{3}{5} < [x] \le \frac{1}{5}$$

Now, $[x] = 0$ as zero is the only integer lying between $-\frac{3}{5}$ and $\frac{1}{5}$

$$\Rightarrow \{x\} = \frac{1}{4}$$

$\Rightarrow x = \frac{1}{4}$ which is less than $\frac{1}{2}$. Hence $\frac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$, i.e., $x > \frac{1}{2}$

$$\Rightarrow 2x - 1 = 3[x] + 2\{x\}$$

$$\Rightarrow 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$$

$$\Rightarrow [x] = -1$$

$\Rightarrow -1 \le x < 0$, which is not a solution as $x > \frac{1}{2}$

$\Rightarrow x = \frac{1}{4}$ is the only solution.

Know Your Progress - 4

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1. Solve following equations

(i) $|x-1| = 5$ (ii) $|x+1| = |2x-1|$

(iii) $|x-1| + 2|x+1| = 10$

(iv) $|2x-1| - |x-3| = 7$

2. Solve the following equations where $x \in R$.

(a) $(x-1)|x^2 - 4x + 3| + 2x^2 + 3x - 5 = 0$

(b) $|x^2 + 4x + 3| + 2x + 5 = 0$

(c) $|x+3|(x+1) + |2x+5| = 0$

3. If $[x]$ and $\{x\}$ denotes the greatest integer function less than or equal to x and fractional part function, respectively, then the number of real x, satisfying the equation $(x-2)[x] = \{x\} - 1$, is:

(A) 0 **(B)** 1

(C) 2 **(D)** infinite

4. Period of $f(x) = \{x\} + \left\{x + \frac{1}{3}\right\} + \left\{x + \frac{2}{3}\right\}$ is equal to (where $\{.\}$ denotes fractional part function)

(A) 1 **(B)** $\frac{2}{3}$

(C) $\frac{1}{2}$ **(D)** $\frac{1}{3}$

5. If $\{x\} = \frac{2}{3}$ and $\left[x + \left\{x + \left[x + \left\{x + \ldots 100 \text{ times}\right\}\right]\right\}\right] = 5$, then

(A) $x = \frac{14}{3}$ **(B)** $[x] = 5$

(C) $x = \frac{17}{3}$ **(D)** $[x] = 4$

(where [.] and $\{.\}$ denotes greatest integer function and fractional part function respectively)

6. The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$ (where [ ] denotes the greatest integer function) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in N$ and $\frac{b}{c}$ is in its lowest form. Find the value of $a + b + c + abc$.

Base System

The number system that we work in is called the 'decimal system'. This is because there are 10 digits in the system 0–9. There can be alternative system that can be used for arithmetic operations. Some of the most commonly used systems are: binary, octal, and hexadecimal.

These systems find applications in computing.

Binary system[2] has 2 digits: 0, 1.

Octal system[8] has 8 digits: 0, 1, …, 7.

Hexadecimal system[16] has 16 digits: 0, 1, 2, …, 9, *A*, *B*, *C*, *D*, *E*, *F*.

After 9, we use the letters to indicate digits. For instance, *A* has a value 10, *B* has a value 11, *C* has a value 12, … so on in all base systems.

The counting sequences in each of the systems would be different though they follow the same principle.

Conversion

Conversion of numbers from

(i) decimal system to other base system.
(ii) other base system to decimal system.

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Conversion from Base 10 to any Other Base

Example 77

Convert $(122)_{10}$ to base 8 system.

Solution

The number in decimal is consecutively divided by the number of the base to which we are converting the decimal number. Then list down all the remainders in the reverse sequence to get the number in that base.

So, here $(122)_{10} = (172)_8$.

Example 78

Convert $(169)_{10}$ in base 7.

Solution

7	169	
7	24	1
7	3	3
	0	3

$(169)_{10} = (331)_7$

Example 79

Convert $(0.3125)_{10}$ to binary equivalent.

Solution

$$0.3125 \times 2 = 0.625 \Rightarrow 0$$

$$0.625 \times 2 = 1.25 \Rightarrow 1$$

$$0.25 \times 2 = 0.5 \Rightarrow 0$$

$$0.5 \times 2 = 1.0 = 1$$

Example 80

Convert $(1987.725)_{10} \rightarrow (.....)_8$

Solution

First convert non–decimal part into base 8.

8	1987	
8	248	3
8	31	0
8	3	7
	0	3

$\therefore \quad (1987)_{10} = (3703)_8$

Now we have to convert $(0.725)_{10} \rightarrow (.....)_8$

Multiply

$0.725 \times 8 = [5.8]$ ….5
$0.8 \times 8 = [6.4]$ ….6
$0.4 \times 8 = [3.2]$ ….3
$0.2 \times 8 = [1.6]$ ….1
$0.6 \times 8 = [4.8]$ ….4

Keep on accomplishing integral parts after multiplication with decimal part till decimal part is zero.

$$\therefore (0.725)_{10} = (0.56314 \ldots)_8$$

$$\therefore (1987.725)_{10} = (3703.56314 \ldots)_8$$

Conversion from any Other Base to Decimal System

Example 81

Convert $(231)_8$ into decimal system.

Solution

$(231)_8$, the value of the position of each of the numbers (as in decimal system) is:

$1 = 8^0 \times 1$
$3 = 8^1 \times 3$
$2 = 8^2 \times 2$

Hence, $(231)_8 = (8^0 \times 1 + 8^1 \times 3 + 8^2 \times 2)_{10}$

$$(231)_8 = (1 + 24 + 128)_{10}$$

$$(231)_8 = (153)_{10}$$

Example 82

Convert $(0.03125)_{10}$ to base 16.

Solution

$$16 \times 0.03125 = 0.50$$

$$16 \times 0.5 = 8.08$$

So $(0.03125)_{10} = (0.08)_{16}$

Example 83

Convert $(761.56)_8 \rightarrow (.....)_{16}$

Solution

In such conversion, which are standard form conversions, it is easier to

$$(761.56)_8 \rightarrow (.....)_2 \rightarrow (.....)_{16}$$

Converting every digit in base 8 to base 2,

$$(111110001.101110)_2 \rightarrow (1F1.B8)_{16}$$

Example 84

Convert $(3C8.08)_{16}$ to decimal

Solution

$$(3C8.08)_{16} = 3 \times 16^2 + C \times 16^1 + 8 \times 16^0 + 0 \times 16^{-1} + 8 \times 16^{-2}$$

$$= 768 + 192 + 8 + 0 + 0.03125 = (968.03125)_{10}$$

So, $(3C8.08)_{16} = (968.03125)_{10}$

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Know Your Progress - 5

1. In which base system 65 can be represented as 41?
2. Represent $0.\overline{n}$ in the form of p/q ($q \neq 0$, p and q are integers). 'n' is the base of the number 2122, which is equal to $(353)_4$.
3. A number of three digits in scale 7, when expressed in scale 9 has its digits reversed in order. Find the number in scale 10.
4. Find the value of when $(25)_x = (85)_{10}$
5. If the decimal number 2^{125} is written in the hexadecimal system then find the 1st digit from left.

Some Miscellaneous Concepts

Sophie Germain Identity

$$a^4 + 4b^4 = (a^2)^2 + (2b^2)^2 + 2 \times a^2 \times 2b^2 - 4a^2b^2$$

$$= (a^2 + 2b^2)^2 - (2ab)^2$$

$$= (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$$

$$= ((a + b)^2 + b^2)((a - b)^2 + b^2)$$

This is very useful for proving whether a given number is a prime or composite.

Perfect Numbers

If the sum of divisors of a numbers n, other than itself, is equal to n, then n is called a perfect number. For example, the first two perfect numbers are 6 and 28.

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

There are 49 perfect numbers known to date (January 26) all even, and it is conjectured that there are no odd perfect numbers.

Euler's Totient Function

The number of positive integers less than or equal to n that are co-prime to n is denote by $\phi(n)$ and is called Euler's totient function.

Euler's totient function is multiplicative, i.e., if $\gcd(m, n) = 1$, then $\phi(mn) = \phi(m) \cdot \phi(n)$.

It is also obvious that for prime p, $\phi(p) = p - 1$, $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p - 1)$.

Also $\phi(1) = 1$.

and it can be shown that

$$\phi(n) = n \cdot \prod_{p|n}\left(1 - \frac{1}{p}\right)$$

i.e., If $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots p_k^{a_k}$ then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right)$$

Euler's Theorem

If $\gcd(n, a) = 1$, then

$$a^{\phi(n)} \equiv 1 \pmod{p}$$

If $n = p$, then we obtain Fermat's little theorem.

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Wilson's Theorem

Natural number $n \geq 2$ is prime number if and only if $(n - 1)! \equiv -1 \pmod{n}$.

Digit Sum Characteristic Theorem

Sum of digits of a number is congruent to the number modulo 9. The same holds for modulo 3.

Proof: Since $10^n \equiv 1 \pmod 9$ for all $n \in N$, any number written in decimal representation such as

$$(a_n a_{n-1} a_{n-2} \ldots a_1 a_0)_{10} = a_n + a_{n-1} + \ldots + a_1 + a_0 \pmod 9.$$

The Digital Sum of *N*

The number of 1's occurring in the digits from 10 to 19 = 11 and from 20 to 99 = 8.

So, total of 1's is 11 + 8 = 19.

Similarly, no. of 2's, 3's, … , 9 are all equal to 19.

So, sum of all the digits = 19 (1 + 2 + 3 + … + 9)

$$= \frac{19 \times 9 \times 10}{2} = 19 \times 5 \times 9 = 855$$

and as 91855, 1011 … 99 is divisible by 9.

When the numbers start from 12, the sum of the digits become 855 – 3 = 852 (since 10, 11 account for the digital sum 3) and hence, is divisible by 3.

(a) For divisibility by 3, it could start from 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, …

(b) For divisibility by $3^2 = 9$ the numbers may start from any of 18, 19, 27, 28, 36, 37,

Example 85

Prove that $n^4 + 4$ is a composite number for all $n > 1$, $n \in N$.

Solution

Since $n^4 + 4 = (n^4 + 4n^2 + 4) - 4n^2$

$= (n^2 + 2)^2 - (2n)^2$

$= (n^2 + 2n + 2)(n^2 - 2n + 2)$.

$= [(n + 1)^2 + 1][(n - 1)^2 + 1]$

For $n > 1$, $(n \pm 1)^2 + 1 > 1$ and hence, $n^4 + 4$ is a composite number for all $n > 1$, $n \in N$.

Example 86

Find the largest positive integer n such that $n^3 + 100$ is divisible by $(n + 10)$.

Solution

Using modulo $(n + 10)$ numbers, we see that

$$n + 10 = 0 \ [\text{mod}(n + 10)]$$

i.e., $n \equiv (-10) \ [\text{mod}(n + 10)]$

$n^3 \equiv (-10)^3 \ [\text{mod}(n + 10)]$

$= -1000[\text{mod}(n + 10)]$

$\therefore$ $n^3 + 100 \equiv (-1000 + 100)[\text{mod}(n + 10)]$

$\equiv -900 \ [\text{mod}(n + 10)]$.

Now, we want $(n + 10)$ to divide $n^3 + 100$, implying that $(n + 10)$ should divide -900.

The largest such n is 900 – 10 = 890, as $(n + 10)$ cannot be greater than 1 – 9001 = 9000 and the greatest divisor of 1 – 9001 is 900.

So the largest positive integer n, such that $n^3 + 100$ is divisible by $(n + 1\ 0)$ is $n = 890$.

Note

$900 = 3^2 \times 2^2 \times 5^2$ has 27 divisors and each divisor greater than 10, gives a corresponding value for n, are 2, 5, 8, 10, 15, 20, 26, 35, 40, 50, 65, 80, 90, 140, 170.

Finding number 'a' such that $5^a \equiv 1 \pmod{n}$ basically allows us to reduce the exponent in a problem, if the a is small such as 2, then the reduction is very drastic as seen in example above.

Example 87

Determine all positive integers n for which $2^n + 1$ is divisible by 3.

Solution

$$2^n + 1 = 2^n + 1^n.$$

If n is odd, then $(2 + 1)$ is a factor, thus for all odd values of n, $2^n + 1$ is divisible by 3.

For n even $= 2m$ say,

$$2n + 1 = 2^{2m} = +1 = 4^m - 1 + 2$$

Now $3 = 4 - 1$ divides $4^m - 1^n$ but $3 \mid 2 \Rightarrow 3 \mid (2^m + 1)$ for n even.

Aliter: $2 \equiv -1 \pmod{3}$

$$\Rightarrow 2^{2m+1} \equiv -1 \pmod{3} \text{ and } 2^{2m} \equiv 1 \pmod{3}$$

So, $2^n + 1 \equiv 0 \pmod{3}$, if n is odd.

and $2^n + 1 \equiv 1 + 1 = 2 \pmod{3}$, if n is even.

Therefore, $2^n + 1$ is divisible by 3, if and only if, n is an odd number.

Example 88

What is the remainder when 2016^{2016} is divided by 2017?

Solution

As $2016 \equiv -1 \pmod{20\,17}$

$$\Rightarrow 2016^{2016} = 1 \pmod{2017}$$

Example 89

Find the remainder when $452^{72^{452}}$ is divided by 3.

Solution

This problem does not require much work, just one insight leads to immediate solution, we note that $452 \equiv -1 \pmod{3}$. Thus

$$452^{72^{452}} \equiv (-1)^{72^{452}} \pmod{3} = 1 \pmod{3}.$$

The last congruence holds because 72^{452} is surely even.

Example 90

Suppose 5^{5555} is divided by 24. Find the remainder.

Solution

It is not hard to find, by inspection, that $5^2 = 25 \equiv 1 \pmod{24}$. Now we can write

$$5^{5555} = 5^{5554} \times 5 = (5^2)^{2777} \times 5 \equiv 12^{777} \times 5 \equiv \mathbf{5}$$

Finding number 'a' such that $5^a \equiv 1 \pmod{n}$ basically allows us to reduce the exponent in problem, if the a is small such as 2, then the reduction is very drastic as seen in Example above.

Example 91

Show that $2^{55} + 1$ is divisible by 11.

Solution

$$2^5 = 32 \equiv (-1) \pmod{11}$$

$$2^{55} = (2^5)^{11} \equiv (-1)^{11} \equiv -1 \pmod{11}$$

So $2^{55} + 1 \equiv 0 \pmod{11}$

$\therefore$ It is a multiple of 11.

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Example 92

Find the sum of all integers n, such that $1 \le n \le 1998$ and that 60 divides

$$n^3 - 30n^2 + 100n.$$

Solution

(i) If $60 = 3 \times 4 \times 5$ and $4 \mid 100n$, then 4 should divide $n^3 + 30n^2$, i.e., 4 should divide $n^2(n + 30)$. This implies that n is even. i.e., $2 \mid n$

(ii) As $5 \mid (30n^2 + 100n)$, 5 should divide n^3. Hence, 5 should divide n.

(iii) As $3 \mid 30n^2$, then 3 should divide $n^3 + 100n$, i.e., 3 should divide $n(n^2 + 100) = n(n^2 + 1 + 99)$

If $n = \pm 1 \pmod{3}$, $n^2 \equiv 1 \pmod{3}$, and $n^2 + 1 \equiv 2 \pmod{3}$, so neither of $(n^2 + 1 + 99)$ and n are divisible by 3.

However, if $n \equiv 0 \pmod{3}$, then $n(n^2 + 1 + 99)$ is divided by 3, i.e., $n(n^2 + 100)$ is divisible by 3 only if n is a multiple of 3.

From (i), (ii), and (iii), we find that n must be a multiple of $2 \times 3 \times 5 = 30$. So, we should find the sum of all multiples of 30 less than 1998

$$s_n = 30 + 60 + \ldots + 1980$$

$$= 30(1 + 2 + \ldots + 66) = 66330.$$

Example 93

Find the last two digits of $(56789)^{41}$.

Solution

$$56789 \equiv 89 (\text{mod } 100)$$
$$\equiv -11 (\text{mod } 100)$$
$$\therefore \quad (56789)^{41} \equiv (-11)^{41} (\text{mod } 100)$$
$$\equiv (-11)^{40} \times (-11) (\text{mod } 100)$$
$$\equiv (11)^{40} \times (-11) (\text{mod } 100)$$
$$11^2 \equiv 21 (\text{mod } 100)$$
$$11^4 \equiv 41 (\text{mod } 100)$$
$$11^6 \equiv 21 \times 41 (\text{mod } 100)$$
$$\equiv 61 (\text{mod } 100)$$
$$11^{10} \equiv 41 \times 61 (\text{mod } 100)$$
$$\equiv 01 (\text{mod } 100)$$
$$11^{40} \equiv (01)^{40} (\text{mod } 100)$$
$$\equiv 1 (\text{mod } 100)$$
$$(-11)^{41} \equiv 11^{40} \times (-11) (\text{mod } 100)$$
$$\equiv 1 \times (-11) (\text{mod } 100)$$
$$\equiv -11 (\text{mod } 100)$$
$$(56789)^{41} = 89 (\text{mod } 100)$$

i.e., the last two digits of $(56789)^{41}$ are 8 and 9 in that order.

Example 94

Prove that $2222^{5555} + 5555^{2222}$is divisible by 7.

Solution

Since $2222^{5555} + 5555^{2222}$

$$= 2222^{5555} + 4^{5555} + 5555^{2222} - 4^{2222} - 4^{5555} + 4^{2222}$$

$$= (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - 4^{2222} (4^{3333} - 1)$$

Now, $2222^{5555} + 4^{5555}$ is divisible by $2222 + 4 = 2226 = 7 \times 318$, $5555^{2222} - 4^{2222}$ is divisible by $5555 - 4 = 5551 = 7 \times 793$ and $4^{3333} - 1 = (4^3)^{1111} - 1 = 64^{1111} - 1$ is divisible by $64 - 1 = 63 = 7 \times 9$.

Thus, $2222^{5555} + 5555^{2222}$ can be split up into three terms each of which is divisible *b*) 7 and hence, the result.

Aliter:

$$2222 \equiv 3 (\text{mod } 7)$$

$$\Rightarrow \quad 2222^2 \equiv 9 \equiv 2 (\text{mod } 7) \qquad \text{(i)}$$

$$\Rightarrow \quad 2222^4 \equiv 4 (\text{mod } 7) \qquad \text{(ii)}$$

$$\Rightarrow \quad 2222^6 \equiv 8 = 1 (\text{mod } 7)$$

From (i) and (ii)

$$\Rightarrow \quad 2222^{5555} = [(2222)^6]^{925} \times 2222^5$$
$$= [(2222)^6]^{925} \times 2222^4 \times 2222^1$$
$$\equiv 1 \times 4 \times 3 (\text{mod } 7) \equiv 12 \equiv 5 (\text{mod } 7)$$

Also $\quad 5555 \equiv 4 (\text{mod } 7)$

$$\Rightarrow \quad 5555^3 = 4^3 (\text{mod } 7) = 1 (\text{mod } 7)$$

$$\Rightarrow \quad (5555)^{2222} = (5555^3)^{740} \times 5555^2 \equiv 1 \times 4 \times 4 (\text{mod } 7)$$
$$\equiv 2 (\text{mod } 7)$$

and hence, $2222^{5555} + 5555^{2222} \equiv 5 + 2 = 0 (\text{mod } 7)$ and hence, the result.

Example 95

All two-digit numbers from 10 to 99 are written consecutively, i.e., $N = 101112 \ldots 99$. Show that $3^2 | N$. From which other two-digit number you should start so that N is divisible by (a) 3 (b) 3^2?

Solution

N is divisible by 9, if the digit sum is divisible by 9.

Example 96

Find the remainder when 33! is divided by 37.

Solution

Notice that 37 is prime, Wilson's theorem states that $36! \equiv -1 \text{ (mod } 37)$, now for simplicity let

$$x = 33!, \text{ then } 34 \cdot 35 \cdot 36 \cdot x \equiv -1 (\text{mod } 37).$$

We have $34 \cdot 35 \cdot 36 = (-3)(-2)(-1) = -6 (\text{mod } 37)$,

So $\quad -6x \equiv -1 (\text{mod } 37) \Leftrightarrow 6x \equiv 1 (\text{mod } 37)$,

i.e., there exists a such that $6x = 37a + 1$, looking at this equation modulo 6 we find $37a \equiv a \equiv -1 \equiv 5 \text{ (mod } 6)$, which is to say that there exists b such that $a = 6b + 5$, thus

$$6x = 37 (6b + 5) + 1 \Leftrightarrow x = 37b + 31. \Rightarrow x = 31 (\text{mod } 37)$$

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Example 97

What is the remainder when 10! is divided by 13?

Solution

By Wilson's Theorem

$$12! \equiv -1 (\text{mod } 13)$$

$$12! \equiv -1 - r\,13 (\text{mod } 13)$$

$$11! \equiv 1 (\text{mod } 13) \text{ by 'dividing' by } 12$$

$$11! \equiv 1 + 5 \times 13 (\text{mod } 13)$$

$$10! = 6 (\text{mod } 13) \text{ by 'dividing' by } 11$$

Example 98

Find the remainder when 4333^{3333} is divided by 9.

Solution

$$4333 \equiv 4 + 3 + 3 + 3 = 13 (\text{mod } 9) = 4 (\text{mod } 9)$$

$$\therefore \quad 4333^3 \equiv 4^3 (\text{mod } 9)$$

$$\equiv 64 (\text{mod } 9)$$

$$\equiv 1 (\text{mod } 9)$$

$$\equiv 4333^{3333} \equiv 1 (\text{mod } 9)$$

i.e., when 4333^{3333} is divided by 9, the remainder is 1.

Example 99

Prove that among 18 consecutive three-digit numbers there is at least one number that is divisible by the sum of its digits.

Solution

Among 18 consecutive integers, there are two numbers which are divisible by 9.

The sum of the digits of these two numbers must be 9, 18 or 27.

If the sum of the digits is 9, then the number is divisible by the sum of the digits, so there is nothing to prove.

If the sum of the digits is 27, then the three-digit numbers should be $999 = 9 \times 111 = 9 \times 3 \times 37$ and hence, the result. Let both the numbers have 18 as the sum of their digits. Let those numbers be a and b with $a < b$.

If a is odd and sum of its digits is 18, it is divisible by 9 but not by 18. However, the other number b is also divisible by 9 and b should be $a + 9 \Rightarrow b$ is even and sum of its digits is 18, and hence, b is an even number as well as divisible by $9 \Rightarrow b$ is divisible by

Example 100

Suppose $\delta(n)$ denotes digit sum of n. Find $\delta(\delta(\delta(5^{2013})))$.

Solution

First repeatedly using modulo 9 yields

$$\delta(\delta(\delta(5^{2013}))) \equiv \delta(\delta(5^{2013})) \equiv \delta(5^{2013}) \equiv 5^{2013} (\text{mod } 9)$$

Thus, finding 5^{2013} in modulo 9 will help us. This can be done by finding $\phi(9) = 6$. So by Euler's theorem we get

$$5^{2013} = (5^6)^{335} \times 5^3 \equiv 5^3 \equiv \delta (\text{mod } 9).$$

Now it suffices to realize that the sought number will probably be very small because digit sum of a big number is much smaller than the number. So it suffices to establish some upper bounds on the number sought. We can of course establish sharp bounds, but it is not needed for this problem, we have $5^{2013} < 10^{2013}$, so

$$\delta(5^{2013}) < 9 \times 2014 = 18126$$

Number less than 18126 with greatest digit sum is 9999, so

$$\delta(\delta(5^{2013})) \le 9 + 9 + 9 + 9 = 36,$$

Again number less than or equal to 36 that has greatest digit sum is 29, thus

$$\delta(\delta(\delta(5^{2013}))) \le 11.$$

But only positive number less than or equal to 11 and congruent to 8 modulo 9 is 8.

Thus, the number sought was 8.

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. The digit at the 100th place in the decimal representation of $\frac{6}{7}$, is:
   (A) 1 **(B)** 2
   (C) 4 **(D)** 5

2. The number of positive n in the range $12 \le n \le 40$ such that the product $(n-1)(n-2) \ldots 3.2.1$ is not divisible by n is:
   (A) 5 **(B)** 7
   (C) 13 **(D)** 14

3. The positive integers A, B, $A - B$ and $A + B$ are all prime numbers. The sum of these four primes is
   (A) even **(B)** divisible by 3
   (C) divisible by 5 **(D)** prime

4. Total number of factors of $(3 \times 5)^{12}\ (2 \times 7)^{10}\ (10)^{25}$ is:
   (A) 47 **(B)** 94
   (C) 72 **(D)** 144144

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Directions (5 to 9): Read the following information carefully and answer the questions given below. In a big hostel, there are 1,000 rooms. In that hostel only even numbers are used for room numbers, i.e., the room numbers are 2, 4, 6, …, 1998, 2000. All the rooms have one resident each. One fine morning, the warden calls all the residents and tells them to go back to their rooms as well as multiples of their room numbers. When a guy visits a room and finds the door open, he closes it, and if the door is closed, he opens it, All 1,000 guys do this operation. All the doors were open initially.

5. The last room that is closed is room number?
   (A) 1936 **(B)** 2000
   (C) 1922 **(D)** None of these

6. The 38^{th} room that is open is room number:
   (A) 80 **(B)** 88
   (C) 76 **(D)** None of these

7. If only 500 guys, i.e., residents of room number 2 to 1000 do the task, then the last room that is closed is room number
   (A) 2000 **(B)** 1936
   (C) 1849 **(D)** None of these

8. In the case of the previous question, how many rooms will be closed in all?
   (A) 513 **(B)** 31
   (C) 13 **(D)** 315

9. If you are a lazy person, you would like to stay in a room whose number is:
   (A) more than 500 **(B)** more than 1000
   (C) 500 **(D)** 2000

10. If $(2^{32} + 1)$ is divisible by a certain number then which of the following is also divisible by that number.
    (A) $(2^{16} - 1)$ **(B)** $2^{16} + 1$
    (C) $2^{96} + 1$ **(D)** None of these

11. $1^{13} + 2^{13} + 3^{13} + \ldots + 60^{13}$ is divisible by:
    (A) 61 **(B)** 63
    (C) 65 **(D)** 59

12. A number when divided by 342 gives a remainder 47. When the same number is divided by 19, what would be the remainder?
    (A) 3 **(B)** 5
    (C) 9 **(D)** None of these

13. In a number system, the product of 44 and 11 is 3414. The number 3111 of this system, when converted to the decimal number system, becomes:
    (A) 406 **(B)** 1086
    (C) 213 **(D)** 691

14. What is remainder when 7^{84} is divided by 2402?
    (A) 1 **(B)** 6
    (C) 2401 **(D)** None of these

15. What is the remainder when 30^{40} is divided by 17?
    (A) 1 **(B)** 16
    (C) 13 **(D)** 4

16. A number when divided successively by 4 and 5 leaves remainders 1 and 4, respectively. When it is successively divided by 5 and 4, then the respective remainders will be:
    (A) 1, 2 **(B)** 2, 3
    (C) 3, 2 **(D)** 4, 1

17. Find out $(A + B + C + D)$ such that $AB \times CB = DDD$, where AB and CB are two-digit numbers and DDD is a three-digit number.
    (A) 21 **(B)** 19
    (C) 17 **(D)** 18

18. If $\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008}5$, then:

(A) $M^9 = \frac{9}{N}$ **(B)** $N^9 = \frac{9}{M}$

(C) $M^3 = \frac{3}{N}$ **(D)** $N^9 = \frac{3}{M}$

19. Three friends, returning from a movie, stopped to eat at a restaurant. After dinner, they paid their bill and noticed a bowl of mints at the front counter. Sita took 1/3 of the mints, but returned four because she had a momentary pang of guilt. Fatima then took 1/4 of what was left but returned three for similar reasons. Eswari then took half of the remainder but threw two back into the bowl. The bowl had only 17 mints left when the raid was over. How many mints were originally in the bowl?

(A) 38 **(B)** 31

(C) 41 **(D)** 48

20. A set of consecutive positive integers beginning with 1 is written on the blackboard. A student came and erased one number. The average of the remaining numbers is $35\frac{7}{17}$. What was the number erased?

(A) 7 **(B)** 8

(C) 9 **(D)** None of these

21. How many even integers n, where $100 \le n \le 200$, are divisible neither by seven nor by nine?

(A) 40 **(B)** 37

(C) 39 **(D)** 38

22. Three pieces of cakes of weights $4\frac{1}{2}$ lbs, $6\frac{3}{4}$ lbs and $7\frac{1}{5}$ lbs, respectively, are to be divided into parts of equal weights. Further, each must be as heavy as possible. If one such part is served to each guest, then what is the maximum number of guests that could be entertained?

(A) 54 **(B)** 72

(C) 20 **(D)** 41

23. The largest natural number by which the product of three consecutive even natural numbers is always divisible, is:

(A) 6 **(B)** 24

(C) 48 **(D)** 96

24. Let $N = 55^3 + 17^3 - 72^3$, then N is divisible by:

(A) both 7 and 13 **(B)** both 3 and 13

(C) both 17 and 7 **(D)** both 3 and 17

25. $7^{6n} - 6^{6n}$, where n is an integer > 0, is divisible by:

(A) 13 **(B)** 127

(C) 559 **(D)** All of these

26. Find the square root of $7 - 4\sqrt{3}$.

(A) $2 - \sqrt{3}$ **(B)** $5 - \sqrt{3}$

(C) $2 - \sqrt{5}$ **(D)** None of these

27. The value of $\sqrt{10 + \sqrt{25 + \sqrt{108 + \sqrt{154 + \sqrt{225}}}}}$ is:

(A) 4 **(B)** 6

(C) 8 **(D)** 10

28. On dividing a natural number by 13, the remainder is 3 and on dividing the same number by 21, the remainder is 11. If the number lies between 500 and 600, then the remainder on dividing the number by 19 is:

(A) 4 **(B)** 6

(C) 9 **(D)** 13

29. Expressing $0.\overline{34} + 0.3\overline{4}$ as a single decimal, we get

(A) $0.67\overline{88}$ **(B)** $0.6\overline{89}$

(C) $0.6\overline{878}$ **(D)** $0.6\overline{87}$

30. The sum of all the possible remainders, which can be obtained when the cube of a natural number is divided by 9, is

(A) 5 **(B)** 6

(C) 8 **(D)** 9

31. What is the smallest positive integer x for which $x^2 - x + 5$ is not a prime?

(A) 155 **(B)** 157

(C) 156 **(D)** 154

32. How many prime numbers satisfy $\sqrt{300} < x < \sqrt{700}$?

(A) 5 numbers **(B)** 3 numbers

(C) 2 numbers **(D)** 1 number

33. If $a + b + c = 66$ and $ab + bc + ca = 107$, where a, b, and c are all prime numbers, find the value of abc.

(A) 1007 **(B)** 2876

(C) 1886 **(D)** 1765

34. Given $1x6y7$ is a five-digit number divisible by 9. The number of ordered pairs (x, y) satisfying this is

(A) 5 **(B)** 6

(C) 8 **(D)** 11

35. The number of three digit numbers that are divisible by 2, but not divisible by 4, is

(A) 200 **(B)** 225

(C) 250 **(D)** 450

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36. If $n = 10^{10} - 1$, the number of digits in n^3 is
(A) 30 **(B)** 28
(C) 32 **(D)** 27

37. The last two digit of 3^{2012}, when represented in decimal notation, will be
(A) 81 **(B)** 01
(C) 41 **(D)** 21

38. Give the units digit of the number 3^{2011}.
(A) 7 **(B)** 8
(C) 10 **(D)** 11

39. When written out completely 16^{2011} has m digits and 625^{2011} has n digits. Find the value of $(m + n)$.
(A) 8000 **(B)** 4045
(C) 8045 **(D)** 7040

40. Find the units digit of $(2013)^{2013}$.
(A) 2 **(B)** 3
(C) 5 **(D)** 7

41. The sum of the digits of the number $10^n - 1$ is 3798. The value of n is
(A) 431 **(B)** 673
(C) 422 **(D)** 501

42. The last digit in the finite decimal representation of the number $\left(\frac{1}{5}\right)^{2004}$ is
(A) 8 **(B)** 6
(C) 4 **(D)** 2

43. The remainder when the number $(2 \times 3 \times 4 \times 2012 \times 2013 \times 2014 \times 2015) - 2008$ is divided by 2015 is
(A) 1 **(B)** 2008
(C) 7 **(D)** 2007

44. Find the value of
$\sqrt{2\sqrt{2\sqrt{2\sqrt{2....}}}} - \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+...}}}}$
(A) 0 **(B)** 1
(C) 2 **(D)** 3

45. Find the smallest positive integer k such that $2^{69} + k$ is divisible by 127.
(A) 45 **(B)** 63
(C) 80 **(D)** 95

46. Simplify $M = \sqrt{2+\sqrt{-2+2\sqrt{5}}} - \sqrt{2-\sqrt{-2+2\sqrt{5}}}$.
(A) $\sqrt{5} - 1$ **(B)** 32
(C) 4 **(D)** $\sqrt{6} + 1$

47. Simplify
$P = \frac{1}{2\sqrt{1}+\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + ... + \frac{1}{100\sqrt{99}+99\sqrt{100}}$
(A) $\frac{4}{10}$ **(B)** $\frac{1}{10}$
(C) $\frac{6}{10}$ **(D)** $\frac{9}{10}$

48. A number is called a palindrome if it reads the same forward or backward. For example 13531 is palindrome. The difference between the biggest 10–digit palindrome and the smallest 9 digit palindrome is
(A) 976666666 **(B)** 9888888888
(C) 9899999998 **(D)** 9777777777

49. The remainder when the number $(2 \times 3 \times 4 \times 2007 \times 2008 \times 2009) - 2008$ is divided by 2009 is
(A) 0 **(B)** 2008
(C) 1 **(D)** 2007

50. The number of primes less than 100 but having 7 as the unit digits is
(A) 6 **(B)** 7
(C) 8 **(D)** 9

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PART – II KVPY Level Problems

1. For how many positive integers a is $a^4 - 3a^2 + 9$ a prime number?
(A) 5 **(B)** 7
(C) 6 **(D)** 2

2. If n is a positive integer greater than 1, then $n^4 + 4$ is a _____ number.
(A) Composite
(B) Composite odd number
(C) Composite even number
(D) Prime

3. What is the smallest positive integer x for which $x^2 - x + 5$ is not a prime?
(A) 5 **(B)** 7
(C) 6 **(D)** 13

4. What is the smallest composite number generated by $p^2 - p - 1$ where p is a prime?
(A) 155 **(B)** 157
(C) 156 **(D)** 154

5. Some prime numbers are 1 more then a square, as in $4^2 + 1 = 17$. How many primes $p \le 100$ are of the from $p = n^2 + 1$?
(A) 5 **(B)** 7
(C) 4 **(D)** 23

6. How many prime numbers satisfy $\sqrt{300} < x < \sqrt{700}$?
(A) 5 numbers **(B)** 3 numbers
(C) 2 numbers **(D)** 1 numbers

7. The number of pairs of prime numbers (p, q) satisfying the condition $\frac{51}{100} < \frac{1}{p} + \frac{1}{q} < \frac{5}{6}$ will be:
(A) 49 **(B)** 24
(C) 50 **(D)** 48

8. The number of positive integral values of n for which $(n^3 - 8n^2 + 20n - 13)$ is a prime number is:
(A) 2 **(B)** 1
(C) 3 **(D)** 4

9. The number of positive integers 'n' for which $3n - 4$, $4n - 5$ and $5n - 3$ are all primes is:
(A) 1 **(B)** 2
(C) 3 **(D)** infinite

10. m, n are natural numbers. The number of pairs (m, n) for which $m^2 + n^2 + 2mn - 2013m - 2013n - 2014 = 0$ is:
(A) 2000 **(B)** 2013
(C) 1023 **(D)** 2007

11. Given $1x6y7$ is a five-digit number divisible by 9. The number of ordered pairs (x, y) satisfying this is:
(A) 5 **(B)** 6
(C) 8 **(D)** 11

12. The number of three digit numbers that are divisible by 2 but not divisible by 4 is:
(A) 200 **(B)** 225
(C) 250 **(D)** 450

13. a, b, c are the digits of a nine-digit number $abcabcabc$. Calculate the quotient when this number is divided by 1001001.
(A) $100a + 10c + b$ **(B)** $100c + 10b + a$
(C) $100a + 10b + c$ **(D)** $10a + c$

14. Find the number of integers n for which $\frac{n}{20 - n}$ is the square of an integer.
(A) 1 **(B)** 2
(C) 3 **(D)** 4

15. The years of 20th century and 21th century are of 4 digits. The number of years that are divisible by the product of the four digits of the year is:
(A) 7 **(B)** 8
(C) 9 **(D)** None of these

16. The number of numbers of the form $30a0b03$ that are divisible by 13, where a, b are digits, is
(A) 5 **(B)** 6
(C) 7 **(D)** 0

17. A natural number n has exactly two divisors and $(n + 1)$ has three divisors. The number of divisors of $(n + 2)$ is:
(A) 2
(B) 3
(C) 4
(D) depends on the value of n

18. a, b, c, d are natural numbers such that $a = bc$, $b = cd$, $c = da$, and $d = ab$. The $(a + b)(b + c)(c + d)(d + a)$ is equal to
(A) $(a + b + c + d)^2$ **(B)** $(a + b)^2 + (c + d)^2$
(C) $(a + d)^2 + (b + c)^2$ **(D)** $(a + c)^2 + (b + d)^2$

19. A three-digit number with digits A, B, C in that order is divisible by 9. A is an odd digit and C is an even digit. B and C are non zero. The number of such three digit numbers is:
(A) 4 **(B)** 8
(C) 16 **(D)** 20

20. Nine numbers are written in ascending order. The middle number is also the average of the nine numbers. The average of the 5 larger numbers is 68 and the average of the 5 smaller numbers is 44. The sum of all the numbers is:
(A) 540 **(B)** 450
(C) 504 **(D)** 501

21. If the average of 20 different positive integers is 20 then the greatest possible number among these 20 numbers can be
(A) 210 **(B)** 200
(C) 190 **(D)** 180

22. If a, b are natural numbers such that $a + b = 2008$, then $(-1)^a + (-1)^b$ is:
(A) 1 **(B)** –1
(C) 2 **(D)** 2 or –2

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23. The number 16^{12} is obtained from the number 8^4 by raising the smaller number to the power n. Then n is

(A) 3 **(B)** 4

(C) $\frac{16}{3}$ **(D)** $\frac{8}{3}$

24. The sum of two natural number is 100. Which of the following can be the greatest product of the numbers.

(A) 1000 **(B)** 4000

(C) 2500 **(D)** 900

25. A computer is printing a list of the seventh powers of all natural numbers, that is sequence $1^7, 2^7, 2^7, \ldots$. The number of terms (or numbers) between 5^{21} and 4^{28} are:

(A) 12 **(B)** 130

(C) 14 **(D)** 150

26. Suppose that $a = \sqrt{6} - 2$ and $b = 2\sqrt{2} - \sqrt{6}$. Then

(A) $a > b$ **(B)** $a = b$

(C) $a < b$ **(D)** $b = \sqrt{2a}$

27. The number of integers x that satisfies the inequality $\frac{3}{1+\sqrt{3}} < x < \frac{3}{\sqrt{5}-\sqrt{3}}$ is:

(A) 2 **(B)** 3

(C) 4 **(D)** 5

PART – III PRMO Level Problems

1. Find the last two digits of $9^{100} - 1$.

2. Find the sum of digits of the largest positive integer n such that $n!$ ends with exactly 100 zeros.

3. Find the last two digits (in order) of $7^{3^{2005}}$.

4. Find the smallest two-digit number N such that the sum of digits of $10^N - N$ is divisible by 170.

5. Find the number of positive integers n such that $n + 2n^2 + 3n^3 + \ldots + 2005n^{2005}$ is divisible by $n - 1$.

6. Let n be the number

$$\underbrace{(999\ 999\ 999\ldots999)^2}_{2006\ 6's} - \underbrace{(666\ 666\ 666\ldots666)^2}_{2006\ 6's}$$

Find the reminder when n is divided by 11.

7. What is the sum of all the digits of the largest positive integer n for which $n^3 + 2006$ is divisible by $n + 26$?

8. Find the smallest positive integer n such that $n(n + 1)(n + 2)$ is divisible by 247.

9. Let $N = 12345678910111213 \ldots 4344$ be the 79-digit number obtained that is formed by writing the integers from 1 to 44 in order, one after the other. The remainder when N is divided by 45 is r. What is r^2?

10. Let $k = 2008^2 + 2^{2008}$. What is the square of the units digit of $k^2 + 2^k$?

11. Anna randomly picked five integers from the following list:

53, 62, 66, 68, 71, 82, 89

and discover that the average value of the five integers she picked is still an integer. If two of the integers she picked were 62 and 89, find the mean of the remaining three integers.

12. Total number of 0's at the end of the value of the product $1 \times 2 \times 3 \times 4 \times \ldots \times 2008$ is n, then find the value of $\sqrt{n + 125}$

13. Find the remainder when $(1! \times 1) + (2! \times 2) + (3! \times 3) + \ldots + (286! \times 286)$ is divided by $2^4 \times 5^3$.

14. What is the remainder when $3^0 + 3^1 + 3^2 + \ldots + 3^{2009}$ is divided by 8?

15. Let n denote the smallest positive integer that is divisible by both 4 and 9, and whose base-10 representation consists of only 4's and 9's, with at least one of each. What will be the sum of the last four digits of n?

16. How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits?

17. Suppose that n is the product of three consecutive integers and that n is divisible by 7. Then largest two digit number with which n is divisible is?

18. An n-digit positive integer is cute if its n digits are an arrangement of the set $\{1, 2, \ldots, n\}$ and its first k digits form an integer that is divisible by k, for $k = 1, 2, \ldots, n$. For example, 321 is a cute 3-digit integer because 1 divides 3, 2 divides 32, and 3 divides 321. How many cute 6-digit integers are there?

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19. Consider the non-decreasing sequence of positive integers:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, …

in which the n^{th} positive integer appears n times. The remainder when the 2019^{th} term is divided by 5 is

20. Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: 1, 2, …, $\frac{n}{2}$, n. At some moment. Mary wrote 323 as a divisor of n. If the smallest possible value of the next divisor written to the right of 323 is N, find the value of $\frac{N}{10}$.

21. Let N = 123456789101112 … 4344 be the 79-digit number obtained that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

22. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $9n^2$ have?

23. What is the last two digits of $2015^{2016} - 2017$?

24. The largest divisor of 2, 014, 000, 000 is itself. If is fifth largest divisor is X. Find sum of all the digits of X.

25. How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

26. For a positive integer k, we write

$(1 + x)\ (1 + 2x)\ (1 + 3x)\ \ldots\ (1 + kx)$

$= a_0 + a_1x + a_2x^2 + \ldots + a_k x^k$,

where a_0, a_1, …, a_k are the coefficients of the polynomial. Find the sum of all the digits of smallest possible value of k if $a_0 + a_1 + a_2 + \ldots + a_{k-1}$ is divisible by 2005.

27. The smallest three-digit number n such that if the three digits are a, b and c, then

$n = a + b + c + ab + bc + ac + abc$.

Evaluate $\frac{n+1}{5}$

28. Why is the largest positive integer n satisfying $n^{200} < 5^{300}$?

29. N pieces of candy are made and packed into boxes, with each box containing 45 pieces. If N is a non-zero perfect cube and 45 is one of its factors, what is the least possible number of boxes that can be packed?

30. For any non-empty finite set A of real numbers, let $s(A)$ be the sum of the elements in A. There are exactly 61, 3-element subsets A of {1, …, 23} with $s(A) = 36$. The total number of 3-element subsets of {1, …, 23} with $s(A) < 36$ is given by n, then evaluate $\frac{n+45}{25}$

31. Let the sum of the digits of the given product is written as $A8B6C$. Find $A + B + C$

$$\underbrace{(111111111\ldots111)}_{2007\ 1's} \times 2007$$

32. When 2007 bars of soap are packed into N boxes of equal size, where N is an integer strictly between 200 and 300, there are extra 5 bars remaining. Find last two digit of N.

33. Let the number of distinct integers among the numbers is N. Find the sum of all digits of N.

$$\left[\frac{1^2}{2007}\right], \left[\frac{2^2}{2007}\right], \ldots, \left[\frac{2007^2}{2007}\right].$$

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34. For each integer $n \geq 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4a_5 \ldots a_{99}$ can be expressed as $\frac{m}{n!}$. where m and n are positive integers and n is as small as possible. What is the value of sum of the digits of m?

35. The square is a multiplicative magic square. That is, the product of the numbers in each row, column, and diagonal is the same. If all the entries are positive integers, what is the sum of the possible values of g?

50	b	c
d	e	f
g	h	2

36. For how many integers n is $\frac{n}{20-n}$ the square of an integer?

37. How many four-digit numbers N have the property that the three-digit number obtained by removing the leftmost digit is one ninth of N?

38. If a, b, and c are positive real numbers such that $a(b + c) = 152$, $b(c + a) = 162$, and $c(a + b) = 170$, then $\frac{abc}{12}$ is

39. If x, y, and z are positive numbers satisfying $x + 1/y = 4$, $y + 1/z = 1$, and $z + 1/x = 7/3$,then find xyz?

40. There are x integer values of n (not necessarily positive) so that value of $4000.\left(\frac{2}{5}\right)^n$ is an integer. What is x^2?

41. Let S be a set of 6 integers taken from $\{1, 2, \ldots 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the square of the least possible value of an element in S?

42. R non-negative integers can be written in the form $a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0$,
where $a_1 \in \{-1, 0, 1\}$ for $0 \le | \le 7$. What is the least two digits of R?

43. How many odd positive 3-digit integers are divisible by 3, but do not contain the digit 3?

44. Let p and q be positive integers such that $\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$ and q is as small as possible. What is $(q - p)^2$?

45. A function f is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n - 1) - f(n - 2) + n$ for all integers $n \ge 3$. What is the sum of the digits of $f(2018)$?

46. Sunita has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she scan pot with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

47. A five-digit palindrome is a positive integer with respective digits $abcba$, where a is non-zero. Let S be the sum of all five-digit palindromes. What is the sum of the digits of S?

48. The products (8) (888….8), where the second factor has k digits. is an integer whose digits have a sum of 1000. What is the sum of digits of k?

49. There are total P positive integers value of n, so that $\frac{n}{30 - n}$ is also a positive integer. What is P^2?

50. Let S be the set of positive integers n for which $\frac{1}{n}$ has the repeating decimal representation $0.\overline{ab} = 0 \cdot ababab\ldots$, with a and b different digits. The sum of the elements of S id a three digit number abc. What is the value of $\frac{abc}{11}$?

51. Let a, b, and c be digit with $a \ne 0$. The three-digit integer abc lies one third of the way from the square of a positive integer to the square of the next larger integer. The integer acb lies two thirds of the way between the same two squares. What is $a + b + c$?

52. There are p pairs of positive integers (a, b) such that $\gcd(a, b) = 1$ and $\frac{a}{b} + \frac{14b}{9a}$ is an integer. What is $7 \times p$?

53. Let a, b, c, d, e, f, g, and h be distinct elements in the set $\{-7, -5, -3, -2, 2, 4, 6, 13\}$. What is the minimum possible value of $(a + b + c + d)^2 + (e + f + g + h)^2$?

54. There are 88 numbers a_1, a_2, a_3, …, a_{88} and each of them is either equals to -3 or -1. Given that $a_1^2 + a_2^2 + \ldots + a_{88}^2 = 280$, and $a_1^4 + a_2^4 + \ldots + a_{88}^4 = 10^3a + b$. Find the product of a and b

55. The difference between the HCF and the LCM of x and 18 is 120. Find the value of x.

56. Given that n is a ten-digit number in the form $\overline{2007x2008y}$ where x and y can be any of the digits 0, 1, 2 …, 9. How many such numbers n are there that are divisible by 33?

57. How many positive integers n, where $10 \le n \le 100$, are there such that $\frac{n^2 - 9}{n^2 - 7}$ is a fraction in its lowest terms?

58. Let n be a positive integer such that $n^2 + 19n + 48$ is a perfect square. Find the value of n.

59. Find the smallest integer n such that $n(\sqrt{101} - 10) > 1$.

60. Find the smallest positive integer N such that $2^n > n^2$ for every integer n in $\{N, N+1, N+2, N+3, N+4\}$.

61. The highest natural number that are divisible by 30 and have exactly 30 different positive divisors can be written as $A1B5C$, then find $A + B + C$.

62. Let p and q represent two consecutive prime numbers. For some fixed integer n, the set $\{n - 1, 3n - 19, 38 - 5n, 7n - 45\}$ represents $\{p, 2p, q, 2q\}$, but not necessarily in that order. Find the value of n.

63. If a, b and c are positive real numbers such that $ab + a + b = bc + b + c = ca + c + a = 35$,

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then the value of $(a + 1)\ (b + 1)\ (c + 1) = x^y$. Find the product of x and y.

64. Find the largest possible value of n such that there exist n consecutive positive integers whose sum is equal to 2010.

65. Let S be the set of all integers n such that $\frac{8n^3 - 96n^2 + 360n - 400}{2n - 7}$ is an integer. Find the value of $\sum_{n \in S} |n|$.

66. Find the sum of all positive integers p such that the expression $(x - p)\ (x - 13) + 4$ can be expressed in the form $(x + q)\ (x + r)$ for distinct integers q and r.

67. Let $P_k = 1 + \frac{1}{k} - \frac{1}{k^2} - \frac{1}{k^3}$, where k is a positive integer. Find the sum of all the digits of least positive integer n such that the product $P_2 P_3 \ldots P_n$ exceeds 2010.

68. What is the sum of the digits of the square of 111, 111, 111?

69. Let a, b, c and d be real numbers with $|a - b| = 2$, $|b - c| = 3$ and $|c - d| = 4$. What is the sum of all possible values of $|a - d|$?

70. Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

71. The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

72. The Saxena family consists of a mother, a father and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

73. How many ordered pairs (m, n) of positive integers, with $m > n$, have the property that their squares differ by 96?

74. For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007$?

75. How many pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and $\frac{a}{b} + \frac{14b}{9a}$ is an integer?

76. If N is the number of four-digit positive integers have at least one digit that is a 2 or a 3 then find the sum of the digits present in N?

77. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

78. How many positive integer cubes divide $3! \cdot 5! \cdot 7!$?

79. The sum of the digits of a two-digit number is subtracted from the number. The unit digit of the result is 6. How many two-digit numbers have this property?

80. For how many positive integers n does $1 + 2 + \ldots + n$ evenly divide from $6n$?

81. For each positive integer $m > 1$, let $P(m)$ denote the greatest prime factor of m. For how many positive integers n is it true that both $P(n) = \sqrt{n}$ and $P(n + 8) = \sqrt{n + 48}$?

82. Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$ and $7^d = 8$. What is $\left(\frac{a.b.c.d}{2}\right)$?

83. All of David's telephone numbers have the form 555 – *abc* – *defg*, where a, b, c, d, e, f, and g are distinct digits and in increasing order and none is either 0 or 1. How many different telephone numbers can David have?

84. For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1 + 2 + \ldots + n$?

85. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. Suppose that the integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer 'm'. What is the value of $\left(\frac{x + y + m}{11}\right)$?

86. A subset 'B' of the set of integers from 1 to 100 inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B?

87. Let n be a 5-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. Let the total number of value of n for which $q + r$ divisible by 11 is P. Then, find the last two digits of P.

88. Let $F(x)$ denote the sum of the digits of the positive integer x. For example, $F(8) = 8$ and $F(123) = 1 + 2 + 3 = 6$. For how many two-digit values of x is $F(F(x)) = 3$?

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89. Given that $3^8 \cdot 5^2 = a^b$, where both a and b are positive integers, find the smallest possible value for $\left(\frac{a+b+93}{25}\right)$.

90. What is the largest integer that is a divisor of
$$(n+1)(n+3)(n+5)(n+7)(n+9)$$
for all positive even integers n?

91. The product of three consecutive positive integers is 8 time their sum. What is the sum of their squares?

92. Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N?

93. The mean of three numbers is 10 more than the least of the numbers and 15 less than the greatest. The median of the three numbers is 5. What is their sum?

94. Two non-zero real numbers, a and b, satisfy $ab = a - b$. Then find the possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

95. There are N number of positive integers not exceeding 2001 and are multiples of 3 or 4 but not 5. Then find the value of $\left(\frac{N-1}{50}\right)$.

96. In the magic square shown, the sums of the numbers in each row, column, and diagonal are the same. Five of these numbers are represented by v, w, x, y, and z. Find $y + z$.

v	24	w
18	x	y
25	z	21

97. A subset of the integers 1, 2, …, 100 has the property that none of its members is 3 times another. What is the largest number of members such a subset can have?

98. A positive integer N is a palindrome if the integer obtained by reversing the sequence of digits of N is equal to N. The year 1991 is the only year in the current century with the following two properties:
(a) It is a palindrome.
(b) It factors as a product of a 2-digit prime palindrome and a 3-digit prime palindrome.
How many years in the millennium between 1000 and 2000 (including the year 1991) have properties (a) and (b)?

99. The two-digit integers from 19 to 92 are written consecutively to form the large integer
$$N = 19202122\ldots909192$$
If 3^k is the highest power of 3 that is a factor of N, then k is

100. For how many n in {1, 2, 3, …, 100} is the tens digit of n^2 odd?

101. If a and b are digits for which

$$\begin{array}{r} 2\ a \\ \times\ b\ 3 \\ \hline 6\ 9 \\ 9\ 2\ \ \\ \hline 9\ 8\ 9 \end{array}$$

then $a + b =$

102. For positive integers n, denote by $D(n)$ the number of pairs of different adjacent digits in the binary (base two) representation of n. For example, $D(3) = D(11_2) = 0$, $D(21) = D(10101_2) = 4$, and $D(97) = D(1100001_2) = 2$. For how many positive integers n less than or equal to 97 does $D(n) = 2$?

103. Call a 7-digit telephone number $d_1d_2d_3$-$d4d_5d_6d_7$ memorable if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). Assuming that each $d1$ can be any of the ten decimal digits 0, 1, 2, … 9. Suppose the total no. of different memorable telephone no.s is 'n'. Then find the sum of digits of 'n'.

104. The number of ordered pairs of integers (m, n) for which $mn \geq 0$ and $m^3 + n^3 + 99mn = 33^3$ is equal to

105. All even numbers from 2 to 98 inclusive, except those ending in 0, are multiplied together. What is the rightmost digit (the units digit) of the product?

106. Given that $xyz = 2014$, and x, y and z are positive integers such that $x < y < z$, how many possible triples (x, y, z) are there?

107. In the following equation, each of the letters represents uniquely a different digit in base ten:
$$(YE) \cdot (ME) = TTT$$
The sum $E + M + T + Y$ equals

108. For how many (not necessarily positive) integer values of n is the value of 4000. $\left(\frac{2}{5}\right)^n$ an integer?

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109. Let S be a set of 6 integers taken from $\{1, 2, \ldots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the least possible values of an element in S?

110. The total number of non-negative integers can be written in the given form are N. Then find sum of all the digits of $Na_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0$,where $a_1 \in \{-1, 0, 1\}$ for $0 \le i \le 7$?

111. Let a, b, c, and d be positive integers such that $\gcd(a, b) = 24$, $\gcd(b, c) = 36$, $\gcd(c, d) = 54$, and $70 < \gcd(d, a) < 100$. Which of the following must be a divisor of a?

112. For a positive integer n and non-zero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b, and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a + b + c$ for which there are at least two values of n such $C_n - B_n = a_n^2$?

113. How many ordered pairs (a, b) of positive integers satisfy the equation $a \cdot b + 63 = 20 \cdot \text{lcm}(a, b) + 12 \cdot \gcd(a, b)$, where $\gcd(a, b)$ denotes the greatest common divisor of a and b, and $\text{lcm}(a, b)$ denotes their least common multiple?

114. There are 10 horses, named Horse 1, Horse 2, …, Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T?

115. How many ordered triples (x, y, z) of positive integers satisfy $\text{lcm}(x, z) = 600$ and $\text{lcm}(y, z) = 900$?

116. For the greatest value of x, if , 2^x is a factor of $10^{1002} - 4^{501}$. Find the value of $\frac{x}{15}$.

117. In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in the digit 3?

118. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

119. The number 2013 has the property that its units digit is the sum of its other digits, that is $2 + 0 + 1 = 3$. How many integers less than 2013 but greater than 1000 share this property?

120. The number 2013 is expressed in the form $2013 = \frac{a_1!a_2 \ldots a_m!}{b_1!b_2 \ldots b_n!}$ where $a_1 \ge a_2 \ge \ldots \ge a_m$ and $b_1 \ge b_2 \ge \ldots b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

121. Two integers have a sum of 26. When two more integers are added to the first two integers the sum is 41. Finally, when two more integers are added to the sum of the previous four integers the sum is 57. What is the minimum number of even integers among the 6 integers?

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122. How many ordered pairs of positive integers (M, N) satisfy the equation $\frac{M}{6} = \frac{6}{N}$?

123. Geeta and Babeeta play the following game. An integer between 0 and 999, inclusive is selected and given to Geeta. Whenever Geeta receives a number, he doubles it and passes the result to Babeeta. Whenever Babeeta receives a number, she adds 50 to it and passes the result to Geeta. The winner is the last person who produces a number less than 1000. Let N be the smallest initial number that results in a win for Geeta. What is the sum of the digits of N?

124. A majority of the 30 students in Mrs. Bhatnagar's class bought pencils at the school bookstore. Each of these students bought the same number of pencils, and this number was greater than 1. The cost of a pencil in paisa was greater than the number of pencils each student bought, and the total cost of all the pencils was ₹17.71. What was the cost of a pencil in paisa?

125. Find last two digit of largest natural number n such that $n + 10$ divides $n^3 + 1000$.

126. Find the last two digits of the greatest integer less than $\left(2+\sqrt{3}\right)^4$.

127. Consider the function $f(x) = \dfrac{1}{3^x+\sqrt{3}}$. Find the value of
$\sqrt{3}\,[f(-5)+f(-4)+f(-3)+f(-2)+f(-1)+f(0)+f(1)+f(2)+f(3)+f(4)+f(5)+f(6)]$.

128. How many ordered pairs of positive integers (x, y) satisfy the equation
$x\sqrt{y}+y\sqrt{x}+\sqrt{2006xy}-\sqrt{2006x}-\sqrt{2006y}-2006=0$?

129. Evaluate X in its simplest form then find sum of all digits of X. Where X is given as
$X=\sqrt{2008+2007\sqrt{2008+2007\sqrt{2008+2007\sqrt{\ldots}}}}$.

130. Find the number of ordered pairs of positive integers (x, y) that satisfy the equation
$x\sqrt{y}+y\sqrt{x}+\sqrt{2009xy}-\sqrt{2009x}-\sqrt{2009y}-2009=0$.

131. Evaluate $\sqrt[3]{77-20\sqrt{13}}+\sqrt[3]{77+20\sqrt{13}}$.

132. Find the value of $\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}$.

133. Let $a=\dfrac{1+\sqrt{2009}}{2}$. Find the value of $(a^3-503a-500)^5$.

134. Let n be the positive integer such that
$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\frac{1}{13\sqrt{15}+15\sqrt{13}}+\ldots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}=\frac{1}{9}$$

135. If $x=\dfrac{1}{2}\left(\sqrt[3]{2009}-\dfrac{1}{\sqrt[3]{2009}}\right)$ and $y=\left(x+\sqrt{1+x^2}\right)^3$, then find the sum of the digits of y.

136. Let x and y be real numbers satisfying $y=\sqrt{\dfrac{2008x+2009}{2010x-2011}}+\sqrt{\dfrac{2008x+2009}{2011-2010x}}+25$. Find the value of y.

137. If the value of fraction $\dfrac{2(\sqrt{2})+\sqrt{6})}{3\sqrt{2+\sqrt{3}}}$ is equal to $12P$. Find P.

138. Let $T=\dfrac{1}{3-\sqrt{8}}-\dfrac{1}{\sqrt{8}-\sqrt{7}}+\dfrac{1}{\sqrt{7}-\sqrt{6}}-\dfrac{1}{\sqrt{6}-\sqrt{5}}+\dfrac{1}{\sqrt{5}-2}$; then value of T is

139. Suppose that real number x satisfies $\sqrt{49-x^2}-\sqrt{25-x^2}=3$.
What is the value of $\sqrt{49-x^2}+\sqrt{25-x^2}$?

140. If the value of $\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}$ is k, find $\dfrac{25}{\sqrt{6}}\times k$.

141. Find the remainder of $47^{37^{27}}$ when it is divided by 11.

142. What is the remainder when 9^{1990} is divided by 11?

143. Find the last two digits of 2^{999}.

144. Find the remainder of $(257^{33}+46)^{26}$ when it is divided by 50.

145. Let p be a prime number such that the next larger number is a perfect square. Find the sum of all such prime numbers. (For example, if you think that 11 and 13 are two such prime numbers, then the sum is 24.)

146. The value of greatest integer N such that $N \le \sqrt{2007^2-20070+31}$. Find $\dfrac{N-2}{100}$

147. Find the sum of all the digit of largest integer N such that both $N+496$ and $N+224$ are perfect squares.

148. The sum of 18 consecutive positive integers is a perfect square. In the smallest possible value of this sum, find the sum of first two integers?

149. Let n be the smallest positive integer such that n is divisible by 20, n^2 is a perfect cube, and n^3 is a perfect square. The number n contains N digits. What is N^2?

150. How many positive integers n are there such that $7n+1$ is a perfect square and $3n+1<2008$?

151. Let $M = 1!\times 2!\times 3!\times 4!\times 5!\times 6!\times 7!\times 8!\times 9!$. The number of factors of M which are perfect squares is $(21b)$. Then find the value of b.

152. Find the number of ordered pairs (x, y), where x is an integer and y is a perfect square, such that $y=(x-90)^2-4907$.

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153. The number $25^{64}.\ 64^{25}$ is the square of a positive integer N. In decimal representation, the sum of the digits of N is.

154. For how many integers n is $\frac{n}{20-n}$ the square of an integer?

155. If 554 is the base b representation of the square of the number whose base b representation is 24, then b, when written in base 10, equals

156. If x is a real number that satisfies $\left[x+\frac{11}{100}\right]+\left[x+\frac{12}{100}\right]+\ldots+\left[x+\frac{99}{100}\right]=765$, find the value of $[900]-[100x]$. Here $[a]$ denotes the largest integer $\le a$.

157. Find the number of consecutive 0's at the end of the base 10 representation of 206!.

158. There are P number of positive integer n less than 2007 such that $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{6}\right]=n$ where $[x]$ is the greatest integer less than or equal to x. Write the last two digit of P.
(For example, $[2.5]=2$; $[5]=5$; $[-2.5]=-3$, etc.)

159. Let $[x]$ denote the greatest integer less than or equal to x. Some P real values of x satisfy the equation $x^2+10{,}000\,[x]=10{,}000x$. What is $P-100$?

160. For any real number x, let $\lceil x\rceil$ denote the smallest integer that is greater than or equal to x and $\lfloor x\rfloor$ denote the largest integer that is less than or equal to x (for example, $\lceil 1.23\rceil 2$ and $\lfloor 1.23\rfloor=1$). Find the value of $2010-\sum_{k=1}^{2010}\left\lceil\frac{2010}{k}-\left\lfloor\frac{2010}{k}\right\rfloor\right\rceil$.

161. Given that $x=[1^{1/3}]+[2^{1/3}]+[3^{1/3}]+\ldots+[7999^{1/3}]$, find the value of $\left[\frac{x}{5000}\right]$, where $[y]$ denotes the greatest integer less than or equal to y. (For example, $[2.1]=2$, $[30]=30$, $[-10.5]=-11$.)

162. What is the greatest integer less than or equal to $\frac{3^{100}+2^{100}}{3^{96}+2^{96}}$?

163. Let $[x]$ denote the greatest integer less than or equal to x. If there are N values of x satisfy the equation $x^2+10{,}000[x]=10{,}000x$. Then find the value of $\frac{N+1}{4}$.

164. Let $f(x)=\sum_{k=2}^{10}([kx]-k[x])$, where $[r]$ denotes the greatest integer less than or equal to r. How many distinct values does $f(x)$ assume for $x\ge 0$?

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. P, Q, and R are three natural numbers such that P and Q are primes and Q divides PR. Then out of the following the correct statement is: **[NSEJS-2008]**
(A) Q divides R **(B)** P divided R
(C) P divides QR **(D)** P divides PQ

2. If $D=a^2+b^2+c^2$ where a and b consecutive integers and $c=ab$, then $\sqrt{D}$ is: **[NSEJS-2009]**
(A) always an even integer
(B) always an odd integer
(C) an integer, odd or even
(D) sometimes an irrational number

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3. In the familiar decimal number system the base is 10. In another number system using base 4, the counting proceeds as 1, 2, 3, 10, 11, 12, 13, 20, 21 … The twentieth number in this system will be **[NSEJS-2010]**
(A) 40 **(B)** 320
(C) 210 **(D)** 110

4. If the eight digit number 2575d568 is divisible by 54 and 87, the value of the digit 'd' is **[NSEJS-2011]**
(A) 4 **(B)** 7
(C) 0 **(D)** 8

5. Given $\sqrt{600}=24.49$, the value of $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ is **[NSEJS-2011]**
(A) 9.978 **(B)** 8.989
(C) 9.989 **(D)** 9.898

6. If $x<0$ and $\log(x^2-5x-65)=0$, then x is **[NSEJS-2011]**
(A) −13 **(B)** −11
(C) −6 **(D)** −5

7. In a certain number system 363 + 1056 = 1452. Find the value of (654 – 456) in the same number system **[NSEJS-2012]**

(A) 156 (B) 165
(C) 178 (D) 198

8. What will be the remainder if the number 7^{2012} is divided by 25? **[NSEJS-2012]**
(A) 24 (B) 18
(C) 7 (D) 1

9. The product of three consecutive natural numbers is 124850054994. What is their average? **[NSEJS-2012]**
(A) 4993 (B) 4994
(C) 4997 (D) 4998

10. The number $5\sqrt{41}$ lies between **[NSEJS-2013]**
(A) 29 and 30 (B) 30 and 31
(C) 31 and 32 (D) 32 and 33

11. Let T be the number of 4-digit integers, each ending in 3 (in units place) and each divisible by 11. Then **[NSEJS-2013]**
(A) $21 \le T \le 79$ (B) $80 \le T \le 89$
(C) $90 \le T \le 99$ (D) $T \ge 100$

12. The sum of 2 digits x and y is divisible by 7. What can one say about a 3 digit number formed by these two digits. **[NSEJS-2014]**
(A) xxy is divisible by 7
(B) xyx is divisible by 7
(C) xyx is divisible by 7^2
(D) yyx is divisible by 7

13. A number x is a rational number is there exists integers p and q such that $x = p/q$. This is definition of rational numbers in which, **[NSEJS-2014]**
(A) both p and q can be zero
(B) both p and q should not be zero
(C) q can be zero but not p
(D) p can be zero but not q

14. The least positive integer, n, such that 2 divides n, 3 divides $n + 1$, 4 divides $n + 2$, 5 divides $n + 3$ and 6 divides $n + 4$ is **[NSEJS-2014]**
(A) 62 (B) 120
(C) 720 (D) 52

15. Number plate of a vehicle consists of 4 digits. The first digit is the square of second. The third digit is thrice the second and the fourth digit is twice the second. The sum of all 4 digits is thrice the first. The number is **[NSEJS-2014]**
(A) 1132 (B) 4264
(C) 1642 (D) 9396

16. If the highest common factor of a, b and c is 1, where a, b and c belong to the set of natural numbers, then the highest common factor of $(a \times b)$ and c is: **[NSEJS-2014]**
(A) c (B) $a \times b$
(C) 1 (D) insufficient data

17. How many three digit numbers are divisible by 13 and having middle digit 5? **[NSEJS-2015]**
(A) 5 (B) 7
(C) 10 (D) 13

18. Select any three distinct digits. Form a three digit number. Form the another number by reversing the digits. Find the difference of these two numbers. What is the G.C.D of all such differences? **[NSEJS-2015]**
(A) 9 (B) 11
(C) 33 (D) 99

19. 5901AB04 is an eight digit number divisible by 792. Find $A + B = ?$ **[NSEJS-2015]**
(A) 6 (B) 7
(C) 8 (D) 9

20. What is the smallest natural number when multiplied by 15 and divided by 63 gives remainder 21? **[NSEJS-2015]**
(A) 13 (B) 14
(C) 17 (D) 20

21. If $\sqrt{338} - \sqrt{228} = m$ then $m = ?$ **[NSEJS-2015]**
(A) $\sqrt{50}$ (B) $\sqrt{32}$
(C) $\sqrt{18}$ (D) $\sqrt{2}$

22. How many four digit numbers divisible by 29 have the sum of their digits 29? **[NSEJS-2015]**
(A) 4 (B) 5
(C) 13 (D) none of these

23. 8888888 * 8888888 this fifteen digit number is divisible by 22. Find the eighth digit in the number. **[NSEJS-2015]**
(A) 1 (B) 3
(C) 5 (D) 8

24. What will be the remainder if the number $(1000000)^3$ is divided by 143? **[NSEJS-2015]**
(A) 9 (B) 6
(C) 1 (D) 0

25. What will be the remainder if the number 7^{2015} is divided by 25? **[NSEJS-2015]**
(A) 1 (B) 7
(C) 18 (D) 24

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26. The number $3^8(3^{10} + 6^5) + 2^3(2^{12} + 6^7)$ is: **[NSEJS-2016]**

(A) A perfect square and a perfect cube
(B) Neither a perfect square nor a perfect cube
(C) a perfect cube but not a perfect square
(D) A perfect square but not a perfect cube

27. Tenth term in the sequence 12, 18, 20, 28, … is: **[NSEJS-2016]**

(A) 336 **(B)** 63
(C) 216 **(D)** 68

28. Number of numbers less than 40 having exactly four divisors is: **[NSEJS-2016]**

(A) 15 **(B)** 12
(C) 11 **(D)** 14

29. Number of integers n such that the number $1 + n$ is a divisor of the number $1 + n^2$ is: **[NSEJS-2016]**

(A) 0 **(B)** 2
(C) 4 **(D)** 2

30. The number of integers a, b, c for which $a^2 + b^2 - 8c = 3$ is: **[NSEJS-2016]**

(A) 2 **(B)** Infinite
(C) 0 **(D)** 4

31. How many four digit numbers are there such that when they are divided by 101, they have 99 as remainder? **[NSEJS-2017]**

(A) 90 **(B)** 98
(C) 100 **(D)** 101

32. If $x = (\sqrt{21} - \sqrt{20})$ and $y = (\sqrt{18} - \sqrt{17})$, then: **[NSEJS-2017]**

(A) $x = y$ **(B)** $x < y$
(C) $x > y$ **(D)** $x + y = 0$

33. By which smallest number we should divide 198396198 to get a perfect square? **[NSEJS-2017]**

(A) 14 **(B)** 18
(C) 22 **(D)** 28

34. What will be the remainder if the number $(7)^{2017}$ is divided by 25? **[NSEJS-2017]**

(A) 1 **(B)** 7
(C) 18 **(D)** 24

35. What is the largest value of the positive integer k such that k divides $n^2(n^2 - 1)(n^2 - n - 2)$ for every natural number n? **[NSEJS-2018]**

(A) 6 **(B)** 12
(C) 24 **(D)** 48

36. Let n be a positive integer not divisible by 6. Suppose n has 6 positive divisors. The number of positive divisors of $9n$ is **[NSEJS-2018]**

(A) 54 **(B)** 36
(C) 18 **(D)** 12

37. The number of distinct integers in the collection $\left[\frac{10^2}{1}\right], \left[\frac{10^2}{2}\right], \left[\frac{10^2}{3}\right], \ldots \left[\frac{10^2}{20}\right]$, where $[x]$ denotes the largest integer not exceeding x, is: **[NSEJS-2018]**

(A) 20 **(B)** 18
(C) 17 **(D)** 15

38. Digits a and b are such that the product b is divisible by 36 (in base 10). The number of ordered pairs (a, b) is: **[NSEJS-2018]**

(A) 11 **(B)** 8
(C) 6 **(D)** 4

39. How many positive integers N give a remainder 8 when 2008 is divided by N? **[NSEJS - 2019]**

(A) 12 **(B)** 13
(C) 14 **(D)** 15

40. LCM of two numbers is 5775. Which of the following cannot be their HCF? **[NSEJS - 2019]**

(A) 175 **(B)** 231
(C) 385 **(D)** 455

41. Mr. X with eight children of different ages is on family trip. His oldest child, who is 9 years old saw a license plate with a 4-digit number in which each of two digits appear two times. "Look daddy!" she exclaims. 'That number is evenly divisible by the age of each of us kids!' 'That's right,' replies Mr. X, 'and the last two digits just happen to be my age'. Which of the following is not the age one of Mr. X's children? **[NSEJS - 2019]**

(A) 4 **(B)** 5
(C) 6 **(D)** 7

42. How many number lie between 11 and 1111 which divided by 9 leave a remainder 6 and when divided by 21 leave a remainder 12 ? **[NSEJS - 2019]**

(A) 18 **(B)** 28
(C) 8 **(D)** None of these

PART – II Previous Years' KVPY Problems

1. If the decimal $0.d25d25d25 \ldots$ is expressible in the form $n/27$, then $d + n$ must be: **[KVPY-2009]**

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(A) 9 **(B)** 28
(C) 30 **(D)** 34

2. Let S_n be the sum of all integers k such that $2^n < k < 2^{n+1}$, for $n \geq 1$. Then, 9 divides S_n if and only if: **[KVPY-2009]**

(A) n is odd **(B)** n is of the form $3k + 1$
(C) n is even **(D)** n is of the form $3k + 2$

3. Let $\log_a b = 4$, $\log_c d = 2$, where a, b, c, d are natural numbers. Given that $b - d = 7$, the value of $c - a$ is: **[KVPY-2009]**

(A) 1 **(B)** –1
(C) 2 **(D)** –2

4. Leela and Madan pooled their music CD's and sold them. They got as many rupees for each CD as the total number of CD's they sold. They share the money as follows: Leela first takes 10 rupees, then Madan takes 10 rupees and they continue taking 10 rupees alternately till Madan is left out with less than 10 rupees to take. Find the amount that is left out for Madan at the end, with justification. **[KVPY-2010]**

5. **(A)** Show that for every natural number n relatively prime to 10, there is another natural number mall of whose digits are 1's such that n divides m.

(B) Hence or otherwise show that every positive rational number can be expressed in the form $\frac{a}{10^b(10^c - 1)}$ for some natural numbers a, b, c. **[KVPY-2010]**

6. Let a, b, c be positive integers such that $\frac{a\sqrt{2} + b}{b\sqrt{2} + c}$ is a rational number, then which of the following is always an integer? **[KVPY-2012]**

(A) $\frac{2a^2 + b^2}{2b^2 + c^2}$ **(B)** $\frac{a^2 + b^2 - c^2}{a + b - c}$
(C) $\frac{a^2 + 2b^2}{b^2 + 2c^2}$ **(D)** $\frac{a^2 + b^2 + c^2}{a + b - c}$

7. How many natural numbers n are there such that $n! + 10$ is a perfect square? **[KVPY-2013]**

(A) 1 **(B)** 2
(C) 4 **(D)** infinitely many

8. The sum of all positive integers n for which $\frac{1^3 + 2^3 + \ldots + (2n)^3}{1^2 + 2^2 + \ldots + n^2}$ is also an integer is: **[KVPY-2014]**

(A) 8 **(B)** 9
(C) 15 **(D)** Infinite

9. The number of 6-digit numbers of the form ababab (in base 10) each of which is a product of exactly 6 distinct primes is: **[KVPY-2014]**

(A) 8 **(B)** 10
(C) 13 **(D)** 15

10. The houses on one side of a road are numbered using consecutive even numbers. The sum of the numbers of all the houses in that row is 170. If there are at least 6 houses in that row and a is the number of the sixth house, then: **[KVPY-2014]**

(A) $2 \leq a \leq 6$ **(B)** $8 \leq a \leq 12$
(C) $14 \leq a \leq 20$ **(D)** $22 \leq a \leq 30$

11. Suppose a_2, a_3, a_4, a_5, a_6, a_7 are integers such that $\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!}$, where $0 \leq a_j < j$ for $j = 2, 3, 4, 5, 6, 7$. The sum of $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$ is: **[KVPY-2014]**

(A) 8 **(B)** 9
(C) 10 **(D)** 11

12. The number of all 3-digit numbers abc (in base 10) for which $(a \times b \times c) + (a \times b) + (b \times c) + (c \times a) + a + b + c = 29$ is: **[KVPY-2014]**

(A) 6 **(B)** 10
(C) 14 **(D)** 18

13. The largest non-negative integer k such that 24^k divides 13! is: **[KVPY-2015]**

(A) 2 **(B)** 3
(C) 4 **(D)** 5

14. The least positive integer n for which **[KVPY-2015]**

$$\sqrt[3]{n+1} - \sqrt[3]{n} < \frac{1}{12}$$

(A) 6 **(B)** 7
(C) 8 **(D)** 9

15. Let $n > 1$ be an integer. Which of the following sets of numbers necessarily contains a multiple of 3? **[KVPY-2015]**

(A) $n^{19} - 1$, $n^{19} + 1$ **(B)** n^{19}, $n^{38} - 1$
(C) n^{88}, $n^{38} + 1$ **(D)** n^{38}, $n^{19} - 1$

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16. The number of distinct primes dividing 12! + 13! + 14! Is **[KVPY-2015]**

(A) 5 **(B)** 6
(C) 7 **(D)** 8

17. For a real number x, let $[x]$ denote the largest integer less than or equal to x, and let $\{x\} = x - [x]$. The number of solutions x to the equation $[x]\{x\} = 5$ with $0 \le x \le 2015$ is **[KVPY-2015]**

(A) 0 **(B)** 3
(C) 2008 **(D)** 2009

18. The number of digits in the decimal expansion of $16^5 5^{16}$is **[KVPY-2016]**

(A) 16 **(B)** 17
(C) 18 **(D)** 19

19. Let $a_1, a_2, \ldots, a_{100}$ be non-zero real numbers such that $a_1 + a_2 + \ldots + a_{100} = 0$ Then, **[KVPY-2016]**

(A) $\sum_{i=1}^{100} a_i 2^{a_i} > 0$ and $\sum_{i=1}^{100} a_i 2^{-a_i} < 0$

(B) $\sum_{i=1}^{100} a_i 2^{a_i} > 0$ and $\sum_{i=1}^{100} a_i 2^{-a_i} < 0$

(C) $\sum_{i=1}^{100} a_i 2^{a_i} > 0$ and < 0

(D) The sign of $\sum_{i=1}^{100} a_i 2^{a_i}$ or $\sum_{i=1}^{100} a_i 2^{-a_i}$ depends on the choice of a_i's

20. The number of all 2-digit numbers n, such that n is equal to the sum of the square of digit in its tens place and the cube of the digit in units place is **[KVPY-2016]**

(A) 0 **(B)** 1
(C) 2 **(D)** 4

21. The largest power of 2 that divides $\frac{200!}{100!}$ is **[KVPY-2017]**

(A) 98 **(B)** 99
(C) 100 **(D)** 101

22. Let $a_1, a_2, \ldots, a_n$ be n non-zero real numbers, of which p are positive and remaining are negative. The number of ordered pairs (j, k), $j < k$, for which $a_j a_k$ is positive, is 55. Similarly, the number of ordered pair (j, k), $j < k$, for which $a_j a_k$ is negative, is 50. Then, the value of $p^2 + (n-p)^2$ is **[KVPY-2017]**

(A) 629 **(B)** 325
(C) 125 **(D)** 221

23. If a, b, c, d are four distinct numbers chosen from the set $\{1, 2, 3, \ldots, 9\}$, then the minimum value of $\frac{a}{b} + \frac{c}{d}$ is **[KVPY-2017]**

(A) $\frac{3}{8}$ **(B)** $\frac{1}{3}$
(C) $\frac{13}{36}$ **(D)** $\frac{25}{72}$

24. It is given that the number 43361 can be written as a product of two distinct prime number p_1, p_2. Further, assume that there are 42900 numbers which are less than 43361 and are coprime to it. Then, $p_1 + p_2$ is **[KVPY-2017]**

(A) 462 **(B)** 464
(C) 400 **(D)** 402

25. The number of positive integers n in the set $\{2, 3, \ldots, 200\}$ such that $\frac{1}{n}$ has a terminating decimal expansion is **[KVPY-2017]**

(A) 16 **(B)** 18
(C) 40 **(D)** 100

26. Consider the following statements: **[KVPY-2017]**
For any integer n,
I. $n^2 + 3$ is never divisible by 17.
II. $n^2 + 4$ is never divisible by 17.
Then,

(A) both I and II are true
(B) both I and II are false
(C) I is false and II is true
(D) I is true and II is false

27. Let S be the set of all ordered pairs (x, y) of positive integers, with HCF$(x, y) = 16$ and LCM$(x, y) = 48000$. The number of elements in S is **[KVPY-2017]**

(A) 4 **(B)** 8
(C) 16 **(D)** 32

28. Consider a three-digit number with the following properties: **[KVPY-2017]**
I. If its digits in units place and tens place are interchanged, the number increases by 36;
II. If its digits in units place and hundreds place are interchanged, the number decreases by 198.
Now, suppose that the digits in tens place and hundreds place are interchanged. Then, the number

(A) increases by 180 **(B)** decreases by 270
(C) increases by 360 **(D)** decreases by 540

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29. Let S be the sum of the digits of the number $15^2 \times 5^{18}$ in base 10. Then, **[KVPY-2018]**

(A) $S < 6$ **(B)** $6 \le S < 140$

(C) $140 \le S < 148$ **(D)** $S \ge 148$

30. What is the sum of all natural numbers n such that the product of the digits of n (in base 10) is equal to $n^2 - 10n - 36$? **[KVPY-2018]**

(A) 12 **(B)** 13

(C) 124 **(D)** 2612

31. Let N be the least positive integer such that whenever a son-zero digit c is written after the last digit of N, the resulting number is divisible by c. The sum of the digits of N is: **[KVPY – 2019]**

(A) 9 **(B)** 18

(C) 27 **(D)** 36

32. A two-digit number $\overline{ab}$ is called almost prime if one obtains a two-digit prime number by changing at most one of its digits a and b. (For example, 18 is an almost prime number because 13 is a prime number.) Then the number of almost prime two-digit numbers is: **[KVPY – 2019]**

(A) 56 **(B)** 75

(C) 87 **(D)** 90

33. Let $N_1 = 2^{55} + 1$ and $N_2 = 165$. Then **[KVPY – 2019]**

(A) N_1 and N_2 are coprime

(B) the HCF of N_1 andN_2 is 55

(C) the HCF of N_1 and N_2 is 11

(D) the HCF of N_1 and N_2 is 33

34. A prime number p is called special if there exist prime $p_1, p_2 p_3, p_4$ such that $p = p_1 + p_2 = p_3 - p_4$. The number of special primes is: **[KVPY – 2019]**

(A) 0

(B) 1

(C) More than one but finite

(D) Infinite

PART – III Previous Years' PRMO Problems

1. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly ? **[PRE-PRMO-2012]**

2. Let $S_n = n^2 + 20n + 12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square? **[PRE-PRMO-2012]**

3. Let $P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$. What is the largest integer that is a divisor of $P(n)$ for all positive even integers n? **[PRE-PRMO-2012]**

4. If $\dfrac{1}{\sqrt{2011+\sqrt{2011^2-1}}} = \sqrt{m} - \sqrt{n}$, where m and n are positive integers, what is the value of $m + n$? **[PRE-PRMO-2012]**

5. How many non-negative integral values of x satisfy the equation $\left[\dfrac{x}{5}\right] = \left[\dfrac{x}{7}\right]$? **[PRE-PRMO-2012]**

6. What is the sum of the squares of the roots of the equations $x^2 - 7[x] + 5 = 0$? **[PRE-PRMO-2012]**

7. What is the smallest positive integer k such that $k(3^3 + 4^3 + 5^3) = a^n$ for some positive integers a and n, with $n > 1$? **[PRE-PRMO-2013]**

8. Let $S(M)$ denote the sum of the digits of a positive integer M written in base 10. Let N be the smallest positive integer such that $S(N) = 2013$. What is the value of $S(5N + 2013)$? **[PRE-PRMO-2013]**

9. Let m be the smallest odd positive integer for which $1 + 2 + \ldots + m$ is a square of an integer and let n be the smallest even positive integer for which $1 + 2 + \ldots + n$ is a square of an integer. What is the value of $m + n$? **[PRE-PRMO-2013]**

10. What is the maximum possible value of k for which 2013 can be written as a sum of k consecutive positive integers? **[PRE-PRMO-2013]**

11. What is the sum (in base 10) of all the natural numbers less than 64 which have exactly three ones in their base 2 representation? **[PRE-PRMO-2013]**

12. A natural number k is such that $k^2 < 2014 < (k+1)^2$, What is the largest prime factor of k? **[PRE-PRMO-2014]**

13. For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \le y$ satisfy the equation $x + y + (x, y)$? **[PRE-PRMO-2014]**

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14. For how many natural numbers n between 1 and 2014 (both inclusive) is $\frac{8n}{9999-n}$ an integer? **[PRE-PRMO-2014]**

15. Let $E(n)$ denote the sum of the even digits of n. For example, $E(1243) = 2 + 4 = 6$. What is the value of $E(1) + E(2) + E(3) + \ldots + E(100)$? **[PRE-PRMO- 2015]**

16. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square ? **[PRE-PRMO- 2015]**

17. Let n be the largest integer that is the product of exactly 3 distinct prime numbers, x, y, and $10x + y$, where x and y are digits. What is the sum of the digits of n? **[PRE-PRMO- 2015]**

18. The digits of a positive integer n are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when n is divided by 37? **[PRE-PRMO- 2015]**

19. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3? **[PRE-RMO-2017]**

20. Find the number of positive integers, n such that $\sqrt{n} + \sqrt{n+1} < 11$. **[PRE-RMO-2017]**

21. Suppose x is a positive real number such that $\{x\}$, $[x]$ and x are in a geometric progression. Find the least positive integer n such that $X^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = -[x]$) **[PRE-RMO-2017]**

22. Suppose an integer 'x', a natural number 'n' and a prime number 'p' satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p? **[PRE-RMO-2017]**

23. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of $3pq$ for all positive integers n. Find the least possible value of $p + q$. **[PRE-RMO-2017]**

24. For each positive integer n, consider the highest common factor h_n of the two numbers $n! + 1$ and $(n + 1)!$. For $n < 100$, find the largest value of h_n. **[PRE-RMO-2017]**

25. Consider all 6-digit numbers of the form $\boldsymbol{a\ b\ c\ c\ b\ a}$ where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7. **[PRE-RMO-2018]**

26. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation. **[PRE-RMO-2018]**

27. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$. **[PRE-RMO-2018]**

28. Let T be the smallest positive integers which, when divided by 11, 13, 15 leaves remainders in the sets $\{7, 8, 9\}$, $\{1, 2, 3\}$, $\{4, 5, 6\}$ respectively. What is the sum of the square of the digits of T? **[PRE-RMO-2018]**

29. Let x_1 be a positive real number and for every integer $n \geq 1$ let $x_{n+1} = 1 + x_1 x_2 \ldots x_{n-1} x_n$. If $x_5 = 43$, what is the sum of digits of the largest prime factor of x_6? **[PRE-RMO-2019]**

30. Let $\overline{abc}$ be a three digit number with nonzero digits such that $a^2 + b^2 = c^2$. What is the largest possible prime factor of $\overline{abc}$? **[PRE-RMO-2019]**

31. Let the rational number p/q be closest to but not equal to 22/7 among all rational numbers with denominator < 100. What is the value of $p - 3q$? **[PRE-RMO-2019]**

32. A natural number $k > 1$ is called good if there exist natural numbers $a_1 < a_2 < \ldots < a_k$ such that $\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_k}} = 1$ Let $f(n)$ be the sum of the first n good numbers, $n \geq 1$. Find the sum of all values of n for which $f(n+5)/f(n)$ is an integer. **[PRE-RMO-2019]**

33. Find the smallest positive integer $n \geq 10$ such that $n + 6$ is a prime and $9n + 7$ is a perfect square. **[PRE-RMO-2019]**

34. How many ordered pairs (a, b) of positive integers with $a < b$ and $100 \leq a, b \leq 1000$ satisfy $\gcd(a, b) : \text{lcm}(a, b) = 1 : 495$? **[PRE-RMO-2019]**

35. Consider the set E of all natural numbers n such that when divided by 11, 12, 13, respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N. **[PRE-RMO-2019]**

36. Positive integers x, y, z satisfy $xy + z = 160$. Compute the smallest possible value of $x + yz$. **[PRE-RMO-2019]**

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37. Let E denote the set of all natural numbers n such that $3 < n < 100$ and the set $\{1, 2, 3, \ldots, n\}$ can be partitioned in to 3 subsets with equal sums. Find the number of elements of E. **[PRE-RMO-2019]**

38. Consider the sequence of number $\left[n + \sqrt{2n} + \frac{1}{2}\right]$ for $n \geq 1$ where $[x]$ denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < \ldots$ then find n_{12}. **[PRE-RMO-2019]**

39. Find the number of positive integers less than 101 that cannot be written as the difference of two squares of integers. **[PRE-RMO-2019]**

40. Let N be the smallest positive integer such that $N + 2N + 3N + \ldots + 9N$ is a number all whose digits are equal. What is the sum of the digits of N? **[PRE-RMO-2019]**

41. Let $s(n)$ denote the sum of the digits of a positive integer n in base 10. If $s(m) = 20$ and $s(33m) = 120$, what is the value of $s(3m)$? **[PRE-RMO-2019]**

42. Let $F_k(a,b) = (a+b)^k - a^k - b^k$ and let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. For how many ordered pairs $(a, b$ with $a, b \in S$ and $a \leq b$ is $\frac{F_5(a,b)}{F_3(a,b)}$ an integer? **[PRE-RMO-2019]**

43. Consider the sequence

$$1, 7, 8, 49, 50, 56, 57, 343, \ldots$$

which consists of sums of distinct powers of 7, that is $7^0, 7^1, 7^0, 7^1, 7^2, \ldots$, in increasing order. At what position will 16856 occur in this sequence? **[PRE-RMO-2019]**

44. In base -2 notation, digits are 0 and 1 only and the places go up in powers of -2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base -2 how many non-zero digits does it contain? **[PRE-RMO-2019]**

45. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N? **[PRE-RMO-2019]**

46. Let a, b, c be distinct positive integers such that $b + c - a$, $c + a - b$ and $a + b - c$ are all perfect squares. What is the largest possible value of $a + b + c$ smaller than 100? **[PRE-RMO-2019]**

47. What is the smallest prime number p such that $p^2 + 4p^2 + 4p$ has exactly 30 positive divisors? **[PRE-RMO-2019]**

48. Consider the set E of all positive integers n such that when divided by 9, 10, 11, respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E. **[PRE-RMO-2019]**

49. For $n \geq 1$, let an be the number beginning with n 9's followed by 744; e.g., $a_4 = 9999744$. Define $f(n) = \max\{m \in N \mid 2^m \text{ divides } a_n\}$, for $n \geq 1$. Find $f(1) + f(2) + f(3) + \ldots + f(10)$. **[PRE-RMO-2019]**

50. For any real number x, let $[x]$ denotes the integer part of x; $\{x\}$ be the fractional part of $x(\{x\} = x \mid x \mid)$. Let A denote the set of all real numbers x satisfying $\{x\} = \frac{x + [x] + [x + (1/2)]}{20}$ If S is the sum of all numbers in A, find $[S]$. **[PRE-RMO-2019]**

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ANSWER KEYS

Know Your Progress - 1

1. $\overline{8}.6686$ **2.** 155 **3.** 32 **4.** 1 **5.** 2 **6.** 3 **7.** 9 **8.** 9 **9.** 24
10. $x = 7^{-4/3}, 7^{-1/2}$ **11.** $x = 10^{10}$ **12.** $x = 343.$ **13.** 15. **15.** 23040

Know Your Progress - 2

1. 16 and 24 **2.** 61 numbers **3.** 645 **4.** 363 **5.** 6 or 8 **6.** $\frac{(a-b)b}{h}$ **7.** 1196
8. 8 **9.** 15 pairs of x and y **10.** (a) 77 min (b) 6, 10

Know Your Progress - 3

1. 111 **2.** 36 **3.** 320 **4.** 12 **5.** 12453 **6.** 6 **7.** 41 **8.** 60 **9.** 222 **10.** 241
11. (a) and (b) 153846 **12.** all number in the form $6k + 1$ or $6k + 2$, $k = 0, 1, 2, \ldots$ **13.** 3 **14.** 0 **15.** 2
16. 8 **17.** 0

Know Your Progress - 4

1. (i) $x = 6, -4$ (ii) $x = 2, 0$ (iii) $x = 3, \frac{-11}{3}$ (iv) $x = 5, -9$
2. (a) 1 (b) $-4, -\sqrt{3}-1$ (c) $-4, -2, -\sqrt{3}-1$ **3.** D **4.** D **5.** A, D **6.** 20

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Know Your Progress - 5

1. A **2.** B **3.** D **4.** B **5.** A

Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. A **2.** B **3.** D **4.** D **5.** C **6.** B **7.** A **8.** A **9.** B **10.** C
11. A **12.** C **13.** A **14.** C **15.** A **16.** C **17.** A **18.** B **19.** D **20.** C
21. C **22.** D **23.** C **24.** D **25.** D **26.** A **27.** A **28.** A **29.** D **30.** D
31. A **32.** C **33.** C **34.** D **35.** B **36.** A **37.** C **38.** A **39.** C **40.** B
41. C **42.** B **43.** C **44.** A **45.** B **46.** A **47.** D **48.** C **49.** C **50.** A

PART – II KVPY Level Problems

1. D **2.** A **3.** A **4.** A **5.** C **6.** C **7.** A **8.** C **9.** A **10.** B
11. D **12.** B **13.** C **14.** C **15.** D **16.** C **17.** B **18.** A **19.** D **20.** C
21. A **22.** D **23.** B **24.** C **25.** B **26.** A **27.** C

PART – III PRMO Level Problems

1. 0	**2.** 13	**3.** 43	**4.** 20	**5.** 16	**6.** 0	**7.** 19	**8.** 37	**9.** 81	**10.** 36
11. 73	**12.** 25	**13.** 8	**14.** 4	**15.** 21	**16.** 30	**17.** 84	**18.** 2	**19.** 4	**20.** 34
21. 9	**22.** 63	**23.** 8	**24.** 20	**25.** 60	**26.** 5	**27.** 40	**28.** 11	**29.** 75	**30.** 36
31. 4	**32.** 86	**33.** 12	**34.** 17	**35.** 35	**36.** 4	**37.** 7	**38.** 60	**39.** 1	**40.** 81
41. 16	**42.** 81	**43.** 96	**44.** 49	**45.** 10	**46.** 17	**47.** 18	**48.** 19	**49.** 49	**50.** 13
51. 16	**52.** 28	**53.** 34	**54.** 16	**55.** 42	**56.** 3	**57.** 46	**58.** 33	**59.** 21	**60.** 5
61. 3	**62.** 7	**63.** 18	**64.** 60	**65.** 50	**66.** 26	**67.** 19	**68.** 81	**69.** 18	**70.** 6
71. 18	**72.** 6	**73.** 4	**74.** 4	**75.** 4	**76.** 16	**77.** 45	**78.** 6	**79.** 10	**80.** 5
81. 1	**82.** 3	**83.** 8	**84.** 16	**85.** 14	**86.** 62	**87.** 81	**88.** 10	**89.** 20	**90.** 15
91. 77	**92.** 9	**93.** 30	**94.** 30	**95.** 2	**96.** 15	**97.** 46	**98.** 76	**99.** 4	**100.** 1
101. 7	**102.** 26	**103.** 28	**104.** 35	**105.** 6	**106.** 5	**107.** 21	**108.** 9	**109.** 4	**110.** 14
111. 13	**112.** 18	**113.** 2	**114.** 3	**115.** 15	**116.** 67	**117.** 13	**118.** 2	**119.** 46	**120.** 2
121. 1	**122.** 9	**123.** 7	**124.** 11	**125.** 90	**126.** 93	**127.** 6	**128.** 8	**129.** 10	**130.** 6
131. 7	**132.** 5	**133.** 32	**134.** 79	**135.** 11	**136.** 25	**137.** 16	**138.** 5	**139.** 8	**140.** 50
141. 9	**142.** 1	**143.** 88	**144.** 29	**145.** 3	**146.** 20	**147.** 17	**148.** 9	**149.** 49	**150.** 18
151. 32	**152.** 4	**153.** 14	**154.** 4	**155.** 12	**156.** 47	**157.** 50	**158.** 34	**159.** 99	**160.** 16
161. 23	**162.** 81	**163.** 50	**164.** 32						

EXERCISE – II

PART – I Previous Years' NSEJS Problems

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1. AD	**2.** B	**3.** D	**4.** B	**5.** D	**6.** C	**7.** B	**8.** D	**9.** D	**10.** D
11. B	**12.** B	**13.** D	**14.** A	**15.** D	**16.** D	**17.** B	**18.** D	**19.** C	**20.** B
21. D	**22.** B	**23.** C	**24.** C	**25.** C	**26.** C	**27.** D	**28.** D	**29.** C	**30.** C
31. A	**32.** B	**33.** C	**34.** B	**35.** D	**36.** C	**37.** D	**38.** A	**39.** D	**40.** D
41. B	**42.** A								

PART – II Previous Years' KVPY Problems

1. D	**2.** C	**3.** A	**4.** 6	**6.** D	**7.** A	**8.** A	**9.** C	**10.** C	**11.** B
12. C	**13.** B	**14.** C	**15.** B	**16.** A	**17.** D	**18.** C	**19.** A	**20.** C	**21.** C
22. C	**23.** D	**24.** A	**25.** B	**26.** D	**27.** B	**28.** D	**29.** B	**30.** B	**31.** A
32. D	**33.** D	**34.** B							

PART – III Previous Years' PRMO Problems

1. 15	**2.** 16	**3.** 15	**4.** 2011	**5.** 9	**6.** 92	**7.** 1	**8.** 18	**9.** 9	**10.** 61
11. 630	**12.** 11	**13.** 3	**14.** 1	**15.** 400	**16.** 8	**17.** 12	**18.** 217	**19.** 28	**20.** 29
21. 10	**22.** 47	**23.** 28	**24.** 97	**25.** 70	**26.** 12	**27.** 17	**28.** 81	**29.** 13	**30.** 29
31. 14	**32.** 18	**33.** 53	**34.** 20	**35.** Bon	**36.** 50	**37.** 64	**38.** 78	**39.** 25	**40.** 37
41. 60	**42.** 22	**43.** 36	**44.** 6	**45.** 69	**46.** 91	**47.** 43	**48.** Bon	**49.** 75	**50.** 21

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Chapter 5 Combinatorics

Learning Outcomes

After reading this chapter, you will be able to understand

- Set Problems
- Permutation and Combinations
- De-arrangement
- Advanced Problems based on PnC
- Probability Problems

Set

A set is a collection of well-defined objects that are distinct from each other. Sets are generally denoted by capital letters A, B, C, … etc. and the elements of the set by a, b, c … etc.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A.

If a does not belong to A then we write $a \notin A$.

Example:
The collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.

Some Important Number Sets

N = Set of all natural numbers

$= \{1, 2, 3, 4, \ldots\}$

W = Set of all whole numbers

$= \{0, 1, 2, 3, \ldots\}$

Z or I set of all integers

$= \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Z^+ = Set of all +ve integers

$= \{1, 2, 3, \ldots\} = N.$

Z^- = Set of all –ve integers

$= (-1, -2, -3, \ldots\}$

Q = The set of all rational numbers.

$$= \left\{\frac{p}{q} : p, q \in I, q \neq 0\right\}$$

R = the set of all real numbers.

Q' or $R-Q$ = The set of all irrational numbers

Example:

$\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots \pi, e$, log2, etc., are all irrational numbers.

Methods to Write a Set

(a) **Roster Method:** In this method, a set is described by listing elements, separated by commas and then enclosed by curly brackets

Example:
The set of vowels of English Alphabet may be described as {a, e, i, o, u}.

(b) **Set Builder Form:** In this case, we write down a property or rule p that gives us all the elements of the set

$A = \{x : P(x)\}$

Example:

$A = \{x : x \in N \text{ and } x = 2n \text{ for } n \in N\}$

i.e., $A = \{2, 4, 6, \ldots\}$

Example:

$B = \{x^2 : x \in z\}$

i.e., $B = \{0, 1, 4, 9, \ldots\}$

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Types of Sets

Null Set or Empty Set

A set having no element in it is called an empty set or a null set or void set it is denoted by ϕ or $\{\ \}$

Example:

$$A = \{x \in N : 5 < x < 6\} = \phi$$

A set consisting of at least one element is called a non-empty set or a non-void set.

Singleton

A set consisting of a single element is called a singleton set.

Example:
Then set $\{0\}$, is a singleton set

Finite Set

A set that has only finite number of elements is called a finite set.

Example:

$$A = \{a, b, c, d, e\}$$

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Order of a Finite Set

The number of elements in a finite set is called the order of the set A and is denoted $O(A)$ or $n(A)$. It is also called cardinal number of the set.

Example:

$$A = \{a, b, c, d\} \Rightarrow n(A) = 4$$

Infinite Set

A set that has an infinite number of elements is called an infinite set.

Example:

$$N = \{1, 2, 3, 4, \ldots\} \text{ is an infinite set}$$

Equal Sets

Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.

If sets A and B are equal, we write $A = B$ and A and B are not equal then $A \neq B$

Example:

$$A = \{1, 2, 6, 7\} \text{ and } B = \{6, 1, 2, 7\} \Rightarrow A = B$$

Equivalent Sets

Two finite sets A and B are equivalent if their number of elements are same

i.e. $n(A) = n(B)$

Example:

$$A = \{1, 3, 5, 7\},\ B = \{a, b, c, d\}$$

$$n(A) = 4 \text{ and } n(B) = 4 \Rightarrow n(A) = n(B)$$

Note

Equal set always equivalent but equivalent sets may not be equal

Subsets

Let A and B be two sets if every element of A is an element B, then A is called a subset of B if A is a subset of B. we write $A \subseteq B$

Example:

$$A = \{1, 2, 3, 4\} \text{ and } B = \{1, 2, 3, 4, 5, 6, 7\} \Rightarrow A \subseteq B$$

The symbol "$\Rightarrow$" stands for "implies"

Proper Subset

If A is a subset of B and $A \neq B$ then A is a proper subset of B. and we write $A \subset B$

Note

1. Every set is a subset of itself i.e., $A \subseteq A$ for all A
2. Empty set ϕ is a subset of every set
3. Clearly $N \subset W \subset Z \subset Q \subset R \subset C$
4. The total number of subsets of a finite set containing n elements is $\mathbf{2^n}$

Universal Set

A set consisting of all possible elements that occur in the discussion is called a universal set and is denoted by U.

Note

All sets in the discussion are contained in the universal set of that discussion.

Example:
If $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$, $C = \{1, 3, 5, 7\}$, then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.

Power Set

Let A be any set. The set of all subsets of A is called power set of A and is denoted by $P(A)$.

Example:
Let $A = \{1, 2\}$ then $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Example:
Let $P(\phi) = \{\phi\}$

$\Rightarrow \quad P(P(\phi)) = \{\phi, \{\phi\}\}$

$\Rightarrow \quad P(P(P(\phi)) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

(Please look this notation very carefully as students often get confused in it)

Note

1. If $A = \phi$ then $P(A)$ has one element
2. Power set of a given set is always non empty

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Some Operation on Sets

(i) **Union of two sets:** $A \cup B = \{x : x \in A \text{ or } x \in B\}$
Example: $A = \{1, 2, 3\}, B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$

(ii) **Intersection of two sets:** $A \cap B = \{x : x \in A \text{ and } x \in B\}$
Example: $A = \{1, 2, 3\}$, $B = \{2, 4, 5\}$ then $A \cap B = \{2\}$

(iii) **Difference of two sets:** $A - B = \{x : x \in A \text{ and } x \notin B\}$
Example: $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$; $A - B = \{1\}$

(iv) **Complement of a set:** A' or $A^c = \{x : x \notin A \text{ but } x \in U\} = U - A$
Example: $U = \{1, 2, \ldots, 10\}$, $A = \{1, 2, 3, 4, 5\}$ then $A' = \{6, 7, 8, 9, 10\}$

(v) **De-Morgan Laws:** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(vi) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(vii) **Distributive Laws:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(viii) **Commutative Laws:** $A \cup B = B \cup A$; $A \cap B = B \cap A$

(ix) **Associative Laws:** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(x) $A \cap \phi = \phi$; $A \cap U = A$
$A \cup \phi = A$; $A \cup U = U$

(xi) $A \cap B \subseteq A$; $A \cap B \subseteq B$

(xii) $A \subseteq A \cup B$; $B \subseteq A \cup B$

(xiii) $A \subseteq B \Rightarrow A \cap B = A$

(xiv) $A \subseteq B \Rightarrow A \cup B = B$

(xv) $(A')' = A$

(xvi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

Disjoint Sets

If $A \cap B = \phi$, then A, B are said to be disjoint.

Example:
If $A = \{1, 2, 3\}$, $B = \{7, 8, 9\}$, then $A \cap B = \phi$

Note

$A \cap A' = \phi$

$\therefore$ A, A' are disjoint.

Symmetric Difference of Sets and Properties Related to Difference

$$A \Delta B = (A - B) \cup (B - A)$$

If A and B are any two sets, then

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$

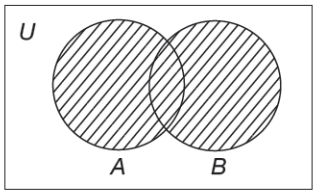
(iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$

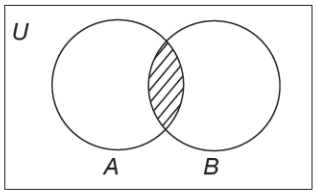
(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Venn Diagram

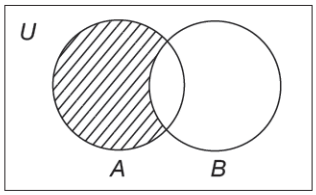
Venn Diagrams are powerful tools to represent a set operation. In these diagrams, universal set is represented as a Rectangle and subsets of universal sets are represented by circles of smaller radius such that they completely lie inside the rectangle. In the diagrams, the common area between any two figures represents elements that are common in the two sets corresponding to those figures. In the figures below, shaded region is representative of set operations written below the figures.



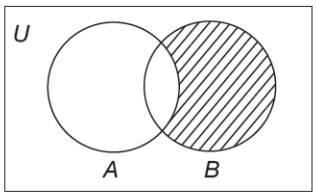
$A \cup B$



$A \cap B$

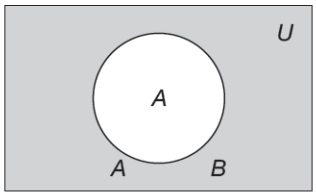


$A - B$ or $A \cap \bar{B}$

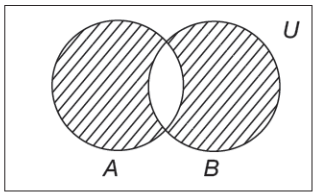


$B - A$

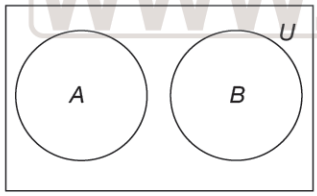
Clearly $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$



$\bar{A}$, A^c or A'



$(A \Delta B) = (A - B) \cup (B - A)$



Disjoint Sets $(A \cap B) = \phi$

Note

$A \cap A' = \phi$, $A \cup A' = U$

Some Important Results on Number of Elements in Sets

If A, B, and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A$, B are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e., $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B)$ = No. of elements that belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$

[$(A - B)$ and $(B - A)$ are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of the sets A, B, $C = n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vii) Number of elements in exactly one of the sets A, B, $C = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

Example 1

In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only? How many can speak Bengali only? How many can speak both Hindi and Bengali?

Solution

Let A and B be the sets of persons who can speak Hindi and Bengali, respectively.

then $n(A \cup B) = 1000$, $n(A) = 750$, $n(B) = 400$.

Number of persons who can speak both Hindi and Bengali

$= n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 750 + 400 - 1000 = 150$

Number of persons who can speak Hindi only

$= n(A - B) = n(A) - n(A \cap B) = 750 - 150 = 600$

Number of persons who can speak Bengali only

$= n(B - A) = n(B) - n(A \cap B) = 400 - 150 = 250$

Tip: Try and understand what a Venn Diagram for four sets would look like.

Example 2

The set $A = \{x : x \in R, x^2 = 16 \text{ and } 2x = 6\}$ equal:

(A) ϕ **(B)** {14, 3, 4}

(C) [3] **(D)** {4}

Solution

$x^2 = 16 \Rightarrow x = \pm 4$

$2x = 6 \Rightarrow x = 3$

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There is no value of x that satisfies both the above equations.

Thus, $A = \phi$

Hence (A) is the correct answer.

Example 3

Let $A = \{x : x \in R, |x| < 1\}$; $B = \{x : x \in R, |x-1| \geq 1\}$ and $A \cup B = R - D$, then the set D is:

(A) $\{x : 1 < x \leq 2\}$ **(B)** $\{x : 1 \leq x < 2\}$
(C) $\{x : 1 \leq x \leq 2\}$ **(D)** none of these

Solution

$$A = \{x : x \in R, -1 < x < 1\}$$

$$B = \{x : x \in R : x - 1 \leq -1 \text{ or } x - 1 \geq 1\}$$

$$= \{x : x \in R : x \leq 0 \text{ or } x \geq 2\}$$

$\therefore \quad A \cup B = R - D$

where $D = \{x : x \in R, 1 \leq x < 2\}$ **Ans. (B)**

Example 4

If $aN = \{ax : x \in N\}$, then the set $6N \cap 8N$ is equal to:

(A) $8N$ **(B)** $48N$ **(C)** $12N$ **(D)** $24N$

Solution

$$6N = \{6, 12, 18, 24, 30, \ldots\}$$

$$8N = \{8, 16, 24, 32, \ldots\}$$

$\therefore \quad 6N \cap 8N = \{24, 48, \ldots\} = 24N$

Short Cut Method

$6N \cap 8N = 24N$ [24 is the L.C.M. of 6 and 8] **Ans. (D)**

Example 5

If $A = \{2, 3\}$. Find value of $n(P(P(P(A))))$.

Solution

$n(A) = 2 \Rightarrow n(P(A)) = 2^2$ thus $n(P(P(P(A)))) = 2^{16}$

Example 6

If $A = \{x, y\}$, then the power set of A is:

(A) $\{x^y, y^x\}$ **(B)** $\{\phi, x, y\}$
(C) $\{\phi, \{x\}\ \{2y\}\}$ **(D)** $\{\phi, \{x\}, \{y\}, \{x, y\}\}$

Solution

Clearly $P(A)$ = Power set of A

= set of all subsets of A

$= \{\phi, \{x\}, \{y\}, \{x, y\}\}$ **Ans. (D)**

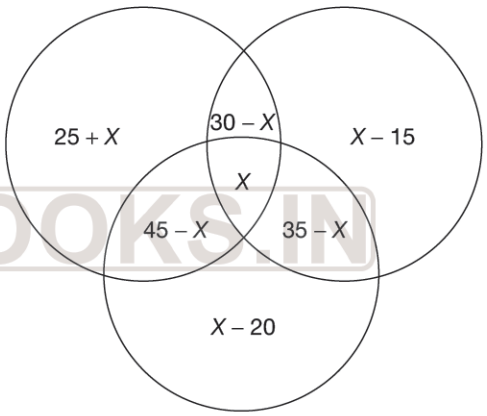
Example 7

These are 200 students in a school. Out of these, 100 students play cricket, 50 students play hockey and 60 students play basketball. 30 students play both cricket and hockey, 35 students play both hockey and basketball, and 45 students play both basketball and cricket.

(a) What is the maximum number of students who play at least one game?
(b) What is the maximum number of students who play all the 3 games?
(c) What is the minimum number of students playing at least one game?
(d) What is the minimum number of students who play all the 3 games?

Solution

Cricket = 100 Hockey = 50



Basketball = 60

Consider the Venn diagram given above:

At first, we will convert all the values in terms of x, which can be seen above.

Since the number of students cannot be negative.

$$x - 15 \geq 0$$

$\therefore \quad x - 20 \geq 0$

For the minimum number of students playing all three games, i.e., $x = 20$.

For the maximum value of x, again none of the categories should have –ve number of students.

$\therefore \quad 30 - x \geq 0$

$$x \leq 30$$

If x is more than 30, $30 - x$ would be –ve that is not possible.

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Total number of students playing at least one game,

$= 100 + x - 15 + 35 - x + x - 20$

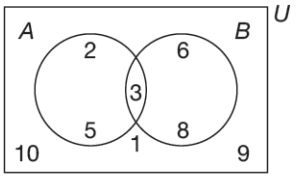
$= 100 + x$

So, the minimum number of students playing at least one game $= 100 + 20 = 120$

Hence, the maximum number of students playing at least one game $= 100 + 30 = 130$.

Know Your Progress - 1

1. Describe the following set by set property method $\{0, 3, 8, 15, 24, 35\}$
2. Write the following set by roster method: The set of all natural numbers 'x' such that $4x + 9 < 50$.
3. If $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 8\}$, $C = \{2, 5, 7, 8\}$ verify that $A - (B \cup C) = (A - B) \cap (A - C)$.
4. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$. Verify that $(A \cap B)' = A' \cup B'$.
5. In a group of 40 students, 26 take tea, 18 take coffee and 8 take neither of the two. How many take both tea and coffee.
6. In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find
   (i) How many drink tea and coffee both?
   (ii) How many drink coffee but not tea?
7. From the adjoining Venn diagram, determine the following sets:



   (i) $A \cup B$ (ii) $A \cap C$
   (iii) $A - B$ (iv) $(A \cap B)'$
8. A and B are two sets such that $n(A - B) = 14 + x$, $n(B - A) = 3x$ and $n(A \cap B) = x$. Draw a Venn diagram to illustrate information and if $n(A) = n(B)$ then find the value of x.

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Permutation and Combination

Fundamental Principle of Counting (Counting Without Actual Counting)

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of:

(a) simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(b) happening exactly one of the events is $m + n$ (known as addition principle).

Example: There are 15 IITs in India and let each IIT has 10 branches, then the IIT-JEE topper can select the IIT and branch in $15 \times 10 = 150$ number of ways.

Example: There are 15 IITs & 20 NITs in India, then a student who cleared both IIT-JEE & AIEEE exams can select an institute in $(15 + 20) = 35$ number of ways.

Permutation

Each of the arrangements in a definite order that can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained. Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

${}^nP_r$ denotes the no. of permutations of n **different** things, taken r at a time ($n \in N$, $r \in W$, $r \leq n$)

$$ {}^nP_r = n\,(n-1)\,(n-2)\,\ldots\,(n-r+1) = \frac{n!}{(n-r)!} $$

Note

(a) ${}^nP_n = n!$, ${}^nP_0 = 1$, ${}^nP_1 = n$

(b) Number of arrangements of n **distinct** things taken all at a time $= n!$

(c) ${}^nP_r$ is also denoted by A_r^n or $P(n, r)$.

Combination

Each of the groups or selections that can be made by taking some or all of the things **without considering the order of the things** in each group is called a COMBINATION. Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

${}^nC_r$ denotes the number of combinations of n different things taken r at a time ($n \in N$, $r \in W$, $r \le n$)

$$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

Note

(a) ${}^nC_r$ is also denoted by $\binom{n}{r}$ or $C(n, r)$.

(b) ${}^nP_r = {}^nC_r \cdot r!$

Properties of ${}^nP_r$ and ${}^nC_r$

1. The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!$. ${}^{n-p}C_{r-p} (p \le r \le n)$
2. The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^r.
3. The number of combinations of n different things taking r at a time,
   (a) when p particular things are always to be included $= {}^{n-p}C_{r-p}$
   (b) when p particular things are always to be excluded $= {}^{n-p}C_r$
   (c) when p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$

Example 8

A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is:

(A) 24 **(B)** 2 **(C)** 12 **(D)** 10

Solution

The student has 6 choices from the morning courses out of which he can select one course in 6 ways. For the evening course, he has 4 choices out of that he can select one in 4 ways. Hence the total number of ways $6 \times 4 = 24$.

Ans.(A)

Example 9

A college offers 6 courses in the morning and 4 in the evening. The number of ways as student can select exactly one course, either in the morning or in the evening:

(A) 6 **(B)** 4 **(C)** 10 **(D)** 24

Solution

The student has 6 choices from the morning courses out of that he can select one course in 6 ways. For the evening course, he has 4 choices out of that he can select one in 4 ways. Hence, the total number of ways $6 + 4 = 10$.

Ans. (C)

Example 10

If a denotes the number of permutations of $(x + 2)$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $(x - 11)$ things taken all at a time such that $a = 182\,bc$, then the value of x is

(A) 15 **(B)** 12 **(C)** 10 **(D)** 18

Solution

$$ {}^{x+2}P_{x+2} = a \quad \Rightarrow \quad a = (x+2)! $$

$$ {}^{x}P_{11} = b \quad \Rightarrow \quad b = \frac{x!}{(x-11)!} $$

and $$ {}^{x-11}P_{x-11} = c \quad \Rightarrow \quad c = (x-11)! $$

$\because$ $$ a = 182bc $$

$$ (x+2)! = 182 \frac{x!}{(x-11)!} (x-11)! $$

$\Rightarrow$ $$ (x+2)(x+1) = 182 = 14 \times 13 $$

$\therefore$ $$ x + 1 = 13 \Rightarrow x = 12 $$ **Ans. (B)**

Example 11

A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are at least two balls of each color?

Solution

The selections of 6 balls, consisting of at least two balls of each color from 5 red and 6 white balls, can be made in the following ways

Red balls (5)	White balls (6)	Number of ways
2	4	${}^5C_2 \times {}^6C_4 = 150$
3	3	${}^5C_3 \times {}^6C_3 = 200$
4	2	${}^5C_4 \times {}^6C_2 = 75$

Therefore, total number of ways = 425 **Ans.**

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Example 12

How many 4 letter words can be formed from the letters of the word 'ANSWER'? How many of these words start with a vowel?

Solution

Number of ways of arranging 4 different letters from 6 different letters are

$$^6C_4 4! = \frac{6!}{2!} = 360.$$

There are two vowels (A and E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A _ _ _ $= {}^5C_3 3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E _ _ _ $= {}^5C_3 3! = \frac{5!}{2!} = 60.$

$\therefore$ Total number of 4 letter words starting with a vowel $= 60 + 60 = 120.$ **Ans.**

Example 13

If all the letters of the word 'RAPID' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RAPID'.

Solution

First of all, arrange all letters of given word alphabetically: 'ADIPR'

Total number of words starting with A _ _ _ _ $= 4! = 24$
Total number of words starting with D _ _ _ _ $= 4! = 24$
Total number of words starting with I _ _ _ _ $= 4! = 24$
Total number of words starting with P _ _ _ _ $= 4! = 24$
Total number of words starting with RAD _ _ $= 2! = 2$
Total number of words starting with RAI _ _ $= 2! = 2$
Total number of words starting with $RAPD$ _ $= 1$
Total number of words starting with $RAPI$ _ $= 1$

$\therefore$ Rank of the word RAPID $= 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 102$ **Ans.**

Example 14

There are 6 pockets in the coat of a person. In how many ways can he put 4 pens in these pockets?

(A) 360 **(B)** 1296
(C) 4096 **(D)** none of these

Solution

First pen can be put in 6 ways.

Similarly, each of second, third and fourth pen can be put in 6 ways.

Hence total number of ways $= 6 \times 6 \times 6 \times 6 = 1296$

Ans. (B)

Example 15

A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected, if:

(a) all the students are equally willing?
(b) two particular students have to be included in the delegation?
(c) two particular students do not wish to be together in the delegation?
(d) two particular students wish to be included together only?
(e) two particular students refuse to be together and two other particular students wish to be together only in the delegation?

Solution

(a) Formation of delegation means selection of 4 out of 12. Hence the number of ways $= {}^{12}C_4 = 495$.

(b) If two particular students are already selected. Here we need to select only 2 out of the remaining 10. Hence the number of ways $= {}^{10}C_2 = 45$.

(c) The number of ways in which both are selected $= 45$. Hence the number of ways in which the two are not included together $= 495 - 45 = 450$

(d) There are two possible cases
(i) Either both are selected. In this case, the number of ways in which the selection can be made $= 45$.
(ii) Or both are not selected. In this case all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.
Hence, the total number of ways of selection $= 45 + 210 = 255$

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.
(i) (A, B, C) selected, (D) not selected
(ii) (A, B, D) selected, (C) not selected
(iii) (A, B) selected, (C, D) not selected
(iv) (C) selected, (A, B, D) not selected
(v) (D) selected, (A, B, C) not selected
(vi) A, B, C, D not selected
For (i) the number of ways of selection $= {}^8C_1 = 8$
For (ii) the number of ways of selection $= {}^8C_1 = 8$
For (iii) the number of ways of selection $= {}^8C_2 = 28$
For (iv) the number of ways of selection $= {}^8C_3 = 56$
For (v) the number of ways of selection $= {}^8C_3 = 56$
For (vi) the number of ways of selection $= {}^8C_4 = 70$
Hence total number of ways $= 8 + 8 + 28 + 56 + 56 + 70 = 226.$ **Ans.**

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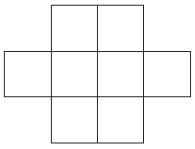
Example 16

In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one 'A'. In how many number of ways is it possible?



(A) 24 **(B)** 25 **(C)** 26 **(D)** 27

Solution

There are 8 squares and 6 'A' in given figure. First we can put 6 'A' in these 8 squares by ${}^8C_6$ number of ways.

(i)

A	A	A	A
	A	A	

(ii)

	A	A	
A	A	A	A

According to question, at least one 'A' should be included in each row. So after subtracting these two cases, number of ways are $= ({}^8C_6 - 2) = 28 - 2 = 26$. **Ans. (C)**

Example 17

There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is:

(A) $3p^2(p-1)+1$ **(B)** $3p^2(p-1)$

(C) $p^2(4p-3)$ **(D)** none of these

Solution

The number of triangles with vertices on different lines $= {}^pC_1 \times {}^pC_1 \times {}^pC_1 = p^3$ The number of triangles with two vertices on one line and the third vertex on any one of the other two lines $= {}^3C_1\{{}^pC_2 \times {}^{2p}C_1\} = 6p \cdot \frac{p(p-1)}{2}$

So, the required number of triangles

$$= p^3 + 3p^2(p-1) = p^2(4p-3)$$ **Ans. (C)**

Example 18

There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive?

Solution

Total number of remaining non-selected points = 6

• • • • • •

Total number of gaps made by these 6 points = 6 + 1 = 7

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x • • x • x • • x •

Total number of ways of selecting 4 gaps out of 7 gaps $= {}^7C_4$ **Ans.**

In general, total number of ways of selection of r points out of n points in a row such that no two of them are consecutive: ${}^{n-r+1}C_r$

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Know Your Progress - 2

1. There are 3 ways to go from A to B, 2 ways to go from B to C, and 1 way to go from A to C. In how many ways can a person travel from A to C?
2. There are 2 red balls and 3 green balls. All balls are identical except color. In how many ways can a person select two balls?
3. Find the number of ways of selecting 4 even numbers from the set of first 100 natural numbers.
4. If all letters of the word 'RANK' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RANK'.
5. How many words can be formed using all letters of the word 'LEARN'? In how many of these words vowels are together?
6. Find the number of ways of selecting 5 members from a committee of 5 men and 2 women such that all women are always included.
7. Out of first 20 natural numbers, 3 numbers are selected such that there is exactly one even number. How many different selections can be made?
8. How many four-letter words can be made from the letters of the word 'PROBLEM'. How many of these start as well as end with a vowel?

Formation of Groups

1. (a) The number of ways in which $(m + n)$ different things can be divided into two groups such that one of them contains m things and other has n things, is $\frac{(m+n)!}{m!n!}$ $(m \neq n)$.

   (b) If $m = n$, it means the groups are equal and in this case the number of divisions is $\frac{(2n)!}{n!n!2!}$, as in any one way it is possible to interchange the two groups without obtaining a new distribution.

   (c) If $2n$ things are to be divided equally between two persons then the number of ways: $\frac{(2n)!}{n!n!2!} \times 2!$.

2. (a) Number of ways in which $(m + n + p)$ different things can be divided into three groups containing m, n and p things, respectively, is: $\frac{(m+n+p)!}{m!n!p!}$, $m \neq n \neq p$.

   (b) If $m = n = p$ then the number of groups $= \frac{(3n)!}{n!\,n!\,n!\,3!}$.

   (c) If $3n$ things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

3. In general, the number of ways of dividing n distinct objects into l groups containing p objects each and m groups containing q objects each is equal to $\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\,\ell!m!}$
   Here $\ell p + mq = n$

Example 19

In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together? Also find the number of ways if these groups are to be sent to three different colleges.

Solution

Here first we separate those two particular students and make 3 groups of 5,5, and 3 of the remaining 13 so that these two particular students always go with the group of 3 students.

$\therefore$ Number of ways $= \frac{13!}{5!5!3!} \cdot \frac{1}{2!}$.

Now if these groups are to be sent to three different colleges, total number of ways $= \frac{13!}{5!5!3!} \cdot \frac{1}{2!} \cdot 3!$ **Ans.**

Example 20

Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

Solution

Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups $= \frac{48!}{(12!)^4\,4!}$

Now, distribute exactly one Ace to each group of 12 cards. Total number of ways $= \frac{48!}{(12!)^4\,4!} \times 4!$

Now, distribute these groups of cards among four players $= \frac{48!}{(12!)^4\,4!} \times 4!\ 4!$

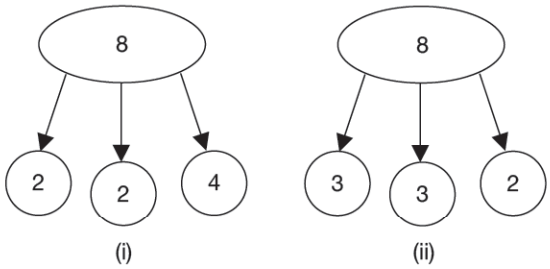
$= \frac{48!}{(12!)^4} \times 4!$ **Ans.**

Example 21

In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books?

Solution

If each receives at least two books, then the division trees would be as shown below:



(i) (ii)

The number of ways of division for tree in figure (i) is $\left[\frac{8!}{(2!)^2\,4!2!}\right]$

The number of ways of division for tree in figure (ii) is $\left[\frac{8!}{(3!)^2\,2!2!}\right]$

The total number of ways of distribution of these groups among 3 students is

$\left[\frac{8!}{(2i)^2\,4!2!} + \frac{8!}{(3!)^2\,2!2!}\right] \times 3!$. **Ans.**

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Know Your Progress - 3

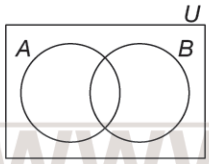
1. Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.
2. In how many ways can 6 different books be distributed among 3 students such that none gets equal number of books?
3. n different toys are to be distributed among n children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.

Principle of Inclusion and Exclusion

In the Venn's diagram (i), we get

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A' \cap B') = n(U) - n(A \cup B)$$



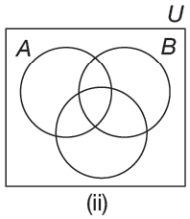
(i)

In the Venn's diagram (ii), we get

$$n(A \cup B \cup C)$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$$



(ii)

In general, we have

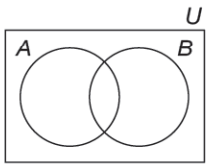
$$n(A_1 \cup A_2 \cup \ldots \cup A_n)$$

$$= \sum n(A_i) - \sum_{i \neq j} n(A_i \cap A_j) + \sum_{i \neq j \neq k} n(A_i \cap A_j \cap A_k)$$

$$+ \ldots + (-1)^n \sum n(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Example 22

Find the number of permutations of letters a, b, c, d, e, f, g taken all at a time if neither '*beg*' nor '*cad*' pattern appear.



Solution

The total number of permutations without any restrictions; $n(U) = 7!$

$\boxed{b\ e\ g}\ a\ c\ d\ f$

Let A be the set of all possible permutations in which '*beg*' pattern always appears: $n(A) = 5!$

$\boxed{c\ a\ d}\ b\ e\ f\ g$

Let B be the set of all possible permutations in which '*cad*' pattern always appears: $n(B) = 5!$

$\boxed{c\ a\ d\ b\ e\ g\ f}$

$n(A \cap B)$: Number of all possible permutations when both '*beg*' and '*cad*' patterns appear. $n(A \cap B) = 3!$.

Therefore, the total number of permutations in which '*beg*' and '*cad*' patterns do not appear

$$n(A' \cap B') = n(U) - n(A \cap B)$$

$$= n(U) - n(A) - n(B) + n(A \cap B)$$

$$= 7! - 5! - 5! + 3!.$$

Ans.

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Know Your Progress - 4

1. Find the number of n digit numbers formed using first 5 natural numbers, which contain the digits 2 and 4 essentially.

Permutations of Alike Objects

Case 1: Taken all at a time:

The number of permutations of n things taken all at a time: when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n-(p+q+r)$ are all different is: $\frac{n!}{p!q!r!}$.

Case 2: Taken some at a time

Example 23

In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative position of vowels and consonants.

Solution

The consonants in their positions can be arranged in $\frac{4!}{2!}$ = 12 ways.

The vowels in their positions can be arranged in $\frac{3!}{2!}$ = 3 ways

$\therefore$ Total number of arrangements = $12 \times 3 = 36$ **Ans.**

Example 24

How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20

Solution

There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh). At these places the odd digits can be arranged in $\frac{4!}{2!2!} = 6$ ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ The required number of numbers = $6 \times 3 = 18$.

Ans. (B)

Example 25

(a) How many permutations can be made by using all the letters of the word HINDUSTAN?

(b) How many of these permutations begin and end with a vowel?

(c) In how many of these permutations, all the vowels come together?

(d) In how many of these permutations, none of the vowels come together?

(e) In how many of these permutations, do the vowels and the consonants occupy the same relative positions as in HINDUSTAN?

Solution

(a) The total number of permutations = Arrangements of nine letters taken all at a time = $\frac{9!}{2!} = 181440$.

(b) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $\frac{7!}{2!}$ ways.

Hence the total number of permutations = $3 \times 2 \times \frac{7!}{2!} = 15120$.

(c) Assume the vowels (I, U, A) as a single letter. The letters (IUA), H, D, S, T, N, N can be arranged in $\frac{7!}{2!}$ ways. Also IUA can be arranged among themselves in 3! = 6 ways.

Hence the total number of permutations = $\frac{7!}{2!} \times 6 = 15120$.

(d) Let us divide the task into two parts. In the first, we arrange the 6 consonants as shown below in $\frac{6!}{2!}$ ways. $\times\, C \times C \times C \times C \times C \times C \times$

(Here C stands for a consonant and $\times$ stands for a gap between two consonants)

Now 3 vowels can be placed in 7 places (gaps between the consonants) in ${}^7C_3 \cdot 3! = 210$ ways.

Hence the total number of permutations = $\frac{6!}{2!} \times 210 = 75600$.

(e) In this case, the vowels can be arranged among themselves in 3! = 6 ways.

Also, the consonants can be arranged among themselves in $\frac{6!}{2!}$ ways.

Hence, the total number of permutations = $\frac{6!}{2!} \times 6 = 2160$. **Ans.**

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Example 26

If all the letters of the word 'PROPER' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'PROPER'.

Solution

First of all, arrange all letters of given word alphabetically: EOPPRR. Total number of words starting with:

$E_____ = \frac{5!}{2!2!} = 30$

$O_____ = \frac{5!}{2!2!} = 30$

$PE____ = \frac{4!}{2!} = 12$

$PO____ = \frac{4!}{2!} = 12$

$PP____ = \frac{4!}{2!} = 12$

PRE _ _ _ = 3! = 6

PROE _ _ = 2! = 2

PROPER = 1= 1

Rank of the word PROPER = 105 **Ans.**

Example 27

Find the total number of 4 letter words formed using four letters from the word 'PARALLELOPIPED'.

Solution

Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No. of ways of selection	No. of ways of arrangements	Total
All distinct	${}^8C_4$	${}^8C_4 \times 4!$	1680
2 alike, 2 distinct	${}^4C_1 \times {}^7C_2$	${}^4C_1 \times {}^7C_2 \times \frac{4!}{2!}$	1008
2 alike, 2 other alike	${}^4C_2$	${}^4C_2 \times \frac{4!}{2!2!}$	36
3 alike, 1 distinct	${}^2C_1 \times {}^7C_1$	${}^2C_1 \times {}^7C_1 \times \frac{4!}{3!}$	56
		Total	2780

Example 28

Find the number of all 6 digit numbers such that all the digits of each number are selected from the set {1, 2, 3, 4, 5} and any digit that appears in the number appears at least twice.

Solution

Cases	No. of Ways of selection	No. of ways of arrangements	Total
All alike	${}^5C_1$	${}^5C_1 \times 1$	5
4 alike + 2 other alike	${}^5C_2 \times 2!$	${}^5C_2 \times 2 \times \frac{6!}{2!4!}$	300
3 alike + 3 other alike	${}^5C_2$	${}^5C_2 \frac{6!}{3!3!}$	200
2 alike + 2 other alike +2 other alike	${}^5C_3$	${}^5C_3 \times \frac{6!}{2!2!2!}$	900
		Total	1405

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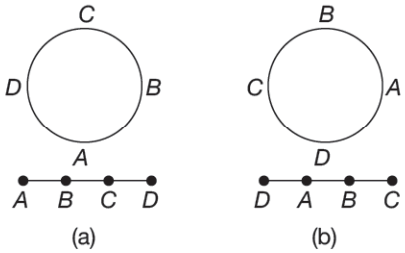
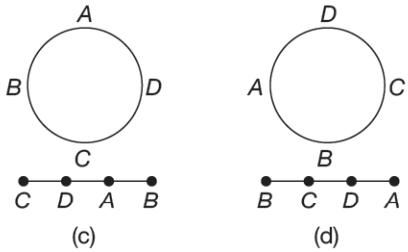
Know Your Progress - 5

1. In how many ways can the letters of the word 'INDIA' be arranged? Also find its rank if all these words are arranged as they are in dictionary.
2. How many numbers greater than 1000 can be formed from the digits 1, 1, 2, 2, 3?

Circular Permutation





Let us consider that persons A, B, C, D are sitting around a round table. If all of them (A, B, C, D) are shifted by one place in anticlockwise order, then we will get Fig. (b) from Fig. (a). Now, if we shift A, B, C, D in anticlockwise order, we will get Fig. (c). Again, if we shift them, we will get Fig. (d) and in the next time, Fig. (a).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A, B, C, D does not change.

But if A, B, C, D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4. Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is $n \times$ (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is

$1/n \times$ (number of linear arrangements of n different things) $= \frac{n!}{n} = (n-1)!$

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Therefore note that:

(i) The number of circular permutations of n different things taken all at a time is: $(n-1)!$
If clockwise and anti-clockwise circular permutations are considered to be same, then it is: $\frac{(n-1)!}{2}$

(ii) The number of circular permutations of n different things taking r at a time distinguishing clockwise and anticlockwise arrangements is: $\frac{{}^nP_r}{r}$

Example 29

In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

(A) $5! \times 5!$ **(B)** $5! \times 4!$

(C) $\frac{1}{2}(5!)^2$ **(D)** $\frac{1}{2}(5! \times 4!)$

Solution

Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls sit in 5! ways. Hence the required number of ways $= 4! \times 5!$

Ans. (B)

Example 30

The number of ways in which 7 girls can stand in a circle so that they do not have same neighbors in any two arrangements?

(A) 720 **(B)** 380

(C) 360 **(D)** none of these

Solution

Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$ number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

$\therefore \quad \frac{(7-1)!}{2!} = 360$ **Ans. (C)**

Example 31

The number of ways in which 20 different pearls of two colors can be set alternately on a necklace, there being 10 pearls of each color, is

(A) $9! \times 10!$ **(B)** $5(9!)^2$

(C) $(9!)^2$ **(D)** none of these

Solution

Ten pearls of one color can be arranged in $\frac{1}{2} \cdot (10-1)!$ ways.

The number of arrangements of 10 pearls of the other color in 10 places between the pearls of the first color = 10!

$\therefore$ The required number of ways $= \frac{1}{2} \times 9! \times 10! = 5(9!)^2$

Ans. (B)

Example 32

A person invites a group of 10 friends at dinner. They sit

(i) 5 on one round table and 5 on other round table,

(ii) 4 on one round table and 6 on other round table.

Find the number of ways in each case in which he can arrange the guests.

Solution

(i) The number of ways of selection of 5 friends for first table is ${}^{10}C_5$. Remaining 5 friends are left for second table.
The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is ${}^{10}C_5 \times 4! \times 4! = \frac{10!4!4!}{5!5!} = \frac{10!}{25}$

(ii) The number of ways of selection of 6 guests is ${}^{10}C_6$. The number of ways of permutations of 6 guests on round table is 5!. The number of permutations of 4 guests on round table is 3!

Therefore, total number of arrangements is: ${}^{10}C_6 5! \times 3! = \frac{(10)!}{6!4!}\, 5!3! = \frac{(10)!}{24}$ **Ans.(B)**

Know Your Progress - 6

1. In how many ways can 3 men and 3 women be seated around a round table such that all men are always together?
2. Find the number of ways in which 10 different diamonds can be arranged to make a necklace.
3. Find the number of ways in which 6 persons out of 5 men and 5 women can be seated at a round table such that 2 men are never together.
4. In how many ways can 8 persons be seated on two round tables of capacity 5 and 3.

Total Number of Combinations

1. Given n different objects, the number of ways of selecting atleast one of them is, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.
2. (a) Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \ldots$ things, where p are alike of one kind, q alike of a second kind, r alike of third kind and so on is given by:

$$(p + 1)\,(q + 1)\,(r + 1)\,\ldots - 1.$$

(b) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by:

$$(p + 1)\,(q + 1)\,(r + 1)\,2^n - 1.$$

Example 33

A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is:

(A) $2^{2n} - {}^{2n}C_n$ **(B)** 2^n

(C) $2^n - 1$ **(D)** 3^n

Solution

Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$. For $a_i \in A$, we have the following choices:

(i) $a_i \in P$ and $a_i \in Q$
(ii) $a_i \in P$ and $a_i \notin Q$
(iii) $a_i \notin P$ and $a_i \in Q$
(iv) $a_i \notin P$ and $a_i \notin Q$

Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, \ldots a_n$ belong to $P \cap Q$ is 3^n. **Ans. (D)**

Example 34

There are 3 books of mathematics, 4 of science and 5 of English. How many different collections can be made such that each collection consists of?

(i) one book of each subject?
(ii) at least one book of each subject?
(iii) at least one book of English?

Solution

(i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 60$
(ii) $(2^3 - 1)\,(2^4 - 1)\,(2^5 - 1) = 7 \times 15 \times 31 = 3255$
(iii) $(2^5 - 1)\,(2^3)\,(2^4) = 31 \times 128 = 3968$ **Ans.**

Example 35

Find the number of groups that can be made from 5 red balls, 3 green balls, and 4 black balls, if at least one ball of all colors is always to be included. Given that all balls are identical except colors.

Solution

After selecting one ball of each color, we have to find total number of combinations that can be made from 4 red, 2 green and 3 black balls. These will be

$$(4 + 1)(2 + 1)(3 + 1) = 60$$ **Ans.**

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Know Your Progress - 7

1. There are p copies each of n different books. Find the number of ways in which at least one book can be selected?

2. There are 10 questions in an examination. In how many ways can a candidate answer the questions, if he attempts at least one question.

Total Distribution

Distribution of Distinct Objects

Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by: p^n

Distribution of Alike Objects

Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Example 36

In how many ways can 5 different mangoes, 4 different oranges and 3 different apples be distributed among 3 children such that each gets at least one mango?

Solution

5 different mangoes can be distributed by following ways among 3 children such that each gets at least 1:

3 1 1
2 2 1

Total number of ways: $\left(\frac{5!}{3!1!1!2!}+\frac{5!}{2!2!2!}\right)\times 3!$

Now, the number of ways of distributing remaining fruits (i.e., 4 oranges + 3 apples) among 3 children = 3^7 (as each fruit has 3 options).

$\therefore$ Total number of ways = $\left(\frac{5!}{3!2!}+\frac{5!}{(2!)^3}\right)\times 3!\times 3^7$

Ans.

Example 37

In how many ways can 12 identical apples be distributed among four children if each gets at least 1 apple and not more than 4 apples.

Solution

Let x, y, z and w be the number of apples given to the children.

$\Rightarrow \quad x+y+z+w=12$

Giving one-one apple to each

Now, $\quad x+y+z+w=8 \quad$ (i)

Here, $0\le x\le 3,\ 0\le y\le 3,\ 0\le z\le 3,\ 0\le w\le 3$

$x=3-t_1,\ y=3-t_2,\ z=3-t_3,\ w=3-t_4.$

Putting value of x, y, z, w in equation (i)

Put $12-8=t_1+t_2+t_3+t_4$

$\Rightarrow t_1+t_2+t_3+t_4=4$

(Here max. value that t_1, t_2, t_3 and t_4 can attain is 3, so we have to remove those cases when any of t_i getting value 4)

$= {}^7C_3 -$ (all cases when at least one is 4)

$= {}^7C_3 - 4 = 35 - 4 = 31$ **Ans.**

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Example 38

Find the number of non-negative integral solutions of the inequation $x+y+z\le 20$.

Solution

Let w be any number $(0\le w\le 20)$, then we can write the equation as:

$x+y+z+w=20$ (here $x, y, z, w\ge 0$)

Total ways = ${}^{23}C_3$ **Ans.**

Example 39

Find the number of integral solutions of $x+y+z+w<25$, where

$x>-2,\ y>1,\ z\ge 2,\ w\ge 0.$

Solution

Given $x+y+z+w<25$

$x+y+z+w+v=25 \quad$ (i)

Let $x=-1+t_1,\ y=2+t_2,\ z=2+t_3,\ w=t_4,\ v=1+t_5$ where $(t_1, t_2, t_3, t_4\ge 0)$

Putting value of x, y, z, w, v in equation (i)

$\Rightarrow \quad t_1+t_2+t_3+t_4+t_5=21.$

Number of solutions = ${}^{25}C_4$ **Ans.**

Example 40

Find the number of positive integral solutions of the in-equation $x + y + z \geq 150$, where $0 < x \leq 60$, $0 < y \leq 60$, $0 < z \leq 60$.

Solution

Let $x = 60 - t_1$, $y = 60 - t_2$, $z = 60 - t_3$ (where $0 \leq t_1 \leq 59$, $0 \leq t_2 \leq 59$, $0 \leq t_3 \leq 59$)

Given $x + y + z \geq 150$

or $x + y + z - w = 150$ (where $0 \leq w \leq 147$) (i)

Putting values of x, y, z in equation (i)

$$60 - t_1 + 60 - t_2 + 60 - t_3 - w = 150$$

$$30 = t_1 + t_2 + t_3 + w$$

Total solutions $= {}^{33}C_3$ **Ans.**

Example 41

Find the number of positive integral solutions of $xy = 12$

Solution

$$xy = 12$$

$$xy = 2^2 \times 3^1$$

(i) 3 has 2 ways either 3 can go to x or y

(ii) 2^2 can be distributed between x and y as distributing 2 identical things between 2 persons (where each person can get 0, 1 or 2 things). Let two person be ℓ_1 and ℓ_2

$\Rightarrow$ $\ell_1 + \ell_2 = 2$

$\Rightarrow$ ${}^{2+1}C_1 = {}^3C_1 = 3$

So total ways $= 2 \times 3 = 6$.

Alternatively:

$$xy = 12 = 2^2 \times 3^1$$

$$x = 2^{a_1}3^{a_2} \qquad 0 \leq a_1 \leq 2$$

$$0 \leq a_2 \leq 1$$

$$y = 2^{b_1}3^{b_2} \qquad 0 \leq b_1 \leq 2$$

$$0 \leq b_2 \leq 1$$

$$2^{a_1+b_1}3^{a_2+b_2} = 2^2 3^1$$

$\Rightarrow$ $a_1 + b_1 = 2 \rightarrow {}^3C_1$ ways

$a_2 + b_2 = 1 \rightarrow {}^2C_1$ ways

Number of solutions $= {}^3C_1 \times {}^2C_1 = 3 \times 2 = 6$ **Ans.**

Example 42

Find the number of solutions of the equation $xyz = 360$ when

(i) $x, y, z \in N$

(ii) $x, y, z \in I$

Solution

(i) $xyz = 360 = 2^3 \times 3^2 \times 5$ $(x, y, z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \leq a_1 \leq 3$, $0 \leq a_2 \leq 2$, $0 \leq a_3 \leq 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \leq b_1 \leq 3$, $0 \leq b_2 \leq 2$, $0 \leq b_3 \leq 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \leq c_1 \leq 3$, $0 \leq c_2 \leq 2$, $0 \leq c_3 \leq 1$)

$\Rightarrow$ $2^{a_1}3^{a_2}5^{a_3} \cdot 2^{b_1}3^{b_2}5^{b_3} \cdot 2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow$ $2^{a_1+b_1+c_1} \cdot 3^{a_2+b_2+c_2} \cdot 5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow$ $a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions $= 10 \times 6 \times 3 = 180$.

(ii) If $x, y, z \in I$ then, (a) all positive (b) 1 positive and 2 negative.

Total number of ways $= 180 + {}^3C_2 \times 180 = 720$

Ans.

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Know Your Progress - 8

1. In how many ways can 12 identical apples be distributed among 4 boys.
   (a) If each boy receives any number of apples.
   (b) If each boy receives at least 2 apples.
2. Find the number of non-negative integral solutions of the equation $x + y + z = 10$.
3. Find the number of integral solutions of $x + y + z = 20$, if $x \geq -4$, $y \geq 1$, $z \geq 2$

De-arrangement

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is $n!\left[1-\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{(-1)^n}{n!}\right]$

Proof: n letters are denoted by 1, 2, 3, …, n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then, $n(A_i) = 1 \times (n-1)!$ [since the remaining $n-1$ letters can be placed in $n-1$ envelops in $(n-1)!$ ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$$(A_i \cap A_j) = 1 \times 1 \times (n-2)!$$

Also $$n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$$

Hence, the required number is

$$n(A_1' \cup A_2' \cup \ldots \cup A_n') = n! - n(A_1 \cup A_2 \cup \ldots \cup A_n)$$

$$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + (-1)^n \sum n(A_i \cap A_2 \ldots \cap A_n)\right]$$

$$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + \ldots + (-1)^{n-1} \times {}^nC_n 1]$$

$$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + \ldots + (-1)^{n-1}\right]$$

$$= n!\left[1-\frac{1}{11}+\frac{1}{2!}+\ldots+\frac{(-1)^n}{n!}\right]$$

Example 43

A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that

(i) all the letters are in the wrong envelopes.
(ii) at least two of them are in the wrong envelopes.

Solution

(i) The number of ways is which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$$= 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

(ii) The number of ways in which at least two of them in the wrong envelopes

$$= {}^6C_4 \cdot 2!\left(1-\frac{1}{1!}+\frac{1}{2!}\right) + {}^6C_3 \cdot 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right)$$

$$+ {}^6C_2 \cdot 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$$

$$+ {}^6C_1 \cdot 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$

$$+ {}^6C_0 \cdot 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$$= 15 + 40 + 135 + 264 + 265 = 719.$$ **Ans.**

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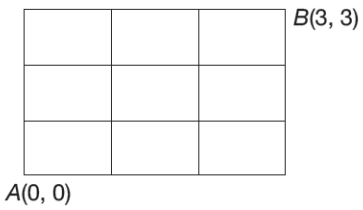
Know Your Progress - 9

1. There are four balls of different colors and four boxes of colors same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own color.

Miscellaneous Examples

Example 44

In how many ways can a person go from point A to point B if he can travel only to theright or upward along the lines (Grid Problem)?



Solution

To reach the point B from point A, a person has to travel along 3 horizontal and 3 vertical strips. Therefore, we have to arrange 3H and 3V in a row. Total number of ways = $\frac{6!}{3!3!}$ = 20 ways **Ans.**

Example 45

Find sum of all numbers formed using the digits 2, 4, 6, 8 taken all at a time and no digit being repeated.

Solution

All possible numbers = 4! = 24

If 2 occupies the unit's place then total numbers = 6

Hence, 2 comes at unit's place 6 times.

Sum of all the digits occurring at unit's place

$$= 6 \times (2 + 4 + 6 + 8)$$

Same summation will occur for ten's, hundred's, and thousand's place. Hence, required sum

$$= 6 \times (2 + 4 + 6 + 8) \times (1 + 10 + 100 + 1000) = 133320$$

Ans.

Example 46

Find the sum of all the numbers greater than 1000 using the digits 0, 1, 2, 2.

Solution

(i) When 1 is at thousand's place, total numbers formed will be = $\frac{3!}{2!} = 3$

(ii) When 2 is at thousand's place, total numbers formed will be = 3! = 6

(iii) When 1 is at hundred's, ten's, or unit's place then total numbers formed will be thousand's place is fixed i.e., only the digit 2 will come here, remaining two places can be filled in 2! ways.

So total numbers = 2!

(iv) When 2 is at hundred's, ten's, or unit's place then total numbers formed will be thousand's place has 2 options and other two places can be filled in 2 ways.

So total numbers = $2 \times 2 = 4$

Sum = $10^3(1 \times 3 + 2 \times 6) + 10^2(1 \times 2 + 2 \times 4) + 10^1(1 \times 2 + 2 \times 4) + (1 \times 2 + 2 \times 4)$

$= 15 \times 10^3 + 10^3 + 10^2 + 10$

$= 16110$ **Ans.**

Example 47

Find the number of positive integral solutions of $x+y+z=20$, if $x \neq y \neq z$.

Solution

$$x \geq 1$$

$$y = x + t_1 \qquad t_1 \geq 1$$

$$z = y + t_2 \qquad t_2 \geq 1$$

$$x + x + t_1 + x + t_1 + t_2 = 20$$

$$3x + 2t_1 + t_2 = 20$$

(i) $x = 1$ $\quad 2t_1 + t_2 = 17$

$t_1 = 1, 2 \ldots 8$ $\quad \Rightarrow 8$ ways

(ii) $x = 2$ $\quad 2t_1 + t_2 = 14$

$t_1 = 1, 2 \ldots 6$ $\quad \Rightarrow 6$ ways

(iii) $x = 3$ $\quad 2t_1 + t_2 = 11$

$t_1 = 1, 2 \ldots 5$ $\quad \Rightarrow 5$ ways

(vi) $x = 4$ $\quad 2t_1 + t_2 = 8$

$t_1 = 1, 2, 3$ $\quad \Rightarrow 3$ ways

(v) $x = 5$ $\quad 2t_1 + t_2 = 5$

$t_1 = 1, 2$ $\quad \Rightarrow 2$ ways

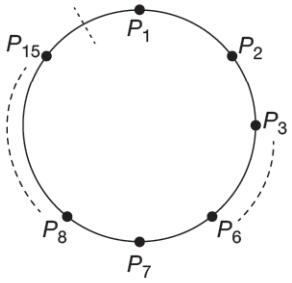
Total = 8 + 6 + 5 + 3 + 2 = 24

But each solution can be arranged by 3! ways.

So total solutions = $24 \times 3! = 144$. **Ans.**

Example 48

A regular polygon of 15 sides is constructed. In how many ways can a triangle be formed using the vertices of the polygon such that no side of triangle is same as that of polygon?



Solution

Select one point out of 15 point, therefore total number of ways = $^{15}C_1$

Suppose we select point P_1. Now we have to choose 2 more point that are not consecutive.

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Since, we cannot select P_2 and P_{15}.

So, total points left are 12.

Now we have to select 2 points out of 12 points that are not consecutive.

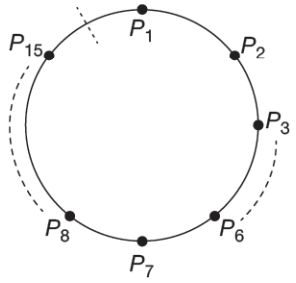
So, total ways $= {}^{12-2+1}C_2 = {}^{11}C_2$

Every select triangle will be repeated 3 times.

So total number of ways $= \dfrac{{}^{15}C_1 \times {}^{11}C_2}{3} = 275$

Alternative:

First of all let us cut the polygon between points P_1 and P_{15}. Now there are 15 points on a straight line and we have to select 3 points out of these, such that the selected points are not consecutive.



$x\ O\ y\ O\ z\ O\ w$

Here bubbles represent the selected points,

x represents the number of points before first selected point,

y represents the number of points between Ist and IInd selected point,

z represents the number of points between IInd and IIIrd selected point and w represents the number of points after IIIrd selected point.

$$x + y + z + w = 15 - 3 = 12$$

here $x \geq 0,\ y \geq 1,\ z \geq 1,\ w \geq 0$

Put $y = 1 + y'$ and $z = 1 + z'$ $(y' \geq 0,\ z' \geq 0)$

$$\Rightarrow \qquad x + y' + z' + w = 10$$

Total number of ways $= {}^{13}C_3$

These selections include the cases when both the points P_1 and P_{15} are selected. We have to remove those cases. Here a represents number of points between P_1 and 3rd selected point and b represents number of points between 3rd selected point and P_{15}

$$\Rightarrow \qquad a + b = 15 - 3 = 12\ (a \geq 1,\ b \geq 1)$$

put $a = 1 + t_1$ and $b = 1 + t_2$

$$t_1 + t_2 = 10$$

Total number of ways $= {}^{11}C_1 = 11$

Therefore, required number of ways $= {}^{13}C_3 - {}^{11}C_1 = 286 - 11 = 275$ **Ans.**

Example 49

Find the number of ways in which three numbers can be selected from the set

$\{5^1, 5^2, 5^3, \ldots 5^{11}\}$ so that they form a G.P.

Solution

Any three selected numbers which are in G.P. have their powers in A.P.

Set of powers is $= \{1, 2, \ldots 6, 7, \ldots 11\}$

By selecting any two numbers from $\{1, 3, 5, 7, 9, 11\}$, the middle number is automatically fixed. Total number of ways $= {}^6C_2$

Now select any two numbers from $\{2, 4, 6, 8, 10\}$ and again middle number is automatically fixed. Total number of ways $= {}^5C_2$

$\therefore$ Total number of ways are $= {}^6C_2 + {}^5C_2 = 15 + 10 = 25$ **Ans.**

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Probability

Definitions

Experiment

An action or operation resulting in two or more well defined outcomes. e.g., tossing a coin, throwing a die, drawing a card from a pack of well-shuffled playing cards etc.

Sample Space

A set S that consists of all possible outcomes of a random experiment is called a sample space and each outcome is called a sample point. Often, there will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information. e.g., in an experiment of 'throwing a die', following sample spaces are possible:

(i) {even number, odd number}

(ii) {a number less than 3, a number equal to 3, a number greater than 3}

(iii) {1, 2, 3, 4, 5, 6}

Here 3rd sample space is the one that provides most information.

If a sample space has a finite number of points it is called finite sample space and if it has an infinite number of points, it is called infinite sample space. e.g., (i) "in a toss of coin"

either a head (H) or tail (T) comes up, therefore sample space of this experiment is $S = \{H, T\}$, which is a finite sample space. (ii) "Selecting a number from the set of natural numbers", sample space of this experiment is $S = \{1, 2, 3, 4, \ldots\}$, which is an infinite sample space.

Event

An event is defined as an occurrence or situation, for example

(i) in a toss of a coin, it shows head,
(ii) scoring a six on the throw of a die,
(iii) winning the first prize in a raffle,
(iv) being dealt a hand of four cards which are all clubs.

In every case it is set of some or all possible outcomes of the experiment. Therefore, event (A) is subset of sample space (S). If outcome of an experiment is an element of A, we say that event A has occurred.

- An event consisting of a single point of S is called a simple or elementary event.
- ϕ is called impossible event and S (sample space) is called sure event.

Note

Probability of occurrence of an event A is denoted by $P(A)$.

Compound Event

If an event has more than one sample points it is called **Compound Event**. If A and B are two given events then $A \cap B$ is called compound event and is denoted by

$$A \cap B \text{ or } AB \text{ or } A \text{ and } B.$$

Complement of an Event

The set of all outcomes that are in S but not in A is called the complement of the event A and denoted by $\overline{A}$, A^c, A' or 'not A'.

Mutually Exclusive Events

Two events are said to be mutually exclusive (or disjoint or incompatible) if the occurrence of one precludes (rules out) the simultaneous occurrence of the other. If A and B are two mutually exclusive events, then $P(A \cap B) = 0$.

Consider, for example, choosing numbers at random from the set {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. If, Event A is the selection of a prime number,

Event B is the selection of an odd number,

Event C is the selection of an even number,

then A and C are mutually exclusive as none of the numbers in this set is both prime and even. But A and B are not mutually exclusive as some numbers are both prime and odd (viz. 3, 5, 7, 11).

Equally Likely Events

Events are said to be equally likely when each event is as likely to occur as any other event. Note that the term 'at random' or 'randomly' means that all possibilities are equally likely.

Exhaustive Events

Events A, B, C … N are said to be exhaustive events if no event outside this set can result as an outcome of an experiment. For example, if A and B are two events defined on a sample space S and A and B are exhaustive $\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$.

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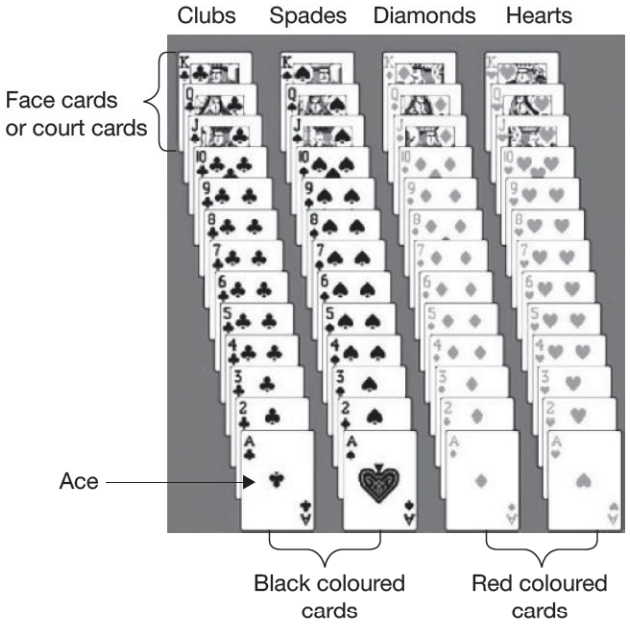
Note

Playing cards: A pack of playing cards consists of 52 cards of 4 suits, 13 in each, as shown in figure.



Comparative study of Equally likely, Mutually exclusive, and Exhaustive events:

Experiment	Events	E/L	M/E	Exhaustive
1. Throwing of a die	A: throwing an odd face $\{1, 3, 5\}$ B: throwing a composite $\{4, 6\}$	No	Yes	No
2. A ball is drawn from an urn containing 2 White, 3 Red, and 4 Green balls	E_1: getting a White ball E_2: getting a Red ball E_3: getting a Green ball	No	Yes	Yes
3. Throwing a pair of dice	A: throwing a doublet $\{11, 22, 33, 44, 55, 66\}$ B: throwing a total of 10 or more $\{46, 64, 55, 56, 65, 66\}$	Yes	No	No
4. From a well-shuffled pack of cards a card is drawn	E_1: getting a heart E_2: getting a spade E_3: getting a diamond E_4: getting a club	Yes	Yes	Yes
5. From a well shuffled pack of cards a card is	A = getting a heart B = getting a face card	No	No	No

Classical Definition of Probability

If n represents the total number of equally likely, mutually exclusive, and exhaustive outcomes of an experiment and m of them are favourable to the happening of the event A, then the probability of happening of the event A is given by $P(A) = m/n$. There are $(n - m)$ outcomes that are favourable to the event that A does not happen. 'The event A does not happen' is denoted by $\bar{A}$ (and is read as 'not A')

Thus $P(\bar{A}) = \dfrac{n-m}{n} = 1 - \dfrac{m}{n}$

i.e. $P(\bar{A}) = 1 - P(A)$

Note

(i) $0 \le P(A) \le 1$

(ii) $P(A) + P(\bar{A}) = 1$,

(iii) If x cases are favourable to A and y cases are favourable to $\bar{A}$ then $P(A) = \dfrac{x}{(x+y)}$ and $P(\bar{A}) = \dfrac{y}{(x+y)}$. We say that Odds In Favour Of A are $x : y$ and Odds Against A are $y : x$

Other Definitions of Probability

Axiomatic Probability

Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval $[0, 1]$ satisfying the following axioms:

(i) For any event E, $P(E) \ge 0$

(ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$.

It follows from (iii) that $P(E \cap F) = P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, \ldots, \omega_n$, i.e., $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$

It follows from the axiomatic definition of probability that:

(i) $0 \le P(\omega_i) \le 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \ldots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(\omega_i)$, $\omega_i \in A$.

Empirical Probability

The probability that you would hit the bull's-eye on a dart-board with one throw of a dart would depend on how much you had practised, how much natural talent for playing darts you had, how tired you were, how good a dart you were using etc. all of which are impossible to quantify. A method which can be adopted in the example given above is to throw the dart several times (each throw is a trial) and count the number of times you hit the bull's-eye (a success) and the number of times you miss (a failure). Then an empirical value of the probability that you hit the bull's-eye with any one throw is

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$$\frac{\text{number of successes}}{\text{number of successes + number of failures}}$$

If the number of throws is small, this does not give a particular good estimate but for a large number of throws the result is more reliable.

When the probability of the occurrence of an event A cannot be worked out exactly, an empirical value can be found by adopting the approach described above, that is:

(i) making a large number of trials (i.e., set up an experiment in which the event may, or may not, occur and note the outcome),

(ii) counting the number of times the event does occur, i.e., the number of successes,

(iii) calculating the value of

$$\frac{\text{number of successes}}{\text{number of trails (i.e. successes + failures)}} = \frac{r}{n}$$

The probability of event A occurring is defined as

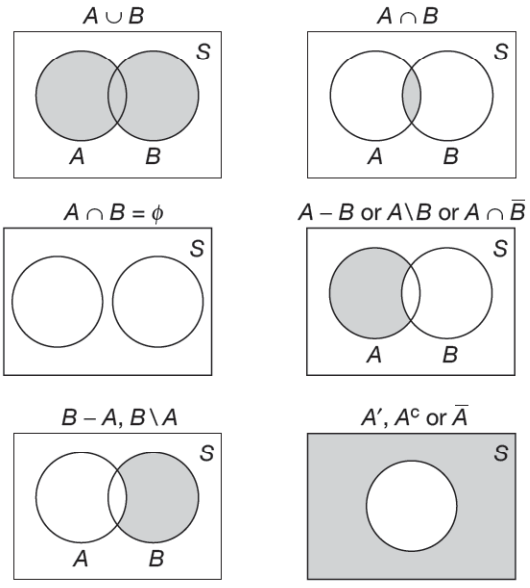
$$P(A) = \lim_{n\to\infty}\left(\frac{r}{n}\right)$$

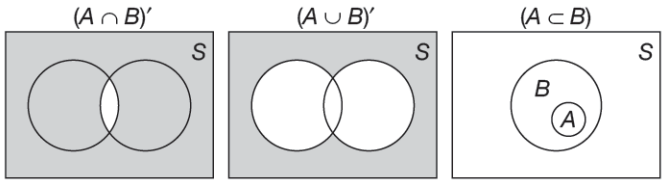
$n \to \infty$ means that the number of trials is large (but what should be taken as 'large' depends on the problem).

Venn Diagrams

A diagram used to illustrate relationships between sets. Commonly, a rectangle represents the universal set and a circle within it represents a given set (all members of the given set are represented by points within the circle). A subset is represented by a circle within a circle and intersection is indicated by overlapping circles.

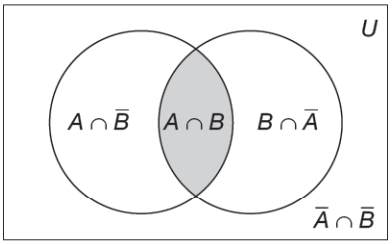
Let S is the sample space of an experiment and A, B are two events corresponding to it:





Addition Theorem

$A \cup B = A + B = A$ or B denotes occurrence of at least A or B.



(This is known as generalized addition theorem)

For two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\left.\begin{array}{ll} P(A \cup B) & \\ P(A + B) & \\ P(A \text{ or } B) & \\ P(\text{occurence of atleast } & \\ A \text{ or } B) & \end{array}\right| = \begin{array}{r} P(A) + P(B) - P(A \cap B) \\ P(A) + P(B \cap \bar{A}) \\ P(B) + P(A \cap \bar{B}) \\ P(A \cap \bar{B}) + P(A \cap B) + P(B \cap \bar{A}) \\ 1 - P(\bar{A} \cap \bar{B}) \\ 1 - P(\overline{A \cup B}) \end{array}$$

Note

1. (i) If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.
   (ii) If A and B are mutually exclusive and exhaustive, then $P(A \cup B) = P(A) + P(B) = 1$
2. P (only A occurs) $= P(A \backslash B) = P(A - B) = P(A \cap B^C) = P(A) - P(A \cap B)$
3. P (either A or B) $= 1 - P$ (neither A nor B)
   i.e. $P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$
4. For any two events A and B
   P (exactly one of A, B occurs) $= P(A \cap \bar{B}) + P(B \cap \bar{A})$
   $\Rightarrow P$(exactly one of A, B occurs) $= P(A) + P(B) - 2P(A \cap B)$
   $= P(A \cup B) - P(A \cap B) = P(A^C \cup B^C) - P(A^C \cap B^C)$
5. $P(A \cap B) \le P(A)$, $P(B) \le P(A \cup B) \le P(A) + P(B)$

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De Morgan's Law

If A and B are two subsets of a universal set U, then

(i) $(A \cup B)^C = A^C \cap B^C$ and
(ii) $(A \cap B)^C = A^C \cup B^C$

Note

(a) $(A \cup B \cup C)^C = A^C \cap B^C \cap C^C$ and $(A \cap B \cap C)^C = A^C \cup B^C \cup C^C$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example 50

A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail, we throw a die. Describe the sample space of this experiment.

Solution

Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is $S = \{HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T_1, T_2, T_3, T_4, T_5, T_6\}$. Here HB_i means head on the coin and ball B_i is drawn, HW_i means head on the coin and ball W_i is drawn. Similarly, T_i means tail on the coin and the number i on the die.

Example 51

Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Solution

In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on. Hence, the desired sample space is $S = \{H, TH, TTH, TTTH, TTTTH, \ldots\}$

Example 52

A coin is tossed three times, consider the following events.

A : 'no head appears'
B : 'exactly one head appears'
C : 'at least two heads appear'

Do they form a set of mutually exclusive and exhaustive events?

Solution

The sample space of the experiment is

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Events A, B, and C are given by

$$A = \{TTT\}$$
$$B = \{HTT, THT, TTH\}$$
$$C = \{HHT, HTH, THH, HHH\}$$

Now,

$$A \cup B \cup C = \{TTT, HTT, THT, TTH, HHT, HTH, THH, HHH\} = S$$

Therefore A, B, and C are exhaustive events. Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$. Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive. Hence, A, B, and C form a set of mutually exclusive and exhaustive events.

Example 53

If the letters of INTERMEDIATE are arranged, then the odds in favour of the event that no two 'E's occur together, are:

(A) $\frac{6}{5}$ **(B)** $\frac{5}{6}$
(C) $\frac{2}{9}$ **(D)** none of these

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Solution

$I \to 2$, $N \to 1$, $T \to 2$, $E \to 3$, $R \to 1$, $M \to 1$, $D \to 1$, A $\to 1$ (3'E's, Rest 9 letters)

First arrange rest of the letters $= \frac{9!}{2!\,2!}$,

Now 3'E's can be placed by ${}^{10}C_3$ ways, so favourable cases $= \frac{9!}{2!\,2!} \times {}^{10}C_3 = 3 \times 10!$

Total cases $= \frac{12!}{2!\,2!\,3!} = \frac{11}{2} \times 10!$; Non-favourable cases $= \left(\frac{11}{2} - 3\right) \times 10! = \frac{5}{2} \times 10!$

Odds in favour of the event $= \frac{3}{5/2} = \frac{6}{5}$ **Ans.**

Example 54

From a group of 10 persons consisting of 5 lawyers, 3 doctors, and 2 engineers, four persons are selected at random. The probability that the selection contains at least one of each category is:

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$
(C) $\frac{2}{3}$ **(D)** none of these

Solution

$$n(S) = {}^{10}C_4 = 210.$$

$$n(E) = {}^5C_2 \times {}^3C_1 \times {}^2C_1 + {}^5C_1 \times {}^3C_2 \times {}^2C_1 + {}^5C_1 \times {}^3C_1 \times {}^2C_2 = 105$$

$$\therefore \quad P(E) = \frac{105}{210} = \frac{1}{2}. \qquad \textbf{Ans.(A)}$$

Example 55

If four cards are drawn at random from a pack of fifty-two playing cards, find the probability that at least one of them is an ace.

Solution

If A is a combination of four cards containing at least one ace (i.e., either one ace, or two aces, or three aces or four aces) then $\bar{A}$ is a combination of four cards containing no aces.

Now

$$P(\bar{A}) = \frac{\text{Number of combinations of four cards with no aces}}{\text{Total number of combinations of four cards}}$$

$$= {}^{48}C_4 / {}^{52}C_4 = 0.72$$

Using $P(A) + P(\bar{A}) = 1$ we have

$$P(A) = 1 - P(\bar{A}) = 1 - 0.72 = 0.28$$

Example 56

A bag contains n white and n red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is $2^n/({}^{2n}C_n)$.

Solution

Let S be the sample space and E be the event that each of the n pairs of balls drawn consists of one white and one red ball.

$$\therefore \quad n(S) = ({}^{2n}C_2)\,({}^{2n-2}C_2)\,({}^{2n-4}C_2) \ldots ({}^4C_2)({}^2C_2)$$

$$= \left\{\frac{(2n)(2n-1)}{1.2}\right\}\left\{\frac{(2n-2)(2n-3)}{1.2}\right\} \left\{\frac{(2n-4)(2n-5)}{1.2}\right\} \ldots \left\{\frac{4.3}{1.2}\right\}.\left\{\frac{2.1}{1.2}\right\}$$

$$= \frac{1.2.3.4 \ldots (2n-1)2n}{2^n} = \frac{2n!}{2^n}$$

and

$$n(E) = ({}^nC_1 \cdot {}^nC_1)\,({}^{n-1}C_1 \cdot {}^{n-1}C_1)\,({}^{n-2}C_1 \cdot {}^{n-2}C_1) \ldots ({}^2C_1 \cdot {}^2C_1)\,({}^1C_1 \cdot {}^1C_1)$$

$$= n^2 \cdot (n-1)^2 \cdot (n-2)^2 \ldots 2^2 \cdot 1^2 = [1.2.3 \ldots (n-1)n]^2$$

$$= (n!)^2$$

$\therefore$ Required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{(n!)^2}{(2n)!/2^n} = \frac{2^n}{\frac{2n!}{(n!)^2}} = \frac{2^n}{{}^{2n}C_n} \qquad \textbf{Ans.}$$

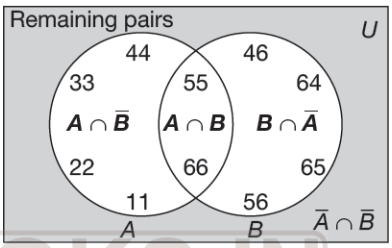
Example 57

Let us conduct an experiment of tossing a pair of dice. Two events defined on the experiment are

Two events defined on the experiment are

A : getting a doublet {11, 22, 33, 44, 55, 66}

B : getting total score of 10 or more {64, 46, 55, 56, 65, 66}



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Example 58

Given two events A and B. If odds against A are as 2 : 1 and those in favour of $A \cup B$ are as 3 : 1, then find the range of $P(B)$.

Solution

Clearly $P(A) = 1/3$, $P(A \cup B) = 3/4$.

Now, $\quad P(B) \le P(A \cup B)$

$\Rightarrow \quad P(B) \le 3/4$

Also, $\quad P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$\Rightarrow \quad P(B) \ge P(A \cup B) - P(A) \qquad (\because P(A \cap B) \ge 0)$

$\Rightarrow \quad P(B) \ge 3/4 - 1/3$

$\Rightarrow \quad P(B) \ge \frac{5}{12}$

$\Rightarrow \quad \frac{5}{12} \le P(B) \le \frac{3}{4}$ **Ans.**

Example 59

If A and B are two events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$ and $P(A^c) = \frac{2}{3}$. Then find:

(i) $P(A)$ **(ii)** $P(B)$

(iii) $P(A \cap B^c)$ **(iv)** $P(A^c \cap B)$

Solution

$$P(A) = 1 - P(A^c) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$P(B) = P(A \cup B) + P(A \cap B) - P(A) = \frac{3}{4} + \frac{1}{4} - \frac{1}{3} = \frac{2}{3}$$

$$P(A \cap B^c) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$P(A^c \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$ **Ans.**

Know Your Progress - 10

1. **(i)** Two balls are drawn from a bag containing 2 Red and 3 Black balls, write sample space of this experiment.

(ii) Out of 2 men and 3 women a team of two persons is to be formed such that there is exactly one man and one woman. Write the sample space of this experiment.

(iii) A coin in tossed and if head comes up, a die is thrown. But if tail comes up, the coin is tossed again. Write the sample space of this experiment.

(iv) In a toss of a die, consider following events:

A : An even number turns up.

B : A prime number turns up.

These events are:

(A) Equally likely events

(B) Mutually exclusive events

(C) Exhaustive events

(D) None of these

2. **(i)** A coin is tossed successively three times. Find the probability of getting exactly one head or two heads.

(ii) A bag contains 5 red and 4 green balls. Four balls are drawn at random then find the probability that two balls are of red and two balls are of green colour.

(iii) Two natural numbers are selected at random, find the probability that their sum is divisible by 10.

(iv) Five card are drawn successively from a pack of 52 cards with replacement. Find the probability that there is at least one Ace.

3. **(i)** Draw Venn diagram of

(a) $(A^C \cap B^C) \cup (A \cap B)$

(b) $B^C \cup (A^C \cap B)$

(ii) If A and B are two mutually exclusive events, then:

(A) $P(A) \leq P(\overline{B})$

(B) $P(\overline{A} \cap \overline{B}) = P(\overline{A}) - P(B)$

(C) $P(\overline{A} \cup \overline{B}) = 0$

(D) $P(\overline{A} \cap B) = P(B)$

(iii) A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

(iv) In a class of 125 students, 70 passed in English, 55 in mathematics and 30 in both. Find the probability that a student selected at random from the class has passed in

(a) at least one subject

(b) only one subject.

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. If A and B are two sets, then $A \cap (A \cup B)'$ is equal to:
(A) A **(B)** B
(C) ϕ **(D)** none of these

2. If A is any set, then:
(A) $A \cup A' = \phi$ **(B)** $A \cup A' = U$
(C) $A \cap A' = U$ **(D)** none of these

3. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 5\}$, $B = \{6, 7\}$ then $A \cap B'$ is:
(A) B' **(B)** A
(C) A' **(D)** B

4. If A and B are two sets, then $A \cup B = A \cap B$ iff:
(A) $A \subseteq B$ **(B)** $B \subseteq A$
(C) $A = B$ **(D)** none of these

5. Let A and B be two sets in the universal set. Then $A - B$ equals:
(A) $A \cap B'$ **(B)** $A' \cap B$
(C) $A \cap B$ **(D)** none of these

6. Two sets A, B are disjoint iff:
(A) $A \cup B = \phi$ **(B)** $A \cap B \neq \phi$
(C) $A \cap B = \phi$ **(D)** None of these

7. If $A \subseteq B$, then $A \cap B$ is equal to:
(A) A **(B)** B
(C) A' **(D)** B'

8. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to:
(A) A **(B)** B
(C) A' **(D)** B'

9. If A and B are not disjoint, then $n(A \cup B)$ is equal to:
(A) $n(A) + n(B)$
(B) $n(A) + n(B) - n(A \cap B)$
(C) $n(A) + n(B) + n(A \cap B)$
(D) $n(A) \cdot n(B)$

10. If $A = \{2, 4, 5\}$, $B = \{7, 8, 9\}$ then $n(A \times B)$ is equal to:
(A) 6 **(B)** 9
(C) 3 **(D)** 0

11. Let A and B be two sets such that $n(A) = 70$, $n(B) = 60$ and $n(A \cup B) = 110$. Then $n(A \cap B)$ is equal to:
(A) 240 **(B)** 20
(C) 100 **(D)** 120

12. Which set is the subset of all given sets?
(A) $\{1, 2, 3, 4, \ldots\}$ **(B)** $\{1\}$
(C) $\{0\}$ **(D)** $\{\ \}$

13. If $Q = \left\{x : x = \frac{1}{y}, \text{ where } y \in N\right\}$, then:
(A) $0 \in Q$ **(B)** $1 \in Q$
(C) $2 \in Q$ **(D)** $\frac{2}{3} \in Q$

14. $A = \{x : x \neq x\}$ represents:
(A) $\{0\}$ **(B)** $\{\ \}$
(C) $\{1\}$ **(D)** $\{x\}$

15. Which of the following statements is true?
(A) $3 \subseteq \{1, 3, 5\}$ **(B)** $3 \in \{1, 3, 5\}$
(C) $\{3\} \in \{1, 3, 5\}$ **(D)** $\{3, 5\} \in \{1, 3, 5\}$

16. Which of the following is a null set?
(A) $A = \{x : x > 1 \text{ and } x < 1]$
(B) $B = \{x : x + 3 = 3\}$
(C) $C = \{\phi\}$
(D) $D = \{x : x \geq 1 \text{ and } x \leq 1\}$

17. $P(A) = P(B) \Rightarrow$
(A) $A \subseteq B$ **(B)** $B \subseteq A$
(C) $A = B$ **(D)** none of these

18. In a recent survey (conducted by *HLL*) of 1,000 houses, washing machine, vacuum cleaners, and refrigerators were counted. Each house had at least one of these products. 400 had no refrigerators, 380 had no vacuum cleaners, 542 had no washing machines. 294 had both a vacuum cleaner and washing machines, 277 had both a vacuum cleaner and a refrigerator, and 120 had both a refrigerator and a washing machine. How many had only a vacuum cleaner?
(A) 132 **(B)** 234
(C) 342 **(D)** 62

19. From 50 students taking examinations in Mathematics, Physics, and Chemistry, 37 passed Mathematics, 24 Physics, and 43 Chemistry. At most, 19 passed Mathematics and Physics, at most 29 passed Mathematics

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and Chemistry and at most 20 passed Physics and Chemistry. The largest possible number that could have passed all three examinations is:

(A) 11 **(B)** 12
(C) 13 **(D)** 14

20. Let Z be the set of all integers and $A = \{(a, b) : a^2 + 3b^2 = 28, a, b \in Z\}$ and $B = \{(a, b) : a > b, a, b \in Z\}$. Then, the number of elements in $A \cap B$, is:

(A) 2 **(B)** 4
(C) 6 **(D)** 5

21. In a class of 25 students, at least one of mathematics or statistics is taken by everybody. 12 have taken mathematics, 8 have taken mathematics but not statistics. Find the difference in the number of students who have taken mathematics and statistics and those who have taken statistics but not maths?

(A) 9 **(B)** 10
(C) 18 **(D)** 8

22. In a class of 200 students, 70 played cricket, 60 played hockey, and 80 played football. Thirty played cricket and football, 30 played hockey and football, and 40 played cricket and hockey. Find the maximum number of people playing all the three games and also the minimum number of people playing at least one game?

(A) 200, 100 **(B)** 30, 110
(C) 30, 120 **(D)** none of these

23. If class with n students is organized into four groups keeping the following conditions:
Each student belongs to exactly two groups and
Each pair of groups has exactly one student in common.
What is the value of n?

(A) $n = 11$ **(B)** $n = 7$
(C) $n = 9$ **(D)** $n = 6$

24. If $A = \{1, 2, 3, 4\}$, then the number of subsets of set A containing element 3, is:

(A) 24 **(B)** 28
(C) 8 **(D)** 16

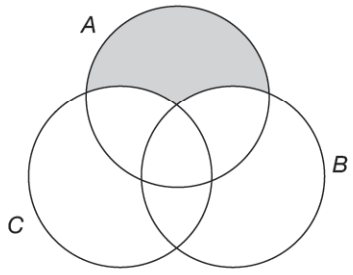
25. Sets A and B have 3 and 6 elements, respectively. What can be the minimum number of elements in $A \cup B$?

(A) 3 **(B)** 6
(C) 9 **(D)** 18

26. In a college of 300 students, every student reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers is:

(A) at least 30 **(B)** at most 20
(C) exactly 25 **(D)** none of these

27. The shaded region in the given figure is:



(A) $A \cap (B \cup C)$ **(B)** $A \cup (B \cap C)$
(C) $A \cap (B - C)$ **(D)** $A - (B \cup C)$

28. Let $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A' \cap B') =$

(A) 400 **(B)** 600
(C) 300 **(D)** 200

29. If $A = \{1, 2, 3, 4, 5\}$, then the number of proper subsets of A is:

(A) 120 **(B)** 30
(C) 31 **(D)** 32

30. If $A = \{(x, y) : x^2 + y^2 = 25\}$ and $B = \{(x, y) : x^2 + 9y^2 = 144\}$ then $A \cap B$ contains:

(A) one point **(B)** three points
(C) two points **(D)** four points

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31. A class has 175 students. The following data shows the number of students obtaining one or more subjects. Mathematics 100; Physics 70; Chemistry 40; Mathematics and Physics 30; Mathematics and Chemistry 28; Physics and Chemistry 23; Mathematics, Physics and Chemistry 18. How many students have offered Mathematics alone ?

(A) 35 **(B)** 48
(C) 60 **(D)** 22

32. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then:

(A) $B = C$ **(B)** $A \cap B = \phi$
(C) $A = B$ **(D)** $A = C$

33. Let $S = \{x \in R : x \geq 0 \text{ and } 2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then S

(A) contains exactly four elements.
(B) is an empty set.
(C) contains exactly one element.
(D) contains exactly two elements.

34. Number of natural numbers between 100 and 1000 such that at least one of their digits is 7, is
(A) 225 **(B)** 243
(C) 252 **(D)** none

35. Number of 4 digit numbers of the form $N = abcd$ which satisfy following three conditions:
(i) $4000 \le N < 6000$
(ii) N is multiple of 5
(iii) $3 \le b < c \le 6$ is equal to
(A) 12 **(B)** 18
(C) 24 **(D)** 48

36. How many of the 900 three digit numbers have at least one even digit?
(A) 775 **(B)** 875
(C) 450 **(D)** 750

37. The number of different seven digit numbers that can be written using only three digits 1, 2 and 3 under the condition that the digit 2 occurs exactly twice in each number is
(A) 672 **(B)** 640
(C) 512 **(D)** none

38. Out of seven consonants and four vowels, the number of words of six letters, formed by taking four consonants and two vowels is (Assume that each ordered group of letter is a word):
(A) 210 **(B)** 462
(C) 151200 **(D)** 332640

39. Number of 5 digit numbers which are divisible by 5 and each number containing the digit 5, digits being all different is equal to $k(4!)$, the value of k is
(A) 84 **(B)** 168
(C) 188 **(D)** 208

40. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6, and 7 so that digits do not repeat and the terminal digits are even is:
(A) 144 **(B)** 72
(C) 288 **(D)** 720

41. If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, then the rank of the word VARUN is:
(A) 98 **(B)** 99
(C) 100 **(D)** 101

42. A new flag is to be designed with six vertical strips using some or all of the colors yellow, green, blue, and red. Then, the number of ways this can be done such that no two adjacent strips have the same color is:
(A) 12×81 **(B)** 16×192
(C) 20×125 **(D)** 24×216

43. Consider the five points comprising of the vertices of a square and the intersection point of its diagonals. How many triangles can be formed using these points?
(A) 4 **(B)** 6
(C) 8 **(D)** 10

44. A 5 digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways this can be done is:
(A) 3125 **(B)** 600
(C) 240 **(D)** 216

45. Number of permutations of 1, 2, 3, 4, 5, 6, 7, 8, and 9 taken all at a time, such that the digit
1 appearing somewhere to the left of 2
3 appearing to the left of 4 and
5 somewhere to the left of 6, is
(e.g. 815723946 would be one such permutation)
(A) $9 \cdot 7!$ **(B)** $8!$
(C) $5! \cdot 4!$ **(D)** $8! \cdot 4!$

46. The number of ways in which 5 different books can be distributed among 10 people if each person can get at most one book is :
(A) 252 **(B)** 10^5
(C) 5^{10} **(D)** ${}^{10}C_5 \cdot 5!$

47. 5 Indian and 5 American couples meet at a party and shake hands. If no wife shakes hands with her own husband and no Indian wife shakes hands with a male, then the number of handshakes that takes place in the party is:
(A) 95 **(B)** 110
(C) 135 **(D)** 150

48. A student has to answer 10 out of 13 questions in an examination. The number of ways in which he can answer if he must answer at least 3 of the first five questions is:
(A) 276 **(B)** 267
(C) 80 **(D)** 1200

49. The number of n digit numbers that consists of the digits 1 and 2 only if each digit is to be used at least once, is equal to 510 then n is equal to
(A) 7 **(B)** 8
(C) 9 **(D)** 10

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50. There are counters available in x different colors. The counters are all alike except for the color. The total number of arrangements consisting of y counters, assuming sufficient number of counters of each color, if no arrangement consists of all counters of the same color is :

(A) $x^y - x$ **(B)** $x^y - y$
(C) $y^x - x$ **(D)** $y^x - y$

51. A question paper on mathematics consists of twelve questions divided into three parts A, B, and C, each containing four questions. In how many ways can an examinee answer five questions, selecting atleast one from each part.

(A) 624 **(B)** 208
(C) 1248 **(D)** 2304

52. If m denotes the number of 5 digit numbers if each successive digits are in their descending order of magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude then $(m - n)$ has the value

(A) ${}^{10}C_4$ **(B)** ${}^{9}C_5$
(C) ${}^{10}C_3$ **(D)** ${}^{9}C_3$

53. A rack has 5 different pairs of shoes. The number of ways in which 4 shoes can be chosen from it, so that there will be no complete pair is:

(A) 1920 **(B)** 200
(C) 110 **(D)** 80

54. Number of ways in which 8 people can be arranged in a line if A and B must be next each other and C must be somewhere behind D, is equal to

(A) 10080 **(B)** 5040
(C) 5050 **(D)** 10100

55. The kindergarten teacher has 25 kids in her class. She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Then the number of visits, the teacher makes to the garden exceeds that of a kid by :

(A) ${}^{25}C_5 - {}^{24}C_5$ **(B)** ${}^{24}C_5$
(C) ${}^{24}C_4$ **(D)** none

56. Seven different coins are to be divided amongst three persons. If no two of the persons receive the same number of coins but each receives at least one coin and none is left over, then the number of ways in which the division may be made is

(A) 420 **(B)** 630
(C) 710 **(D)** none

57. The number of ways in which 8 distinguishable apples can be distributed among 3 boys such that every boy should get at least 1 apple and at most 4 apples is $K \cdot {}^{7}P_3$ where K has the value equal to

(A) 14 **(B)** 66
(C) 44 **(D)** 22

58. An old man while dialing a 7 digit telephone number remembers that the first four digits consists of one 1's, one 2's and two 3's. He also remembers that the fifth digit is either a 4 or 5 while has no memorizing of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. Maximum number of distinct trials he has to try to make sure that he dials the correct telephone number, is

(A) 360 **(B)** 240
(C) 216 **(D)** none

59. Number of ways in which 9 different toys be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more, is:

(A) $\dfrac{(5!)^2}{8}$ **(B)** $\dfrac{9!}{2}$
(C) $\dfrac{9!}{3!(2!)^3}$ **(D)** none

60. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row , if no two of them are consecutive. If, $P_{n+1} - P_n = 15$ then the value of 'n' is:

(A) 7 **(B)** 8
(C) 9 **(D)** 10

61. There are six periods in each working day of a school. Number of ways in which 5 subjects can be arranged if each subject is allotted at least one period and no period remains vacant is

(A) 210 **(B)** 1800
(C) 360 **(D)** 3600

62. Number of ways in which 7 green bottles and 8 blue bottles can be arranged in a row if exactly 1 pair of green bottles is side by side, is (assume all bottles to be alike except for the color).

(A) 84 **(B)** 360
(C) 504 **(D)** none

63. There are 10 red balls of different shades and 9 green balls of identical shades. Then the number of arranging them in a row so that no two green balls are together is

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(A) $(10!) \cdot {}^{11}P_9$ **(B)** $(10!) \cdot {}^{11}C_9$
(C) 10! **(D)** 10! 9!

64. A gentleman invites a party of $m + n$ $(m \neq n)$ friends to a dinner and places m at one table T_1 and n at another table T_2, the table being round. If not all people shall have the same neighbor in any two arrangement, then the number of ways in which he can arrange the guests, is

(A) $\frac{(m+n)!}{4mn}$ **(B)** $\frac{1}{2}\frac{(m+n)!}{mn}$
(C) $2\frac{(m+n)!}{mn}$ **(D)** none

65. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter, then the number of ways in which they can travel, is
(A) 91 **(B)** 182
(C) 126 **(D)** 3920

66. One hundred management students who read at least one of the three business magazines are surveyed to study the readership pattern. It is found that 80 read Business India, 50 read Business world, and 30 read Business Today. Five students read all the three magazines. How many read exactly two magazines?
(A) 50 **(B)** 10
(C) 95 **(D)** 65

67. Number of ciphers at the end of ${}^{2002}C_{1001}$ is
(A) 0 **(B)** 1
(C) 2 **(D)** 200

68. Three vertices of a convex n sided polygon are selected. If the number of triangles that can be constructed such that none of the sides of the triangle is also the side of the polygon is 30, then the polygon is a
(A) Heptagon **(B)** Octagon
(C) Nonagon **(D)** Decagon

69. Number of 5 digit numbers divisible by 25 that can be formed using only the digits 1, 2, 3, 4, 5, and 0 taken five at a time is
(A) 2 **(B)** 32
(C) 42 **(D)** 52

70. Let P_n denotes the number of ways of selecting 3 people out of 'n' sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, then 'n' is equal to:
(A) 8 **(B)** 9
(C) 10 **(D)** 12

71. Let m denote the number of ways in which 4 different books are distributed among 10 persons, each receiving none or one only and let n denote the number of ways of distribution if the books are all alike. Then:
(A) $m = 4n$ **(B)** $n = 4m$
(C) $m = 24n$ **(D)** none

72. Number of 7 digit numbers the sum of whose digits is 61 is:
(A) 12 **(B)** 24
(C) 28 **(D)** none

73. There are 2 identical white balls, 3 identical red balls, and 4 green balls of different shades. The number of ways in which they can be arranged in a row so that at least one ball is separated from the balls of the same color, is:
(A) 6 (7! – 4!) **(B)** 7 (6! – 4!)
(C) 8! – 5! **(D)** none

74. Number of ways in which four different toys and five indistinguishable marbles can be distributed between Amar, Akbar, and Anthony, if each child receives at least one toy and one marble, is
(A) 42 **(B)** 100
(C) 150 **(D)** 216

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75. In a unique hockey series between India and Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw is:
(A) 126 **(B)** 252
(C) 225 **(D)** none

76. There are 100 different books in a shelf. Number of ways in which 3 books can be selected so that no two of which are neighbors is
(A) ${}^{100}C_3 - 98$ **(B)** ${}^{97}C_3$
(C) ${}^{96}C_3$ **(D)** ${}^{98}C_3$

77. Number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$, is
(A) 150 **(B)** 270
(C) 420 **(D)** 1024

78. There are counters available in 3 different colors (at least four of each color). Counters are all alike except for the color. If 'm' denotes the number of arrangements of four counters if no arrangement consists of counters of same color and 'n' denotes the corresponding figure when every arrangement consists of counters of each color, then:
(A) $m = 2n$ **(B)** $6m = 13n$
(C) $3m = 5n$ **(D)** $5m = 3n$

79. There are 12 books on Algebra and Calculus in our library, the books of the same subject being different. If the number of selections each of which consists of 3 books on each topic is greatest then the number of books of Algebra and Calculus in the library are, respectively:

(A) 3 and 9 **(B)** 4 and 8
(C) 5 and 7 **(D)** 6 and 6

80. There are $(p + q)$ different books on different topics in Mathematics. $(p \neq q)$
If L = The number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.
M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.
N = The number of ways in which these books are divided into two groups of p books and q books then,

(A) $L = M = N$ **(B)** $L = 2M = 2N$
(C) $2L = M = 2N$ **(D)** $L = M = 2N$

81. Number of ways in which 5 A's and 6 B's can be arranged in a row that reads the same backwards and forwards, is

(A) 6 **(B)** 8
(C) 10 **(D)** 12

82. A person writes letters to his 5 friends and addresses the corresponding envelopes. Number of ways in which the letters can be placed in the envelope, so that at least two of them are in the wrong envelopes, is,

(A) 1 **(B)** 2
(C) 118 **(D)** 119

83. **MATCH THE COLUMN:**

Column–I	Column–II
(A) Number of increasing permutations of m symbols are there from the n set numbers $\{a_1, a_2, \ldots, a_n\}$ where the order among the numbers is given by $a_1 < a_2 < a_3 < \ldots a_{n-1} < a_n$ is	(P) n^m
(B) There are m men and n monkeys. Number of ways in which every monkey has a master, if a man can have any number of monkeys	(Q) ${}^mC_n$
(C) Number of ways in which n red balls and $(m - 1)$ green balls can be arranged in a line, so that no two red balls are together, is (balls of the same color are alike)	(R) ${}^nC_m$
(D) Number of ways in which 'm' different toys can be distributed in 'n' children if every child may receive any number of toys, is	(S) m^n

(*Continued*)

84. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidate and 4 are to be elected. If a voter votes for at least one candidates, then the number of ways in which he can vote is

(A) 385 **(B)** 1110
(C) 5040 **(D)** 6210

85. The set $S = \{1, 2, 3, \ldots, 12\}$ is to partitioned into three sets A, B, C of equal size. Thus, $A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$, then number of ways to partition S are:

(A) $\dfrac{12!}{3!(3!)^4}$ **(B)** $\dfrac{12!}{(4!)^3}$

(C) $\dfrac{12!}{(3!)^4}$ **(D)** $\dfrac{12!}{3!(4!)^3}$

86. In a shop there are five types of ice-creams available. A child buys six ice-creams.

Statement-1: The number of different ways the child can buy the six ice-creams is ${}^{10}C_5$.

Statement-2: The number of different ways the child can buy the six ice-creams is equal to the number of different ways of arranging 6 A's and 4 B's in a row.

(A) Statement-1 is false, Statement-2 is true
(B) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(D) Statement-1 is true, Statement-2 is false

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87. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangements is:

(A) At least 750 but less than 1000
(B) At least 1000
(C) Less than 500
(D) At least 500 but less than 750

88. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is:

(A) 3 **(B)** 36
(C) 66 **(D)** 108

89. **Statement-1:** The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^{9}C_{3}$.
Statement-2: The number of ways of choosing any 3 places from 9 different places is ${}^{9}C_{3}$.

(A) Statement-1 is true, Statement-2 is false.
(B) Statement-1 is false, Statement-2 is true
(C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(D) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

90. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then:

(A) $N > 190$ **(B)** $N \leq 100$
(C) $100 < N \leq 140$ **(D)** $140 < N \leq 190$

91. Assuming the balls to be identical except for difference in colors, the number of ways in which one or more balls can be selected from 10 white, 9 green, and 7 black balls is:

(A) 879 **(B)** 880
(C) 629 **(D)** 630

92. Let A and B be two sets containing 2 elements and 4 elements, respectively. The number of subsets of $A \times B$ having 3 or more elements is

(A) 256 **(B)** 220
(C) 219 **(D)** 211

93. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is:

(A) 7 **(B)** 5
(C) 10 **(D)** 8

94. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41), and (41, 0) is:

(A) 820 **(B)** 780
(C) 901 **(D)** 861

95. Let A and B be two sets containing four and two elements, respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is:

(A) 275 **(B)** 510
(C) 219 **(D)** 256

96. The number of integers greater than 6000 that can be formed, using the digits 3,5,6,7 and 8 without repetition, is:

(A) 120 **(B)** 72
(C) 216 **(D)** 192

97. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary; then the position of the word SMALL is:

(A) 58^{th} **(B)** 46^{th}
(C) 59^{th} **(D)** 52^{nd}

98. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is:

(A) 484 **(B)** 485
(C) 468 **(D)** 469

PART – II KVPY Level Problems

1. In a certain strange language, words are written with letters from the following six-letter alphabet: A, G, K, N, R, U. Each word consists of six letters and none of the letters repeat. Each combination of these six letters is a word in this language. The word "KANGUR" remains in the dictionary at,

(A) 248^{th} **(B)** 247^{th}
(C) 246^{th} **(D)** 253^{rd}

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2. All possible three-digit even numbers that can be formed with the condition that if 5 is one of the digits, then 7 is the next digit is:
(A) 5 **(B)** 325
(C) 345 **(D)** 365

3. Number of three-digit numbers in which the digit at hundredth's place is greater than the other two digit is:
(A) 285 **(B)** 281
(C) 240 **(D)** 204

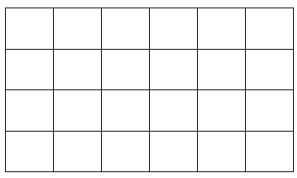
4. The number of three digit numbers having only two consecutive digits identical is:
(A) 153 **(B)** 162
(C) 180 **(D)** 161

5. A committee of 5 is to be chosen from a group of 9 people. Number of ways in which it can be formed if two particular persons either serve together or not at all and two other particular persons refuse to serve with each other, is
(A) 41 **(B)** 36
(C) 47 **(D)** 76

6. There are m points on a straight-line AB and n points on the line AC none of them being the point A. Triangles are formed with these points as vertices, when
(i) A is excluded
(ii) A is included. The ratio of number of triangles in the two cases is:
(A) $\frac{m+n-2}{m+n}$ **(B)** $\frac{m+n-2}{m+n-1}$
(C) $\frac{m+n-2}{m+n+1}$ **(D)** $\frac{m(n-1)}{(m+1)(n+1)}$

7. There are 10 straight lines in a plane, such that no 3 are concurrent and no 2 are parallel to each other. If points of intersection of above lines are joined, then maximum number of lines thus formed are (including old lines)
(A) 610 **(B)** 620
(C) 630 **(D)** 640

8. Number of rectangles in the grid shown that are not squares is



(A) 160 **(B)** 162
(C) 170 **(D)** 185

9. Six people are going to sit in a row on a bench. A and B are adjacent, C does not want to sit adjacent to D. E and F can sit anywhere. Number of ways in which these six people can be seated, is
(A) 200 **(B)** 144
(C) 120 **(D)** 56

10. Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the maximum number of circles that can be drawn so that each contains atleast three of the given points is:
(A) 216 **(B)** 156
(C) 172 **(D)** none

11. Number of ways in which these 16 players can be divided into four equal groups, such that when the best player is selected from each group, P_6 is one among them, is $(k)\ \frac{12!}{(4!)^3}$. The value of k is:
(A) 36 **(B)** 24
(C) 18 **(D)** 20

12. The number of ways of choosing a committee of 2 women and 3 men from 5 women and 6 men, if Mr. A refuses to serve on the committee if Mr. B is a member and Mr. B can only serve, if Miss C is the member of the committee, is
(A) 60 **(B)** 84
(C) 124 **(D)** none

13. Product of all the even divisors of $N = 1000$, is
(A) $32 \cdot 10^2$ **(B)** $64 \cdot 2^{14}$
(C) $64 \cdot 10^{18}$ **(D)** $128 \cdot 10^6$

14. Two classrooms A and B having capacity of 25 and $(n - 25)$ seats, respectively. A_n denotes the number of possible seating arrangements of room 'A', when 'n' students are to be seated in these rooms, starting from room 'A' that is to be filled up full to its capacity. If $A_n - A_{n-1} = 25!\ ({}^{49}C_{25})$, then '$n$' equals:
(A) 50 **(B)** 48
(C) 49 **(D)** 51

15. A guardian with 6 wards wishes everyone of them to study either Law or Medicine or Engineering. Number of ways in that he can make up his mind with regard to the education of his wards if every one of them be fit for any of the branches to study, and at least one child is to be sent in each discipline is:
(A) 120 **(B)** 216
(C) 729 **(D)** 540

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For Question Nos. 16 to 18
Consider the word W = MISSISSIPPI

16. If N denotes the number of different selections of 5 letters from the word W = MISSISSIPPI then N belongs to the set
(A) {15, 16, 17, 18, 19}
(B) {20, 21, 22, 23, 24}
(C) {25, 26, 27, 28, 29}
(D) {30, 31, 32, 33, 34}

17. Number of ways in which the letters of the word W can be arranged if at least one vowel is separated from rest of the vowels
(A) $\frac{8!\cdot 161}{4!\cdot 4!\cdot 2!}$ **(B)** $\frac{8!\cdot 161}{4\cdot 4!\cdot 2!}$
(C) $\frac{8!\cdot 161}{4!\cdot 2!}$ **(D)** $\frac{8!}{4!\cdot 2!}\cdot\frac{165}{4!}$

18. If the number of arrangements of the letters of the word W if all the S's and P's are separated is $(K)\left(\frac{10!}{4!4!}\right)$, then K equals:
(A) $\frac{6}{5}$ **(B)** 1
(C) $\frac{4}{3}$ **(D)** $\frac{3}{2}$

Paragraph for Question Nos. 19 to 21
16 players P_1, P_2, P_3, … P_{16}take part in a tennis tournament. Lower suffix player is better than any higher suffix player. These players are to be divided into 4 groups each comprising of 4 players and the best from each group is selected for semifinals.

19. Number of ways in which 16 players can be divided into four equal groups, is
(A) $\frac{35}{27}\prod_{r=1}^{8}(2r-1)$ **(B)** $\frac{35}{24}\prod_{r=1}^{8}(2r-1)$
(C) $\frac{35}{52}\prod_{r=1}^{8}(2r-1)$ **(D)** $\frac{35}{6}\prod_{r=1}^{8}(2r-1)$

20. Number of ways in which they can be divided into 4 equal groups if the players P_1, P_2, P_3, and P_4 are in different groups, is:
(A) $\frac{(11)!}{36}$ **(B)** $\frac{(11)!}{72}$
(C) $\frac{(11)!}{108}$ **(D)** $\frac{(11)!}{216}$

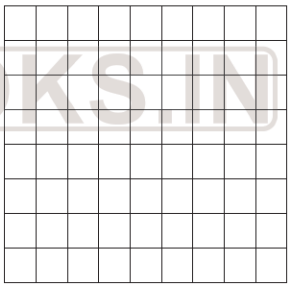
21. How many five digits numbers divisible by 3 can be formed using the digits 0, 1, 2, 3, 4, 7, and 8 if each digit is to be used at most once.
(A) 744 **(B)** 747
(C) 645 **(D)** 644

22. Find the number of four-digit positive integers if the product of their digits is divisible by 3.
(A) 7714 **(B)** 7704
(C) 7707 **(D)** 7717

23. Find the number of words each consisting of 3 consonants and 3 vowels that can be formed from the letters of the word **"Circumference"**.
(A) 22000 **(B)** 22200
(C) 22100 **(D)** 22350

24. Find the number of three elements sets of positive integers $\{a, b, c\}$ such that $a \times b \times c = 2310$.
(A) 36 **(B)** 40
(C) 44 **(D)** 45

Paragraph for Question Nos. 25 and 26
If 10 vertical equispaced (1 cm) lines and 9 horizontal equispaced lines (1 cm) are drawn in a plane as shown in the given figure.



On the basis of above information, answer the following questions:

25. Total number of rectangles with one side odd and one side even are given by:
(A) 600 **(B)** 700
(C) 800 **(D)** 900

26. If squares of odd side length are selected from the above grid, then sum of their areas is equal to:
(A) $\sum_{r=1}^{4}(11-2r)\ (10-2r)\ (2r-1)^2\ \text{cm}^2$
(B) $\sum_{r=1}^{8}(9-2r)(7-2r)(2r+1)^2\ \text{cm}^2$
(C) $\sum_{r=1}^{8}(11-2r)(9-2r)\ (2r+1)^2\ \text{cm}^2$
(D) $\sum_{r=1}^{5}(11+2r)\ (9+2r)\ (2r-1)^2\ \text{cm}^2$

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27. How many 4 digit numbers are there which contains not more than 2 different digits?
(A) 576 **(B)** 574
(C) 572 **(D)** 570

Instruction for Question Nos. 28 to 30:
2 American men; 2 British men; 2 Chinese men and one each of Dutch, Egyptian, French and German persons are to be seated for a round table conference.

28. If the number of ways in which they can be seated if exactly two pairs of persons of same nationality are together is $p(6!)$, then find p.
(A) 50 **(B)** 56
(C) 60 **(D)** 62

29. If the number of ways in which only American pair is adjacent is equal to $q(6!)$, then find q.
(A) 60 **(B)** 64
(C) 67 **(D)** 69

30. If the number of ways in which no two people of the same nationality are together given by $r(6!)$, find r.
(A) 244 **(B)** 247
(C) 241 **(D)** 237

31. For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence 1, 4, 7, 10 …. Find the number of values of k for which S_k contain the term 361.
(A) 36 **(B)** 24
(C) 18 **(D)** 20

Instruction for Question Nos. 32 to 35:
A shop sells 6 different flavors of ice-cream. In how many ways can a customer choose 4 ice-cream cones if

32. They are all of different flavors
(A) 12 **(B)** 15
(C) 18 **(D)** 20

33. They are non necessarily of different flavors
(A) 120 **(B)** 122
(C) 126 **(D)** 128

34. They contain only 3 different flavors
(A) 52 **(B)** 57
(C) 55 **(D)** 60

35. They contain only 2 or 3 different flavors?
(A) 105 **(B)** 103
(C) 101 **(D)** 99

36. How many different ways can 15 Candy bars be distributed between Ram, Shyam, Ghanshyam, and Balram, if Ram cannot have more than 5 candy bars and Shyam must have at least two. Assume all Candy bars to be alike.
(A) 440 **(B)** 435
(C) 438 **(D)** 432

Instruction for Question Nos. 37 to 38:
There are 3 cars of different make available to transport 3 girls and 5 boys on a field trip. Each car can hold up to 3 children. Find

37. The number of ways in which they can be accommodated.
(A) 1650 **(B)** 1670
(C) 1680 **(D)** 1690

38. The numbers of ways in which they can be accommodated if 2 or 3 girls are assigned to one of the cars.
In both the cases internal arrangement of children inside the car is considered to be immaterial.
(A) 1140 **(B)** 1150
(C) 1155 **(D)** 1172

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PART – III PRMO Level Problems

1. A telephone number has the form $ABC - DEF - GHIJ$, where each letter represents a different digit. The digits in each part of the numbers are in decreasing order; that is, $A > B > C$, $D > E > F$, and $G > H > I > J$. Furthermore, D, E, and F are consecutive even digits; G, H, I and J are consecutive odd digits; and $A + B + C = 9$. Find A.

2. Define $n_a!$ for n and a positive to be
$n_a! = n(n-a)(n-2a)(n-3a) \ldots (n-ka)$, where k is the greatest integer for which $n > ka$. Then the quotient $72_8!/18_2!$ is equal to 2^x. Find x.

3. How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory in a 6-period day if no two mathematics courses can be taken in consecutive periods? (what courses the student takes during the other 3 periods is of no concern here).

4. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes,

but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

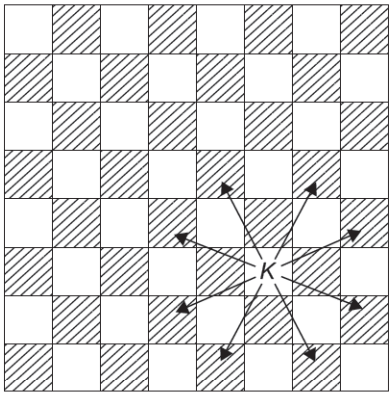
5. Five friends sat in a movie theatre in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions "left" and "right" are from the point of view of the people as they sit in the seats.) During the movie Asha went to the lobby to get some popcorn. When she returned, she found that Bhawna had moved two seats to the right, Chetna had moved one seat to the left, and Divya and Esha had switched seats, leaving an end seat for Asha. In which seat had Asha been sitting before she got up?

6. A palindrome number is the same either read from left to right or right to left, for example. 121 is a palindrome number. Total possible 5-digit palindrome numbers can be written as $10a$. Find the value of a.

7. Five students A, B, C, D, and E form a team to take part in a 5-leg relay competition. If A cannot run the first leg and D cannot run the last leg, how many ways can we arrange them to run the relay?

8. Different positive 3-digit integers are formed from the five digits 1, 2, 3, 5, 7, and repetitions of the digits are allowed as an example, such positive 3-digit integers include 352, 577, 111, etc. Now the sum of all the distinct positive 3-digit integers formed in this way is written as $ABC50$. Find $A + B + C$.

9. As shown in the picture, the knight can move to any of the indicated squares of the 8×8 chessboard in 1 move. If the knight starts form the position shown, find the number of possible landing positions after 20 consecutive moves.



10. There are N number of 11-digit positive integers such that the digits from left to right are non-decreasing. (For example, 12345678999, 55555555555, 23345557889). Find sum of all the digits of N.

11. A four digit number consists of two distinct pairs of repeated digits (for example 2211, 2626, and 7007). Find the total number of such possible numbers that are divisible by 7 or 101 but not both.

12. m and n are two positive integers satisfying $1 < m < n \leq 40$. Find the number of pairs of (m, n) such that their product mn is divisible by 33.

13. How many 7-digit palindromes (numbers that read the same backward as forward) can be formed using the digits 2, 2, 3, 3, 5, 5, 5?

14. Ajay refuses to sit next to either Bharat or Chandan. Deepak refuses to sit next to Enees. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

15. If there are N integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999? For example, both 121 and 211 have this property. Then find the value of $\frac{N-1}{5}$.

16. The numbers 1, 2, 3, 4, 5 are to be arranged in a circle. An arrangement is bad if it is not true that for every n from 1 to 15 one can find a subset of the numbers that appear consecutively on the circle that sum to n. Arrangements that differ only by a rotation or a reflection are considered the same. How many different bad arrangements are there?

17. A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

18. What is the sum of the last two digits of the integer $1! + 2! + 3! + \ldots + 2005!$?

19. Total number of different ways in which 7 different prizes be awarded to 5 students in such a way that each student has at least one prize is P. Find the value of $\frac{\sqrt{P+100}}{2}$.

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20. There are N three digit numbers, lying between 100 and 999 inclusive, have two and only two consecutive digits identical. Then find the value of $\frac{N}{9}$.

21. A set of 16 square blocks in arranged into a 4×4 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

22. There are 100 players in a singles tennis tournament. The tournament is single elimination, meaning that a player who loses a match is eliminated. In the first round, the strongest 28 players are given a bye, and the remaining 72 players are paired off to play. After each round, the remaining players play in the next round. The match continues until only one player remains unbeaten. Find the total number of matches played.

23. A restaurant offers three desserts and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, as main course, and a dessert. What is the least number of main courses that the restaurant should offer so that a customer could have a different dinner each night in the year 2003?

24. A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

25. And the least n such that whenever the elements of the set $\{1, 2, \ldots, n\}$ are coloured red or blue, there always exist x, y, z, w (not necessarily distinct) of the same colour such that $x + y + z = w$.

26. There are n balls in a box, and the balls are numbered 1, 2, 3, …, n respectively. One of the balls is removed from the box, and it turns out that the sum of the numbers on the remaining balls in the box Is 5048. If the number on the ball removed from the box is m, find the value of m.

27. Consider the 800-digit integer

234523452345……0 2345.

The first m digits and the last n digits of the above integer are crossed out so that the sum of the remaining digits $m + n$ is 2345. And the value of $\frac{m+n}{10}$.

28. The total number of Integers between 0 and 10^5 having the digit sum equal to 8 is n then find $5(500 - n)$.

29. How many ordered pairs of Integers (m, n) where $0 < m < n < 2008$ satisfy the equation $2008^2 + m^2 = 2007^2 + n^2$?

30. A set of tiles numbered 1 through 100 is modified repeatedly by the following operation: remove all tiles numbered with a perfect square, and renumber the remaining tiles consecutively starting with 1. How many times must the operation be performed to reduce the number of tiles in the set to one?

31. When 15 is appended to a list of integers, the mean is increased by 2. When list appended to the enlarged list, the mean of the enlarged list is decreased by 1. How many integers were in the original list?

32. Mr. and Mrs. Yadav want to name their baby Yadav so that its monogram (first, middle, and last initials) will be in X alphabetical order with no letters repeated. If total number of possible monograms are X, find $\frac{X}{30}$.

33. Four whole numbers, when added three at a time, give the sums 180, 197, 208 and 222. What is the largest of the four numbers?

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34. Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six tents and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose the chairs?

35. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. Then the least number of distinct values that can occur in the list is N. Find the value of $\frac{N}{5}$.

36. Let n be a positive integer greater than 4 such that the decimal representation of $n!$ ends in k zeros and the decimal representation of $(2n)!$ ends in $3k$ zeros. Lets denote the sum of the four last possible values of n. What is the sum of the digits of s?

37. Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that or each $2 \le i \le 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. If number of such lists are 2^k find k.

38. Anshul, Yogesh, and Sunil listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the boys, there is at least one song liked by those two boys but disliked by the third. If it is possible in $2^k + 4$ different ways, then find k.

39. The players on a basketball team made some three-point shots, some two-point shots, and some one-point free throws. They scored as many points with two-point shots as with three-point shots. Their number of successful free throws was one more than their number of successful two-point shots. The team's total score was 61 points.
How many free throws did they make?

40. There are 52 people in a room. What is the largest value of n such that the statement "At least n people in the room have birthdays falling in the same month" is always true?

41. How many ways can the word MATHEMATICS be partitioned so that each part contains at least one vowel? For example, MA-THE-MATICS, MATHE-MATICS, MATHEM-A-TICS, and MATHEMATICS are such partitions.

42. Four visitors A, B, C, and D arrive at a town that has 5 hotels. In how many ways can they disperse themselves among 5 hotels, if 4 hotels are used to accommodate them.

43. There are 6 roads between A and B and 4 roads between B and C.
(i) In how many ways can one drive from A to C by way of B?
(ii) In how many ways can one drive from A to C and back to A, passing through B on both trips?
(iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.

44. (i) Find the number of four letter word that can be formed from the letters of the word HISTORY. (each letter to be used at most once)
(ii) How many of them contain only consonants?
(iii) How many of them begin and end in a consonant?
(iv) How many of them begin with a vowel?
(v) How many contain the letters Y?
(vi) How many begin with T and end in a vowel?
(vii) How many begin with T and also contain S?
(viii) How many contain both vowels?

45. If repetitions are not permitted
(i) How many 3 digit numbers can be formed from the six digits 2, 3, 5, 6, 7, and 9?
(ii) How many of these are less than 400?
(iii) How many are even?
(iv) How many are odd?
(v) How many are multiples of 5?

46. How many two digit numbers are there in which the tens digit and the units digit are different and odd?

47. Every telephone number consists of 7 digits. How many telephone numbers are there that do not include any other digits but 2, 3, 5, and 7?

48. (a) In how many ways can four passengers be accommodated in three railway carriages, if each carriage can accommodate any number of passengers.
(b) In how many ways four persons can be accommodated in 3 different chairs if each person can occupy only one chair.

49. How many odd numbers of five distinct digits can be formed with the digits 0, 1, 2, 3, 4?

50. Number of ways in which 7 different colors in a rainbow can be arranged if green is always in the middle.

51. Find the number of ways in which the letters of the word "MIRACLE" can be arranged if vowels always occupy the odd places.

52. A letter lock consists of three rings each marked with 10 different letters. Find the number of ways in which it is possible to make an unsuccessful attempt to open the lock.

53. Find the number of ways in which letters of the word VALEDICTORY be arranged so that the vowels may never be separated.

54. If x_1, x_2, $x_3$are the whole numbers and gives remainders 0, 1, 2, respectively, when divided by 3 then total number of different solutions of the equation $x_1 + x_2 + x_3 = 33$ are k, then $\frac{k}{11}$ is equal to

55. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. Find the number of participants and the total numbers of games played in the tournament.

56. 5 boys and 4 girls sit in a straight line. Find the number of ways in which they can be seated if 2 girls are together and the other 2 are also together but separate from the first 2.

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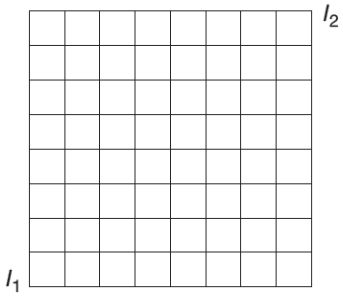
57. An examination paper consists of 12 questions divided into parts A and B-Part-A contains 7 questions and Part-B contains 5 questions. A candidate is required to attempt 8 questions selecting at least 3 from each part. In how many maximum ways can the candidate select the questions?

58. In how many ways can a team of 6 horses be selected out of a stud of 16, so that there shall always be 3 out of $A\ B\ C\ A'\ B'\ C'$, but never $A\ A'$, $B\ B'$ or $C\ C'$ together.

59. During a draw of lottery, tickets bearing numbers 1, 2, 3, …, 40, 6 tickets are drawn out and then arranged in the descending order of their numbers. In how many ways, it is possible to have 4^{th} ticket bearing number 25.

60. Find the number of distinct natural numbers up to a maximum of 4 digits and divisible by 5, which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occurring more than once in each number.

61. On the normal chess board as shown, I_1 and I_2 are two insects that starts moving towards each other. Each insect moving with the same constant speed. Insect I_1 can move only to the right or upward along the lines while the insect I_2 can move only to the left or downward along the lines of the chess board. Find the total number of ways the two insects can meet at same point during their trip.



62. A firm of Chartered Accountants in Bombay has to send 10 clerks to 5 different companies, two clerks in each. Two of the companies are in Bombay and the others are outside. Two of the clerks prefer to work in Bombay while three others prefer to work outside. In how many ways can the assignment be made if the preferences are to be satisfied.

63. In a certain algebraically exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation-combination, and 6 examples on binomial theorem. Number of ways a teacher can select for his pupils at least one but not more than 2 examples from each of these sets, is _____.

64. Find the number of ways in which two squares can be selected from an 8 by 8 chess board of size 1×1 so that they are not in the same row and in the same column.

65. There are 10 seats in a double decker bus, 6 in the lower deck, and 4 on the upper deck. Ten passengers board the bus, of them 3 refuse to go to the upper deck and 2 insist on going up. The number of ways in which the passengers can be accommodated is _____ (assume all seats to be duly numbered)

66. Find the number of permutations of the word "AUROBIND" in which vowels appear in an alphabetical order.

67. In how many different ways a grandfather along with two of his grandsons and four granddaughters can be seated in a line for a photograph so that he is always in the middle and the two grandsons are never adjacent to each other.

68. Define a 'good word' as a sequence of letters that consists only of the letters A, B, and C and in which A never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. If the number of n-letter good words are 384, find the value of n.

69. In how many other ways can the letters of the word **MULTIPLE** be arranged;

(i) without changing the order of the vowels
(ii) keeping the position of each vowel fixed and
(iii) without changing the relative order/position of vowels and consonants.

70. Each of 3 committees has 1 vacancy which is to be filled from a group of 6 people. Find the number of ways the 3 vacancies can be filled if;

(i) Each person can serve on at most 1 committee.
(ii) There is no restriction on the number of committees on which a person can serve.
(iii) Each person can serve on at most 2 committees.

71. How many 6 digits odd numbers greater than 60,0000 can be formed from the digits 5, 6, 7, 8, 9, 0 if

(i) repetitions are not allowed
(ii) repetitions are allowed.

72. Find the number of ways in which the letters of the word 'KUTKUT' can be arranged so that no two alike letters are together.

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73. If as many more words as possible be formed out of the letters of the 'DOGMATIC', then find the number of words in which the relative order of vowels and consonants remain unchanged.

74. Find the number of ways in which 3 distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, \ldots 3^{100}, 3^{101}\}$ so that they form a G.P.

75. (a) How many divisors are there of the number $x = 21600$. Find also the sum of these divisors.
(b) In how many ways the number 7056 can be resolved as a product of 2 factors.
(c) Find the number of ways in which the number 300300 can be split into 2 factors that are relatively prime.
(d) Find the number of positive integers that are divisors of at least one of the numbers 10^{10}; 15^7; 18^{11}.

76. There are 10 different books in a shelf. Find the number of ways in which 3 books can be selected so that exactly two of them are consecutive.

77. Determine the number of paths from the origin to the point (9, 9) in the Cartesian plane that never pass through (5, 5) in paths consisting only of steps going 1 unit North and 1 unit East.

78. There are 20 books on Algebra and Calculus in our library. Prove that the greatest number of selections each of which consists of 5 books on each topic is possible only when there are 10 books on each topic in the library.

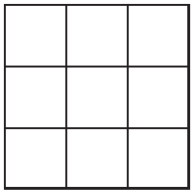
EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. If $U = \{X \mid X$ is a point on straight line $AB\}$, $P = \{M \mid M$ is a point on ray $AB\}$, $Q = \{N \mid N$ is a point on ray $BA\}$ and $R = \{L \mid L$ is point on segment $AB\}$, then which of the following statements is/are true? **[NSEJS-2009]**

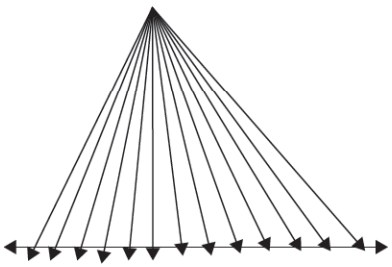
(A) $P \cap Q = R$ **(B)** $P \cap' Q' = Q \cap P$
(C) $[P' \cup Q'] = R$ **(D)** $P \cap Q' = P' \cup Q$

2. Figure shows a square gird of order 3, which of the following is correct formula for the total number of squares in a similar grid of order n. **[NSEJS-2014]**



(A) $\frac{n(n+1)}{2}$ **(B)** $\frac{n^2(n+1)^2}{4}$
(C) $\frac{n(n+1)(2n+1)}{6}$ **(D)** $\frac{n(n+1)(n+2)}{6}$

3. How many triangles are there in this figure? **[NSEJS-2015]**



(A) 50 **(B)** 70
(C) 84 **(D)** 91

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4. Let the number of rectangles formed by 6 horizontal and 4 vertical lines be n and those formed by 5 vertical and 5 horizontal lines be m then we have **[NSEJS-2016]**

(A) $n = m$ **(B)** $n \geq m + 1$
(C) $m \geq n$ **(D)** $m > n + 5$

5. The probability of a point within an equilateral triangle with side 1 – unit lying outside its in-circle (inscribed circle) is: **[NSEJS-2016]**

(A) $1 - \frac{1}{2(\sqrt{3})}$ **(B)** $1 - \frac{\pi}{3\sqrt{3}}$
(C) $1 - \frac{\pi}{2\sqrt{3}}$ **(D)** $1 - \frac{2\pi}{3\sqrt{3}}$

6. If set X consists of three elements then the number of elements in the power set of power set of X is **[NSEJS-2016]**

(A) 3^3 **(B)** 2^3
(C) 3^8 **(D)** 2^8

7. The number of squares formed by 5 vertical and 4 horizontal lines (all are equispaced) is **[NSEJS-2016]**

(A) 60 **(B)** 20
(C) 40 **(D)** 46

8. A $5 \times 5 \times 5$ cube is built using unit cubes. How many different cuboids (that differ in at least one unit cube) can be formed using the same number of unit cubes? **[NSEJS-2018]**

(A) 1000 **(B)** 1728
(C) 2730 **(D)** 3375

9. Two regular polygons of different number of sides are taken. In one of them, its sides are colored red and diagonals are colored green; in the other, sides are colored green and diagonals are colored red. Suppose there are 103 red lines and 80 green lines. The total number of sides the two polygons together have is: **[NSEJS-2018]**

(A) 23 **(B)** 28
(C) 33 **(D)** 38

10. Two unbiased dice are rolled .What is the probability of getting a sum which is neither 7 nor 11? **[NSEJS-2019]**

(A) 7/9 **(B)** 7/18
(C) 2/9 **(D)** 11/18

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PART – II Previous Years' KVPY Problems

1. Let $(1 + 2x)^{20} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$. Then, $3a_0 + 2a_1 + 3a_2 + 2a_3 + 3a_4 + 2a_5 + \ldots + 2a_{19} + 3a_{20}$ equals **[KVPY-2009]**

(A) $\frac{5 \cdot 3^{20} - 3}{2}$ **(B)** $\frac{5 \cdot 3^{20} + 3}{2}$

(C) $\frac{5 \cdot 3^{20} + 1}{2}$ **(D)** $\frac{5 \cdot 3^{20} - 1}{2}$

2. A woman has 10 keys out of which only one opens a lock. She tries the keys one after the another (keeping aside the failed ones) till she succeeds in opening the lock. What is the chance that it is the seventh key that works? **[KVPY-2009]**

(A) $\frac{7}{10}$ **(B)** $\frac{1}{2}$

(C) $\frac{3}{10}$ **(D)** $\frac{1}{10}$

3. In a certain school, 74% students like cricket, 76% students like football, and 82% like tennis. Then, all the three sports are liked by at least **[KVPY-2009]**

(A) 68% **(B)** 32%
(C) 77% **(D)** 36%

4. Let $S = \{1, 2, 3, \ldots, 40\}$ and let A be a subset of S such that no two elements in A have their sum divisible by 5. What is the maximum number of elements possible in A? **[KVPY-2012]**

(A) 10 **(B)** 13
(C) 17 **(D)** 20

5. Ten points lie in a plane so that no three of them are collinear. The number of lines passing through exactly two of these points and dividing the plane into two regions each containing four of the remaining points is: **[KVPY-2013]**

(A) 1
(B) 5
(C) 10
(D) dependent on the configuration of points

6. How many ways are there to arrange the letters of the word **EDUCATION** so that all the following three conditions hold? **[KVPY-2015]**

– the vowels occur in the same order (EUAIO),
– the consonants occur in the same order (DCTN),
– no two consonants are next to each other.

(A) 15 **(B)** 24
(C) 72 **(D)** 120

7. Let $A_1, A_2, \ldots, A_m$ be non-empty subsets of $\{1, 2, 3, \ldots, 100\}$ satisfying the following conditions:

1. The numbers $|A_1|, |A_2|, \ldots, |A_m|$ are distinct.
2. $A_1, A_2, \ldots, A_m$ are pair wise disjoint.

(Here $|A|$ denotes the number of elements in the set A) Then, the maximum possible value of m is **[KVPY-2016]**

(A) 13 **(B)** 14
(C) 15 **(D)** 16

8. If a 3-digit number is randomly chosen. What is the probability that either the number itself or some permutation of the number (which is a 3-digit number) is divisible by 4 and 5? **[KVPY-2016]**

(A) $\frac{1}{45}$ **(B)** $\frac{29}{180}$

(C) $\frac{11}{60}$ **(D)** $\frac{1}{4}$

9. Consider the set $A = \{1, 2, 3, \ldots, 30\}$. The number of ways in which one can choose three distinct number from A so that the product of the chosen numbers is divisible by 9 is **[KVPY-2017]**

(A) 1590 **(B)** 1505
(C) 1110 **(D)** 1025

10. Consider the set A of natural numbers n whose units digit is non-zero, such that if this units digit is erased, then the resulting number divides n. If K is the day corresponding to February 15 in that year is **[KVPY-2017]**

(A) K is infinite
(B) K is infinite but $K > 100$
(C) $25 \le K \le 10$
(D) $K < 25$

11. The least value of a natural number n such that $\binom{n-1}{5}+\binom{n-1}{6}<\binom{n}{7}$, where $\binom{n}{r}=\frac{n!}{(n-r)!r!}$ is **[KVPY-2017]**

(A) 12 **(B)** 13
(C) 14 **(D)** 15

12. Let S be the set of all points $\left(\frac{a}{b},\frac{c}{d}\right)$ on the circle with radius 1 centered at (0, 0) where a and b are relatively prime integers, c and d are relatively prime integers (that is HCF (a, b) = HCF $(c, d) = 1$), and the integers b and d are even. Then the set S **[KVPY-2017]**

(A) is empty **(B)** has four elements
(C) has eight elements **(D)** is infinite

13. Let m (respectively, n) be the number of 5-digit integers obtained by using the digits 1, 2, 3, 4, 5 with repetitions (respectively, without repetitions) such that the sum of any two adjacent digits is odd. Then $\frac{m}{n}$ is equal to **[KVPY-2018]**

(A) 9 **(B)** 12
(C) 15 **(D)** 18

14. Let $x_1, x_2, \ldots, x_{11}$ be 11 distinct positive integers. If we replace the largest of these integers by the median of the other 10 integers, then **[KVPY-2019]**

(A) the median remains the same
(B) the mean increases
(C) the median decreases
(D) the mean remains the same

PART – III Previous Years' PRMO Problems

1. A postman has to deliver five letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how many different way could he do this so that exactly two of the five house receive the correct letter? **[PRE-RMO-2012]**

2. There are $n - 1$ red balls, n green balls, and $n + 1$ blue balls in a bag. The number of ways of choosing two ball from the bag that have different colors is 299. What is the value of n? **[PRE-RMO-2013]**

3. To each element of the set $S = \{1, 2, \ldots, 1000\}$ a color is assigned. Suppose that for any two elements a, b of S, if 15 divides $a + b$ the n they are both assigned the same color. What is the maximum possible number of distinct colors used? **[PRE-RMO-2013]**

4. What is the number of ordered pairs (A, B) where A and B are subsets of $\{1, 2, \ldots, 5\}$ such that neither $A \subseteq B$ nor $B \subseteq A$? **[PRE-RMO-2014]**

5. Let S be a set of real numbers with mean M. If the mean of the sets $S \cup \{15\}$ and $S \cup \{15, 1\}$ are $M + 1$ and $M + 2$, respectively then how many elements does S have? [04] **[PRE-RMO-2014]**

6. How many line segments have both their endpoints located at the vertices of a given cube? **[PRE-RMO-2015]**

7. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party ? **[PRE-RMO-2015]**

8. A subject B of the set of first 100 positive integers has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? **[PRE-RMO-2015]**

9. There are three kinds of fruits in the market. How many possible ways are there to purchase 25 fruits from among them if each kind has at least 25 of its fruits available? [351] **[PRE-RMO-2016]**

10. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once? (The order in which he visits the cities also matters: e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different) **[PRE-RMO-2017]**

11. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room.) Four guests have to be accommodated in four of the eight room (that is, one is each) such that no

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two guests are in adjacent room or in opposite rooms. In how many way can the guests be accommodated?
[PRE-RMO-2017]

12. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$. **[PRE-RMO-2017]**

13. There are several tea cups in the kitchen, some with handle and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?
[PRE-RMO-2018]

14. Determine the number of 8-tuples $(\in_1, \in_2, \ldots, \in_8)$such that $\in_1, \in_2, \ldots, \in_8 \in \{1, -1\}$ and $\in_1 + 2\in_2 + 3\in_3 + \ldots + 8\in_8$ is a multiple of 3. **[PRE-RMO-2018]**

15. A positive integer k is said to be good if there exists a partition of $\{1, 2, 3, \ldots, 20\}$ in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there?
[PRE-RMO-2018]

16. If N is the number of triangles of different shapes (i.e. not similar) whose angles are all integers (in degrees), what is N/100? **[PRE-RMO-2018]**

17. Let T be the smallest positive integers which, when divided by 11, 13, 15 leaves remainders in the sets $\{7, 8, 9\}$, $\{1, 2, 3\}$, $\{4, 5, 6\}$, respectively. What is the sum of the square of the digits of T? **[PRE-RMO-2018]**

18. What is the number of ways in which one can choose 60 units square from a 11×11 chessboard such that no two chosen square have a side in common?
[PRE-RMO-2018]

19. What is the number of ways in which one can color the square of a 4×4 chessboard with colors red and blue such that each row as well as each column has exactly two red squares and blue squares?
[PRE-RMO-2018]

20. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N. **[PRE-RMO-2018]**

21. Five persons wearing badges with numbers 1, 2, 3, 4, 5 are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other? (Two arrangements obtained by rotation around the table are considered different.)
[PRE-RMO-2019]

22. In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected?
[PRE-RMO-2019]

23. Consider the set $E = \{5, 6, 7, 8, 9\}$. For any partition $\{A, B\}$ of E, with both A and B non empty. Consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of the digits of N **[PRE-RMO-2019]**

24. A $1 \times n$ rectangle $(n \geq 1)$ is divided into n unit (1×1) squares. Each square of this rectangle is colored red, blue or green. Let $f(n)$ be the number of colorings of the rectangle in which there are an even number of red squares. What is the largest prime factor of $f(9)/f(3)$? (The number of red squares can be zero)
[PRE-RMO-2019]

25. We will say that a rearrangement of the letters of a word has no fixed letters if. When the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, H B R A T A is a rearrangement with no fixed letters of B H A R A T. How many distinguishable rearrangement with no fixed letters does B H A R A T have? (The two A's are considered identical) **[PRE-RMO-2019]**

26. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set $\{10a + b : 1 \leq a \leq 5, 1 \leq b \leq 5\}$ where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?
[PRE-RMO-2019]

27. How many 4-digit numbers $\overline{abcd}$ are there such that $a < b < c < d$ and $b - a < c - b < d - c$? $abcd$
[PRE-RMO-2019]

28. Let E denote the set of all natural numbers n such that $3 < n < 100$ and the set $\{1, 2, 3, \ldots, n\}$ can be partitioned into 3 equal subsets with equal sums. Find the number of elements of E **[PRE-RMO-2019]**

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ANSWER KEYS

Know Your Progress - 1

1. $\{x : x = n^2 - 1, n \in N, n \le 6\}$
2. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
3. $\{4\}$
4. $A' \cup B' = \{1, 3, 4, 5, 6, 7, 8, 9\}$
5. 12
6. (i) 16 (ii) 20
7. (i) $A \cup B = \{0, 2, 3, 5, 6, 8\}$ (ii) $A \cap C = \{3\}$
   (iii) $A - B = \{2, 5\}$ (iv) $(A \cap B)' = \{0, 1, 2, 5, 6, 8, 9, 10, 12\}$
8. $x = 7$

Know Your Progress - 2

1. 7 2. 3 3. ${}^{50}C_4$ 4. 20 5. 120, 48 6. 10 7. 450 8. 840, 40

Know Your Progress - 3

1. $\frac{16!}{(2!)^8\, 8!} \times 8!$ 2. 360 3. ${}^{n}C_2 \cdot n!$

Know Your Progress - 4

1. $5^n - 4^n - 4^n + 3^n$

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Know Your Progress - 5

1. 60 2. 60

Know Your Progress - 6

1. 36 2. $\frac{9!}{2} = 181440$ 3. 5400 4. 2688

Know Your Progress - 7

1. $(p + 1)^n - 1$ 2. $2^{10} - 1$

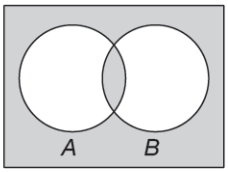
Know Your Progress - 8

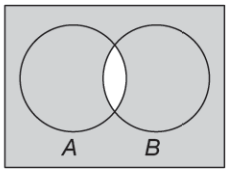
1. (a) ${}^{15}C_3$ (b) ${}^{7}C_3$
2. ${}^{12}C_2$
3. ${}^{23}C_2$

Know Your Progress - 9

1. 9

Know Your Progress - 10

1. **(i)** $\{B_1R_1, B_2R_1, B_3R_1, B_1R_2, B_2R_2, B_3R_2, B_1B_2, B_2B_3, B_1B_3, R_1R_2\}$
   (ii) $\{M_1W_1, M_2W_1, M_1W_2, M_2W_2, M_1W_3, M_2W_3\}$
   (iii) $\{H1, H2, H3, H4, H5, H6, TH, TT\}$
   (iv) A
2. **(i)** $\frac{3}{4}$ **(ii)** $\frac{10}{21}$ **(iii)** $\frac{1}{10}$ **(iv)** $\frac{(13)^5 - (12)^5}{(13)^5}$
3. **(i) (a)** 

(b) 

   (ii) A, B, D **(iii)** $11/15$
   (iv) **(a)** $\frac{19}{25}$ **(b)** $\frac{13}{25}$

Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

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1. C	2. B	3. B	4. C	5. A	6. C	7. A	8. A	9. B	10. B
11. B	12. D	13. B	14. B	15. B	16. A	17. C	18. D	19. D	20. C
21. A	22. C	23. D	24. C	25. B	26. C	27. D	28. C	29. C	30. D
31. C	32. A	33. D	34. C	35. C	36. A	37. A	38. C	39. B	40. D
41. C	42. A	43. C	44. D	45. A	46. D	47. C	48. A	49. C	50. A
51. A	52. B	53. D	54. B	55. B	56. B	57. D	58. B	59. C	60. B
61. B	62. A	63. B	64. A	65. C	66. A	67. B	68. C	69. C	70. C
71. C	72. C	73. A	74. D	75. A	76. D	77. C	78. B	79. D	80. C
81. C	82. D	83. (A) R; (B) S; (C) Q; (D) P				84. A	85. B	86. A	87. B
88. D	89. C	90. B	91. A	92. C	93. B	94. B	95. C	96. D	97. A
98. B									

PART – II KVPY Level Problems

1. A	2. D	3. A	4. B	5. A	6. A	7. D	8. A	9. B	10. B
11. D	12. C	13. C	14. A	15. D	16. C	17. B	18. B	19. A	20. C
21. A	22. B	23. C	24. B	25. C	26. A	27. A	28. C	29. B	30. A
31. B	32. B	33. C	34. D	35. A	36. A	37. C	38. A		

PART – III PRMO Level Problems

1. 8	2. 18	3. 24	4. 3	5. 2	6. 90	7. 78	8. 22	9.32	10. 27
11. 97	12. 64	13. 6	14. 28	15. 45	16. 2	17. 9	18. 4	19. 65	20. 18
21. 24	22. 99	23. 8	24. 10	25. 11	26. 2	27. 13	28. 25	29. 3	30. 18
31. 4	32. 10	33. 89	34. 60	35. 45	36. 8	37. 9	38. 7	39. 13	40. 5

41. 36 **42.** 120 **43.** (i) 24; (ii) 576; (iii) 360
44. (i) 840; (ii) 120; (iii) 400; (iv) 240; (v) 480; (vi) 40; (vii) 60; (viii) 240
45. (i) 120; (ii) 40 (iii) 40; (iv) 80; (v) 20 **46.** 20 **47.** 4^7 **48.** (a) 3^4; (b) 24
49. 36 **50.** 720 **51.** 576 **52.** 999 **53.** 967680 **54.** 6 **55.** 13,156 **56.** 43200 **57.** 420 **58.** 960
59. ${}^{24}C_2 \cdot {}^{15}C_3$ **60.** 1106 **61.** 12870 **62.** 5400 **63.** 3150 **64.** 1568 **65.** 172800 **66.** ${}^{8}C_4 \cdot 4!$ **67.** 528
68. $n = 7$ **69.** (i) 3359; (ii) 59; (iii) 359 **70.** 120, 216, 210 **71.** 240,15552 **72.** 30
73. 719 **74.** 2500 **75.** (a) 72; 78120; (b) 23; (c) 32; (d) 435 **76.** 56 **77.** 30980

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. C **2.** C **3.** D **4.** D **5.** B **6.** D **7.** B **8.** D **9.** B **10.** A

PART – II Previous Years' KVPY Problems

1. C **2.** D **3.** B **4.** C **5.** B **6.** A **7.** A **8.** B **9.** A **10.** D
11. C **12.** A **13.** C **14.** C

PART – III Previous Years' PRMO Problems

1. 20 **2.** 10 **3.** 8 **4.** 570 **5.** 4 **6.** 28 **7.** 12 **8.** 62 **9.** 351 **10.** 60
11. 48 **12.** 60 **13.** 29 **14.** 88 **15.** 6 **16.** 27 **17.** 81 **18.** 62 **19.** 90 **20.** 24
21. 10 **22.** 45 **23.** 17 **24.** 37 **25.** 84 **26.** 47 **27.** 7 **28.** 64

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Chapter

6 Plane Geometry-I

Learning Outcomes

After reading this chapter, you will be able to understand

- Introduction to Euclid Geometry
- Lines and Angles
- Triangles
- Similar Triangles
- Quadrilaterals
- Area of Parallelograms and Triangles

Introduction to Euclid Geometry and Lines and Angles

Introduction to Euclid Geometry

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The credit for introducing geometrical concepts goes to the distinguished Greek mathematician **'Euclid'** who is known as the **"Father of Geometry"** and the word **'geometry'** comes from the Greek words **'geo'** which means **'Earth'** and **'metreon'** which means '**measure**'.

Euclid's Definitions

(i) A point is that which has no part.
(ii) A line is breadth-less length.
(iii) The ends of a line segment are points.
(iv) A straight line is that which has length only.
(v) A surface is that which has length and breadth only.
(vi) The edges of a surface are lines.
(vii) A plane surface is that which lies evenly with the straight lines on itself.

Some Undefined Terms

There are three basic concepts in geometry, namely, 'point', 'line' and 'plane'. It is not possible to define these three concepts precisely. We can, however, have a good idea of these concept by considering examples given below

Point: A point is represented by a fine dot made by sharp pencil on a sheet of paper. A Point is denoted by capital letter A, *B*, *P*, *Q*, *R* etc.

Plane: The surface of a smooth wall, the surface of a sheet of paper, or the surface of a smooth black board are close examples of a plane.

Line: If we fold a piece of paper, the crease in the paper represents a geometrical straight line. The edge of ruler, the edge of the top of a table, the meeting place of two walls of a room are close examples of a geometrical straight line.

Some Geometrical Concepts

Axioms: The basic facts that are taken for granted, without proof, are called axioms.
E.g. Halves of equal are equal.

Statements: A sentence that can be judged to be true or false is called a sentence.
E.g. The sum of the angles of a triangle is 180°, is a true statement.

Theorems: A statement that requires a proof is called a theorem. Establishing the truth of a theorem is known as proving the theorem.
E.g. The sum of all angles around a point is 360°.

Corollary: A statement whose truth can easily be decided from a theorem is called its corollary.

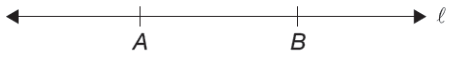
Conjecture: In mathematics, a **conjecture** is a conclusion or proposition based on incomplete information, for which no proof has been found. E.g. **Goldbach Conjecture** states that every even integer greater than two is the sum of two prime numbers.

Some Terms Related to Geometry

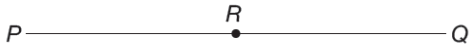
Point: A point is an exact location. A fine dot represents a point. Point is denoted by capital letter A, B, P, Q, R etc.

Line segment: The straight path between two points A and B is called the line segment $\overline{AB}$. The points A and B are called the end points of the line segment $\overline{AB}$. A line segment has a definite length.

Line: A line segment $\overline{AB}$ when extended indefinitely in both direction is called the line $\overleftrightarrow{AB}$. A line has no end points. A line has no definite length. Sometimes, we label lines by small letters l, m, n etc.



Interior point of a line segment: A point R is called an interior point of a line segment PQ if R lies between P and Q but R is neither P nor Q.

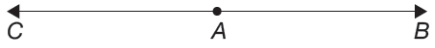


Congruence of line segment: Two line segments AB and CD are congruent if trace copy of one can be superposed on the other so as to cover it completely and exactly in this case we write $AB \cong CD$. In other words, we can say two lines are congruent if their lengths are same.

Distance between two points: The distance between two points P and Q is the length of line segment PQ.

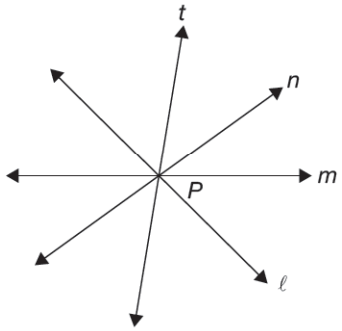
Ray: Directed line segment is called a ray. If AB is a ray then it is denoted by $\overrightarrow{AB}$. Point A is called initial point of ray.

Opposite rays: Two rays AB and AC are said to be opposite rays if they are collinear and point A is the only common point of the two rays.

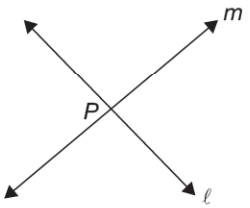


Collinear points: Three or more points are said to be collinear if there is a line which contains all of them.

Concurrent Lines: Three or more lines are said to be concurrent if there is a point which lies on all of them.



Intersecting lines: Two lines are intersecting if they have a common point. The common point is called the 'point of intersection'.



Parallel lines: Two lines l and m in a plane are said to be parallel lines if they do not have a common point.

Plane: A plane is a surface such that every point of the line joining any two points on it lies on it.

Euclid's Axioms

(i) Things which are equal to the same thing are equal to one another.
(ii) If equal are added to equals, the wholes are equal.
(iii) If equal are subtracted from equals, the remainders are equal.
(iv) The things that coincide with one another are equal to one another.
(v) The whole is greater than the part.
(vi) The things that are double of the same thing are equal to one another.
(vii) The thing that are halves of the same thing are equal to one another.

Euclid's Five Postulates

(i) A straight line may be drawn from any one point to any other point.
(ii) A terminated line or a line segment can be produced infinitely.

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(iii) A circle can be drawn with any centre and of any radius.

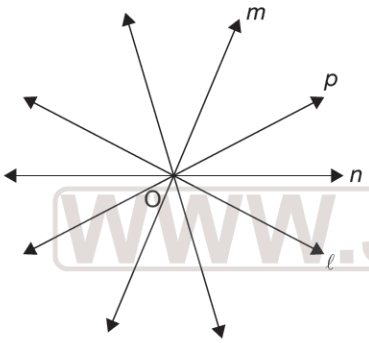
(iv) All right angles are equal to one another.

(v) If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines if produced infinitely meet on that side on which the sum of angles are less than two right angles.

Incidence Axioms

(i) A line is the collection of infinite number of points.

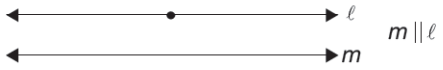
(ii) Through a given point, an infinite line can be drawn.



(iii) Given two distinct points, there is one and only one line that contains both the points.

Parallel Axioms

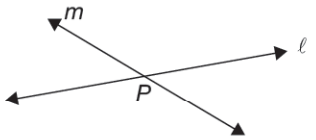
If P is a point outside a line ℓ, then one and only one line can be drawn through P that is parallel to ℓ.



The above axioms may also be re-stated as follows: "Two intersecting lines cannot both be parallel to the same line".

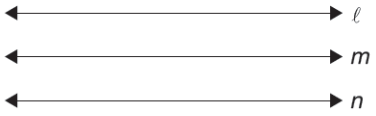
(i) Some more results

(a) Two distinct lines cannot have more than one point in common.



(b) Two lines that are both parallel to the same line, are parallel to each other.

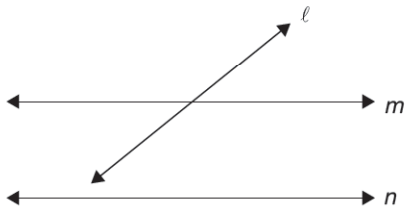
i.e., $\ell \,||\, n$, $m \,||\, n \Rightarrow \ell \,||\, m$



Theorem: If ℓ, m, n are lines in the same plane such that ℓ intersects m and $n \,||\, m$, then ℓ intersects n also.

Given: Three lines ℓ, m, n in the same plane s.t. ℓ intersects m and $n \,||\, m$.

To prove: Lines l and n are intersecting lines.



Proof: Let ℓ and n be non-intersecting lines. Then, $\ell \,||\, n$.

But, $n \,||\, m$ [Given]

$\therefore \ell \,||\, n$ and $n \,||\, m \Rightarrow \ell \,||\, m$

ℓ and m are non-intersecting lines.

This is a contradiction to the hypothesis that ℓ and m are intersecting lines.

So our supposition is wrong.

Hence, ℓ intersects line n.

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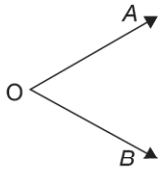
Lines and Angles

Line

A line has length but no width and no thickness.

Angle

An angle is the union of two non–collinear rays with a common initial point. The common initial point is called the **'vertex'** of the angle and two rays are called the **'arms'** of the angles.



Note

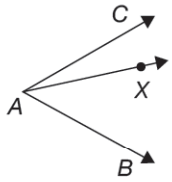
Every angle has a measure and unit of measurement is degree.

One right angle = 90°

1° = 60′ (minutes)

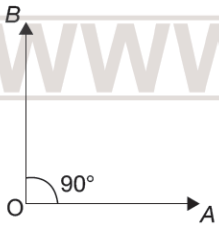
1′ = 60″ (Seconds)

Angle addition axiom: If X is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAX + m\angle XAC$.

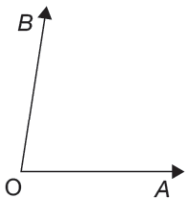


Types of Angles

(i) Right angle: An angle whose measure is 90° is called a **right angle**.

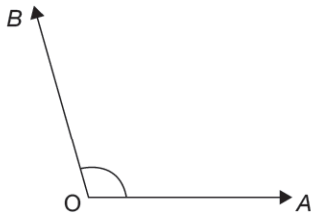


(ii) Acute angle: An angle whose measure is less than 90° is called an **acute angle**.



$0° < \angle BOA < 90°$

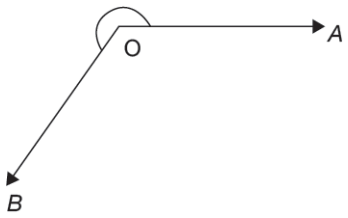
(iii) Obtuse angle: An angle whose measure is more than 90°, but less than 180°, is called an **obtuse angle**.



$90° < \angle AOB < 180°$.

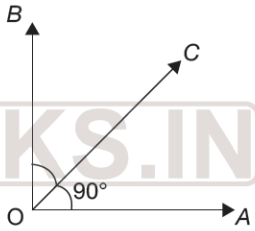
(iv) Straight angle: An angle whose measure is 180° is called a **straight angle**.

(v) Reflex angle: An angle whose measure is more than 180°, but less than 360° is called a **reflex angle**.



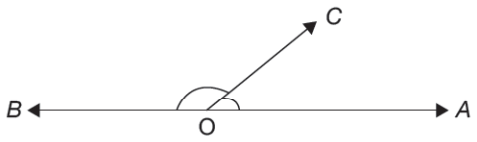
$180° < \angle AOB < 360°$.

(vi) Complementary angles: Two angles, the sum of whose measures is 90° are called **complementary angles**.



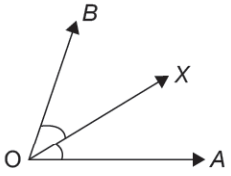
$\angle AOC$ and $\angle BOC$ are complementary as their sum is 90°.

(vii) Supplementary angles: Two angles, the sum of whose measures is 180°, are called the **supplementary angles.**



$\angle AOC$ and $\angle BOC$ are supplementary as their sum is 180°.

(viii) Angle Bisectors: A ray OX is said to be the bisector of $\angle AOB$, if X is a point in the interior of $\angle AOB$, and $\angle AOX = \angle BOX$.



(ix) Adjacent angles: Two angles are called **adjacent angles**, if

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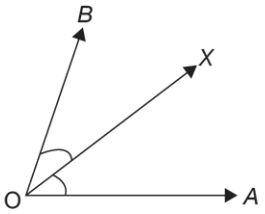
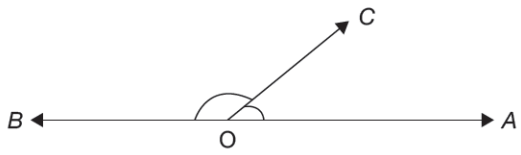
(A) they have the same vertex,
(B) they have a common arm,
(C) non-common arms are on either side of the common arm.



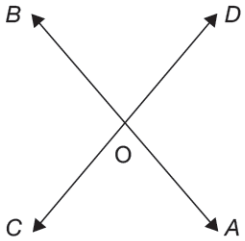
$\angle AOX$ and $\angle BOX$ are adjacent angles, OX is common arm, OA and OB are non-common arms and lie on either side of OX.

(x) Linear pair of angles: Two adjacent angles are said to form a linear pair of angles, if their non-common arms are two opposite rays.



$\angle AOC + \angle BOC = 180°$.

(xi) Vertically opposite angles: Two angles are called a pair of **vertically opposite angles**, if their arms form two pairs of opposite rays.

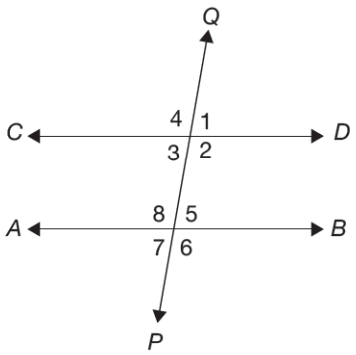


$\angle AOC$ and $\angle BOD$ form a pair of vertically opposite angles. Also $\angle AOD$ and $\angle BOC$ form a pair of vertically opposite angles.

If two lines intersect, then the vertically opposite angles are equal, i.e., $\angle AOC = \angle BOD$ and $\angle BOC = \angle AOD$.

Angles made by a Transversal

Transversal: A line that intersects two or more given lines at distinct points is called a transversal of the given lines.



(i) Corresponding angles: Two angles on the same side of a transversal are known as the corresponding angles if both lie either above the two lines or below the two lines, in figure $\angle 1$ and $\angle 5$, $\angle 4$ and $\angle 8$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$ are the pairs of corresponding angles.

If a transversal intersects two parallel lines then the corresponding angles are equal i.e., $\angle 1 = \angle 5$, $\angle 4 = \angle 8$, $\angle 2 = \angle 6$ and $\angle 3 = \angle 7$.

(ii) Alternate interior angles: $\angle 3$ and $\angle 5$, $\angle 2$ and $\angle 8$, are the pairs of alternate interior angles.

If a transversal intersects two parallel lines then the each pair of alternate interior angles are equal i.e., $\angle 3 = \angle 5$ and $\angle 2 = \angle 8$.

(iii) Co-interior angles: The pair of interior angles on the same side of the transversal are called pairs of consecutive or cointerior angles. In figure $\angle 2$ and $\angle 5$, $\angle 3$, and $\angle 8$, are the pairs of co–interior angles.

If a transversal intersects two parallel lines then each pair of consecutive interior angles are supplementary, i.e., $\angle 2 + \angle 5 = 180°$ and $\angle 3 + \angle 8 = 180°$.

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Example 1

If a point C lies between two points A and B such that $AC = BC$, then prove that $AC = \frac{1}{2} AB$. Explain by drawing the figure.

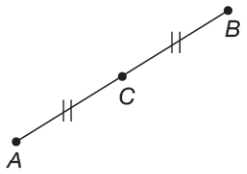
Solution

According to the given statement, the figure will be as shown alongside in which the point C lies between two points A and B such that $AC = BC$.

Clearly, $AC + BC = AB$

$\Rightarrow$ $AC + AC = AB$ $[\because AC = BC]$

$\Rightarrow$ $2AC = AB$



And, $AC = \frac{1}{2} AB$

Example 2

Give a definition for each of the following terms. Are there other terms that need to be defined first? What are they, and how might you define them?

(i) Parallel lines
(ii) Perpendicular lines
(iii) Line segment
(iv) Radius

Solution

(i) Parallel lines: Lines that do not intersect anywhere are called parallel lines.

(ii) Perpendicular lines: Two lines that are at a right angle to each other are called perpendicular lines.

(iii) Line segment: It is a terminated line.

(iv) Radius: The length of the line segment joining the centre of a circle to any point on its circumference is called its radius.

Example 3

How would you rewrite Euclid's fifth postulate so that it would be easier to understand?

Solution

Two distinct intersecting lines cannot be parallel to the same line.

Example 4

Does Euclid's fifth postulate imply the existence of parallel lines? Explain.

Solution

If a straight line ℓ falls on two straight lines m and n such that sum of the interior angles on one side of ℓ is two right angles, then by Euclid's fifth postulate, the line will not meet on this side of ℓ. Next, we know that the sum of the interior angles on the other side of line ℓ also be two right angles. Therefore, they will not meet on the other side. So, the lines m and n never meet and are, therefore, parallel.

Example 5

If lines AB, AC, AD, and AE are parallel to a line ℓ, then prove that points A, B, C, D, and E are collinear.

Solution

Lines AB, AC, AD, and AE are parallel to a line ℓ.

To prove: A, B , C, D, E are collinear.

Proof: Since AB, AC, AD, and AE are all parallel to a line ℓ. Therefore, point A is outside ℓ and lines AB, AC, AD, AE are drawn through A and each line is parallel to ℓ.

But by parallel lines axiom, one and only one line can be drawn through the point A outside it and parallel to ℓ.

This is possible only when A, B, C, D, and E all lie on the same line. Hence, A, B, C, D, and E are collinear.

Example 6

Find the measure of an angle, if six times its complement is 12° less than twice its supplement.

Solution

Let the measure of the required angle be $x°$.

Then, measure of its complement = $(90 - x)°$.

Measure of its supplement = $(180 - x)°$.

$\therefore \quad 6(90 - x) = 2(180 - x) - 12$

$\Rightarrow \quad 540 - 6x = 360 - 2x - 12$

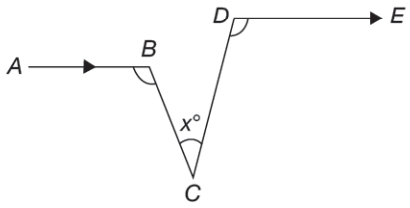
$\Rightarrow \quad 4x = 192$

$\Rightarrow \quad x = 48°$.

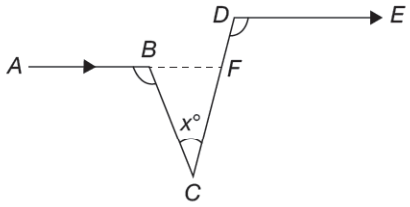
Hence, the required angle is 48°.

Example 7

In the adjoining figure, $\angle ABC = 100°$, $\angle EDC = 120°$, and $AB \parallel DE$. Then, find $\angle BCD$.



Solution



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Produce AB to meet CD at F.

$\angle BFD = \angle EDF = 120°$ [alternate interior $\angle s$]

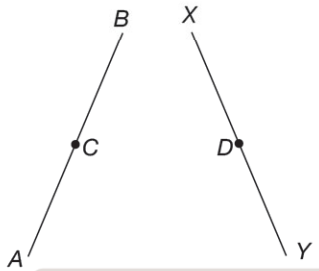
$\Rightarrow$ $\angle BFC = (180° - 120°) = 60°$

$\Rightarrow$ $\angle CBF = (180° - 100°) = 80°$

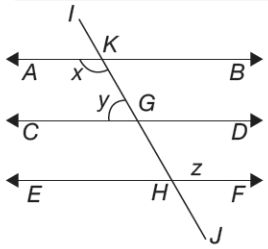
$\therefore$ $\angle BCF = 180° - (60° + 80°) = 40°$.

Know Your Progress - 1

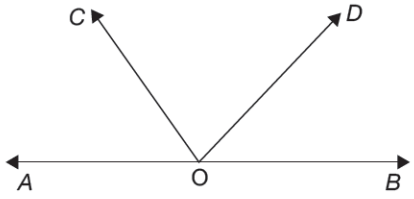
1. Solve the equation $a - 15 = 25$ and state which axiom do you use here.
2. In figure, we have $AC = XD$, C is the mid-point of AB and D is the mid-point of XY. Using an Euclid's axiom, show that $AB = XY$.



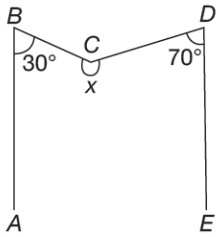
3. In figure, if $AB \parallel CD \parallel EF$ and $y : z = 3 : 7$, find x.



4. Two complementary angles are such that two times the measure of one is equal to three times measure of the other. Find the measure of the large angle.
5. How many lines can pass through
   (i) One point (ii) Two distinct point
6. Prove that every line segment has one and only one mid-point.
7. Find the supplement of $100° 48'$.
8. In figure, find $\angle COD$ when $\angle AOC + \angle BOD = 100°$.



9. Two complementary angles are such that two times the measure of one is equal to three times measure of the other. Find the measure of the larger angle.
10. In figure, $AB \parallel ED$ and $\angle ABC = 30°$, $\angle EDC = 70°$ then find $x°$.



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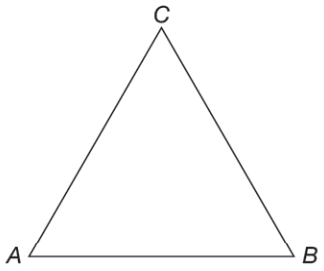
Triangle

A plane figure bounded by three lines in a plane is called a triangle. Every triangle have three sides and three angles. If ABC is any triangle then AB, BC, and CA are three sides and $\angle A$, $\angle B$, and $\angle C$ are three angles.



Types of Triangles

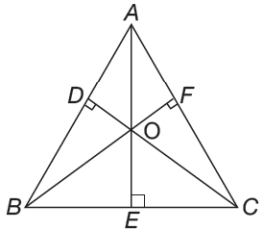
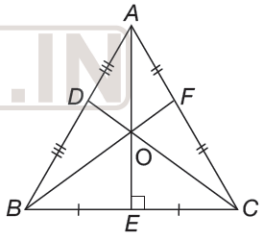
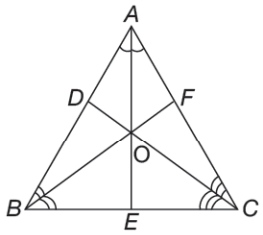
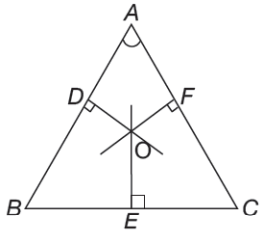
(a) On the basis of sides, we have three types of triangles.

1. **Scalene triangle:** A triangle in which no two sides are equal is called a scalene triangle.
2. **Isosceles triangle:** A triangle having two sides equal is called an isosceles triangle.
3. **Equilateral triangle:** A triangle in which all sides are equal is called an equilateral triangle.

(b) On the basis of angles, we have three types:

1. **Right triangle:** A triangle in which any one angle is right angle is called right triangle.
2. **Acute triangle:** A triangle in which all angles are acute is called an acute triangle.
3. **Obtuse triangle:** A triangle in which any one angle is obtuse is called an obtuse triangle.

Some Important Terms

Nomenclature	Property / Definition	Diagram
Altitude (or height)	The perpendicular drawn from the opposite vertex of a side in a triangle called an altitude of the triangle. • There are three altitudes in a triangle.	 *AE*, *CD*, and *BF* are the altitudes
Median	The line segment joining the mid-point of a side to the vertex opposite to the side is called a median. • There are three medians in a triangle. • A median bisects the area of the triangle i.e, $\text{Ar}(\Delta ABE) = \text{Ar}(\Delta AEC) = \frac{1}{2}\ \text{Ar}(\Delta ABC)$ etc. • Point of intersection is called Centroid.	 *AE*, *CD*, and *BF* are the medians ($BE = CE$, $AD = BD$, $AF = CF$)
Angle bisector	A line segment that originates from a vertex and bisects the same angle is called an angle bisector $(\angle BAE = \angle CAE = \frac{1}{2}\ \angle BAC)$ etc. • Point of intersection of angle bisectors is called incentre.	 *AE*, *CD*, and *BF* are the angles' bisectors
Perpendicular bisector	A line segment that bisects a side perpendicularly (i.e., at right angle) is called a perpendicular bisector of a side of triangle. • All points on the perpendicular bisector of a line are equidistant from the ends of the line. • Point of intersection of perpendicular bisectors is called circumcentre.	 *DO*, *EO*, and *FO* are the perpendicular bisectors

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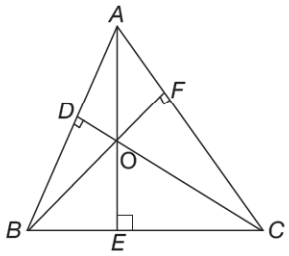
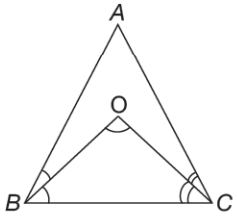
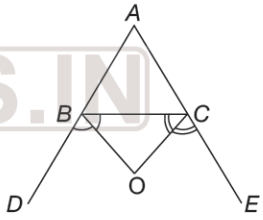
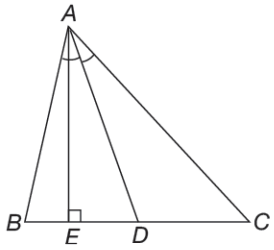
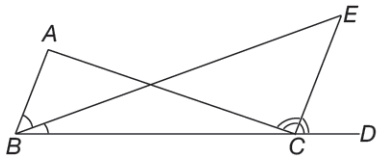
(*Continued*)

Nomenclature	Property / Definition	Diagram
Orthocentre	The point of intersection of the three altitudes of the triangle is called as the orthocentre. $\angle BOC = 180° - \angle A$ $\angle COA = 180° - \angle B$ $\angle AOB = 180° - \angle C$	 'O' is the orthocentre

Some Important Results

S.No.	Statement	Diagram
1.	In a ΔABC, if the bisectors of $\angle B$ and $\angle C$ meet at O then $\angle BOC = 90° + (\angle A)/2$	
2.	In a ΔABC, if sides AB and AC are produced to D and E, respectively, and the bisectors of $\angle DBC$ and $\angle ECB$ intersect at O, then $\angle BOC = 90° - (\angle A)/2$	
3.	In a ΔABC, if AD is the angle bisector of $\angle BAC$ and $AE \perp BC$, $\angle DAE = \frac{1}{2}(\angle ABC - \angle ACB)$	
4.	In a ΔABC, if side BC is produced to D and bisectors of $\angle ABC$ and $\angle ACD$ meet at E, then $\angle BEC = \frac{1}{2}\angle BAC$	

Congruent Figures

The figures are called congruent if they have same shape and same size. In other words, two figures are called congruent if they are having equal length, width, and height.

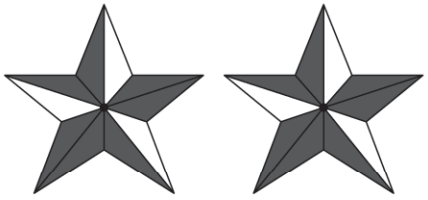


Fig.(i) Fig.(ii)

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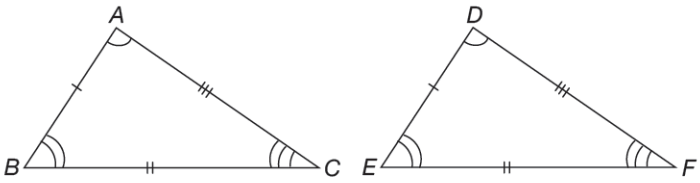
In the above figures {fig.(i) and fig.(ii)}, both are equal in length, width, and height, so these are congruent figures.

Congruent Triangles

Two triangles are congruent if and only if one of them can be made to superimposed on the other, so as to cover it exactly.



If two triangles ΔABC and ΔDEF are congruent then $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and $AB = DE$, $BC = EF$, $AC = DF$.

If two ΔABC and ΔDEF are congruent then we write $\Delta ABC \cong \Delta DEF$, we cannot write as $\Delta ABC \cong \Delta DFE$ or $\Delta ABC \cong \Delta EDF$.

Hence, we can say that two triangles are congruent if and only if there exists a one–one correspondence between their vertices such that the corresponding sides and the corresponding angles of the two triangles are equal.

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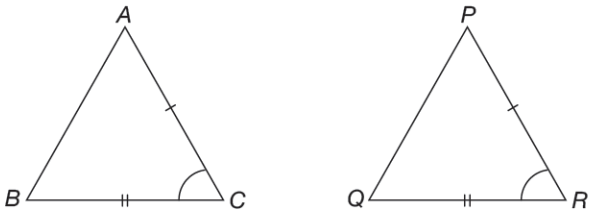
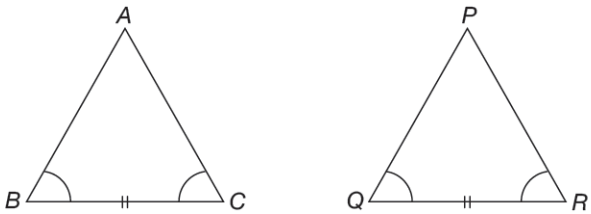
Sufficient Conditions for Congruence of Two Triangles

(i) *SAS* Congruence Criterion:



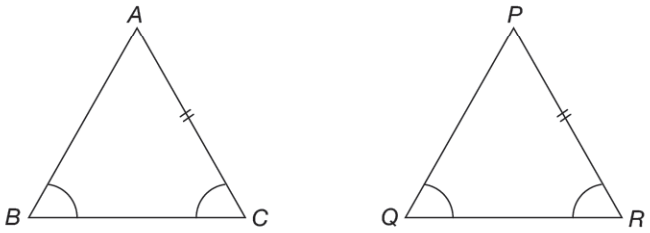
Two triangles are congruent if two sides and the included angle of one are equal to the corresponding sides and the included angle of the other triangle.

(ii) *ASA* Congruence Criterion:



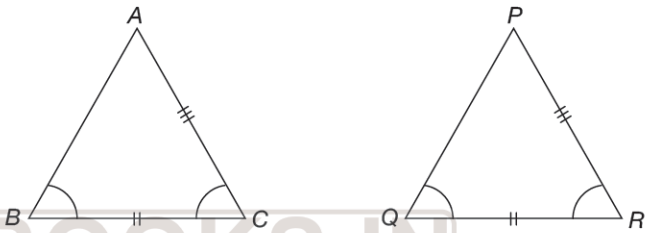
Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

(iii) *AAS* Congruence Criterion:



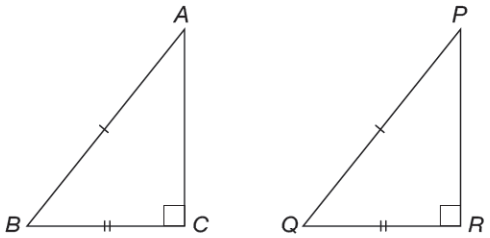
If any two angles and a non-included side of one triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

(iv) *SSS* Congruence Criterion:



Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

(v) *RHS* Congruence Criterion:



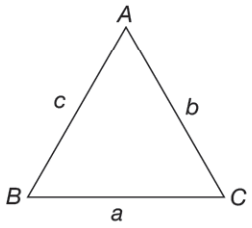
Two right triangles are congruent if the hypotenuse and one side of one triangle are, respectively, equal to the hypotenuse and one side of the other triangle.

Note

If two triangles are congruent, then their corresponding sides and angles are also congruent by *CPCT* (corresponding parts of congruent triangles are also congruent).

Inequality in a Triangle

In triangle ABC, $AB = c$, $BC = a$, and $CA = b$, then,



(i) The sum of any two sides of a triangle is greater than its third side, i.e., in ΔABC,
(A) $a + b > c$ (B) $b + c > a$ (C) $a + c > b$

(ii) If two sides of a triangle are unequal, then the longer side has greater angle opposite to it,
i.e., in ΔABC, if $AB > AC$ then $\angle C > \angle B$.

(iii) Of all the line segments that can be drawn to a given line, from a point, not lying on it, the perpendicular line segment is the shortest

(iv) The difference between any two sides of a triangle is less than its third side, i.e., in ΔABC,
(A) $a - b < c$ (B) $b - c < a$ (C) $a - c < b$

(v) In a right-angled triangle, the sum of squares of two smaller sides is equal to the square of its third side.
i.e., in ΔABC, $a^2 + b^2 = c^2$ (where a and b are smaller as compared to c).

(vi) If sum of squares of two smaller sides is greater than the square of its third side, then that triangle is acute-angled triangle.
i.e., in ΔABC, $a^2 + b^2 > c^2$ (where a and b are smaller as compared to c).

(vii) If sum of squares of two smaller sides is lesser than the square of its third side, then that triangle is obtuse-angled triangle.
i.e., in ΔABC, $a^2 + b^2 < c^2$ (where a and b are smaller as compared to c).

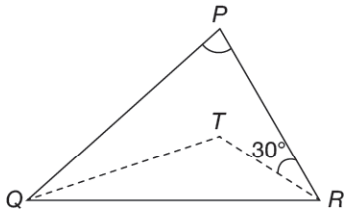
Example 8

In figure, TQ and TR are the bisectors of $\angle Q$ and $\angle R$, respectively. If $\angle QPR = 80°$ and $\angle PRT = 30°$, determine $\angle TQR$ and $\angle QTR$.

Solution

Since the bisectors of $\angle Q$ and $\angle R$ meet at T.

$$\therefore \quad \angle QTR = 90° + \frac{1}{2}\angle QPR$$



$$\Rightarrow \quad \angle QTR = 90° + \frac{1}{2}(80°)$$

$$\Rightarrow \quad \angle QTR = 90° + 40° = 130°$$

In ΔQTR, we have

$$\angle TQR + \angle QTR + \angle TRQ = 180°$$

$$\Rightarrow \quad TQR + 130° + 30° = 180°$$

$$[\because \angle TRQ = \angle PRT = 30°]$$

$$\Rightarrow \quad \angle TQR = 20°$$

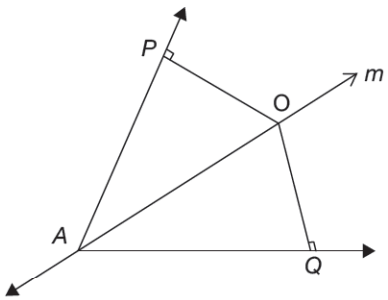
Thus, $\angle TQR = 20°$ and $\angle QTR = 130°$.

Example 9

In figure, line m is the bisector of an angle $\angle A$ and O is any point on m. OQ and OP are perpendiculars from O to the arms of $\angle A$. Prove that:

(i) $\Delta AOQ \cong \Delta AOP$

(ii) $OQ = OP$.



Solution

(i) In ΔAOQ and ΔAOP,

$\angle OAQ = \angle OAP$ [$\because$ line m is the bisector of $\angle A$]

$\angle AQO = \angle APO = 90°$ [Given]

and $AO = AO$ [Common]

By AAS congruency

$\therefore \quad \Delta AOQ \cong \Delta AOP$

(ii) $OQ = OP$ [C.P.C.T.]

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Example 10

If D is the mid–point of the hypotenuse AC of a right triangle ABC, prove that $BD = \frac{1}{2} AC$.

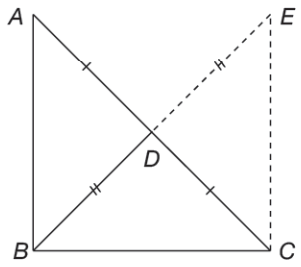
Solution

Given: ΔABC is a right triangle such that $\angle B = 90°$ and D is mid-point of AC.

To prove: $BD = \frac{1}{2} AC$.

Construction: Produce BD to E such that $BD = DE$ and join EC.

Proof:



In ΔADB and ΔCDE

$AD = DC$ [Given]

$BD = DE$ [By construction]

And, $\angle ADB = \angle CDE$ [Vertically opposite angles]

$\therefore$ By *SAS* criterion of congruence we have

$\Delta ADB \cong \Delta CDE$

$\Rightarrow$ $EC = AB$ and $\angle CED = \angle ABD$ …(i)

[By CPCT]]

But $\angle CED$ and $\angle ABD$ are alternate interior angles

$\therefore$ $CE \parallel AB$

$\Rightarrow$ $\angle ABC + \angle ECB = 180°$

[Consecutive interior angles]

$90 + \angle ECB = 180° \Rightarrow \angle ECB = 90°$.

Now, In ΔABC and ΔECB, we have

$AB = EC$ [By (i)]

$BC = BC$ [Common]

And, $\angle ABC = \angle ECB = 90°$

$\therefore$ By *SAS* criterion of congruence

$\Delta ABC \cong \Delta ECB$

$\Rightarrow$ $AC = EB$ [By CPCT]

$\Rightarrow$ $\frac{1}{2} AC = \frac{1}{2} EB \Rightarrow BD = \frac{1}{2} AC$.

Hence proved.

Example 11

From which given triplet we can make the sides a triangle.

(i) {15, 7, 8}
(ii) {3.5, 4.5, 5.5}

Solution

As we know sum of two sides is always greater than third side.

(i) We cannot make the triangle because here the sum of two sides is equal to third side
i.e., $7 + 8 = 15$

(ii) We can make the triangle because sum of two sides is always greater then third side
$3.5 + 4.5 > 5.5$
$3.5 + 5.5 > 4.5$
$4.5 + 5.5 > 3.5$

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Example 12

Prove that any two sides of the triangle are together greater than twice the median drawn to the third side.

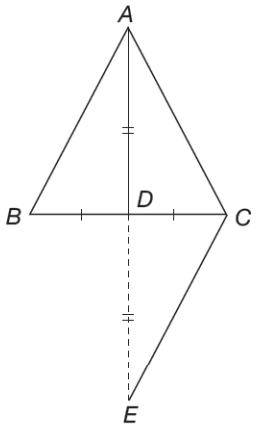
Solution

Given: ΔABC and AD is the median.

To prove: $AB + AC > 2AD$

Construction: Produce AD to E such that $AD = DE$. Join EC.

Proof: In ΔADB and ΔCDE



$AD = DE$ [By construction]

$BD = DC$ [AD is the median]

$\angle ADB = \angle CDE$ [Vertically opposite angles]

$\Delta ADB \cong \Delta CDE$ [By SAS congruency]

So, by CPCT

$$AB = EC$$

In ΔAEC

$$AC + EC > 2AD.$$

[Sum of two sides of a triangle is always greater than the……,

So, $AC + AB > 2AD.$ [As $EC = AB$].

Example 13

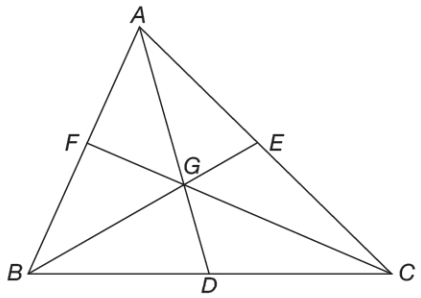
AD, BE, and CF be medians of a ΔABC.
Prove that $2(AD + BE + CF) < 3(AB + BC + CA) < 4(AD + BE + CF)$.

Solution

In ΔABC, AD, BE, and CF are medians. We know that these are concurrent. Let these meet at G.

Also $AG : GD = BG : GE = CG : GF = 2 : 1$

So, $AG = \frac{2}{3} AD$, $BG = \frac{2}{3} BE$, $CG = \frac{2}{3} CF$.



Because in triangle, the sum of two sides is greater than the third,

So, in ΔAGB, $AG + BG > AB$

in ΔBGC, $BG + CG > BC$

in ΔCGA, $CG + AG > CA$

Adding these, we get

$$2(AG + BG + CG) > AB + BC + CA$$

or $$2\left(\frac{2}{3}AD + \frac{2}{3}BE + \frac{2}{3}CF\right) > AB + BC + CA$$

or $$\frac{4}{3}(AD + BE + CF) > AB + BC + CA$$

or $$4(AD + BE + CF) > 3(AB + BC + CA) \quad \ldots(i)$$

Now, in ΔABD

$$AB + BD > AD$$

or, $$AB + \frac{1}{2} BC > AD$$ (Since AD is a median)

Similarly, $BC + \frac{1}{2} CA > BE$ and $CA + \frac{1}{2} AB > CF$

Adding these, we get

$$\frac{3}{2}(AB + BC + CA) > AD + BE + CF$$

or $$3(AB + BC + CA) > 2(AD + BE + CF) \quad \ldots(ii)$$

Combining (i) and (ii), we get

$$2(AD + BE + CF) < 3(AB + BC + CA) < 4(AD + BE + CF).$$

Example 14

In a ΔABC, $AC > AB$. The bisector of $\angle A$ meets BC at E. Prove that $CE > BE$.

Solution

Given: A ΔABC, in which $AC > AB$ and AE bisects $\angle A$

To prove: $CE > BE$.

Construction: Take a point F on AC, such that $AF = AB$. Join BF and EF. Let BF intersect AE at O.

Proof: ΔABF is an isosceles (Since $AB = AF$, by construction)

AE bisects $\angle A$ (given)

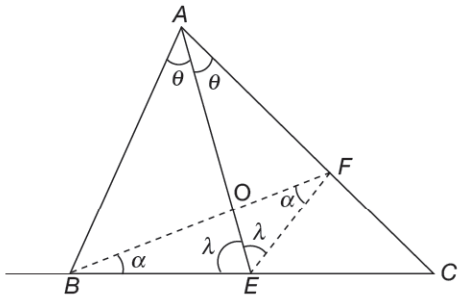
$\therefore$ AE is the perpendicular bisector of BF, so any point on AE is equidistant from B and F. In particular,

$$BE = FE \quad \ldots(i)$$

So, in ΔBEF,

$\angle EBF = \angle EFB = \alpha$, say (angles opposite equal sides)

So, $\angle BEO = 90° - \alpha = \angle FEO$ ($= \lambda$, say)



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Thus, AE also bisects $\angle BEF$.

Now, because in a triangle, exterior angle is the sum of interior opposite angles, so

$$\angle EFC = \theta + \lambda \text{ and ext } \angle B = \theta + \lambda$$

So, $\angle EFC$ = ext $\angle B$. But $\angle B = 2\theta + \angle C$, so $\angle EFC = 2\theta + \angle C$.

Therefore, $\angle EFC > \angle C$

So, $CE >$ FE

Hence $CE > BE$ (using (i)).

Example 15

The perimeter of a triangle is 20. The lengths of three sides are all integers. How many triangle are there are not congruent?

Solution

Method–1:

Let x be the length of the shortest side.

$x = 2$:	(2, 9, 9)	1 triangle
$x = 3$:	(3, 8, 9)	1 triangle
$x = 4$:	(4, 7, 9), (4, 8, 8)	2 triangle
$x = 5$:	(5, 6, 9), (5, 7, 8)	2 triangle
$x = 6$:	(6, 6, 8), (6, 7, 7)	2 triangle

In total, there are 8 such triangles.

Method – 2:

Let x be the length of the longest side.

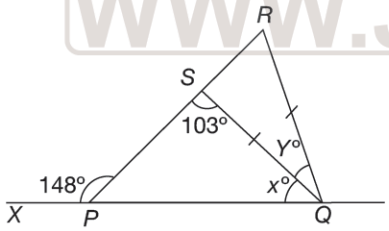
$x = 9$:	(9, 9, 2), (9, 8, 3), (9, 7, 4), (9, 6, 5),	4 triangle
$x = 8$:	(8, 8, 4), (8, 7, 5), (8, 6, 6),	3 triangle
$x = 7$:	(7, 7, 7)	1 triangle

In total, there are 8 such triangles.

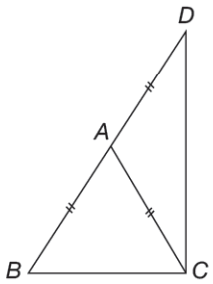
Know Your Progress - 2

1. How many different isosceles triangles have integer side lengths and perimeter of 81 units?

2. In given figure, find the values of x and y, if $QS = RQ$.



3. In the figure, $AB = AC = AD$. Prove that $\angle BCD$ is a right angle.

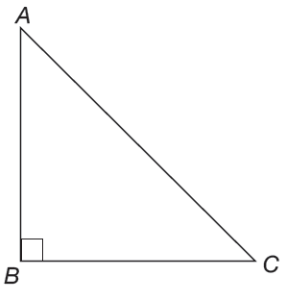


4. Prove that the perimeter of a triangle is greater than the sum of its three medians.

5. O is any point in the interior of a triangle ABC. Prove that:
   (i) $AB + AC > OB + OC$
   (ii) $AB + BC + CA > OA + OB + OC$
   (iii) $OA + OB + OC > \frac{1}{2}(AB + BC + AC)$.

6. In how many ways can square be cut into two congruent triangles?

7. In Figure, ABC is a right triangle and right angled at B such that $\angle BCA = 2\angle BAC$. Show that hypotenuse $AC = 2BC$.



8. Line segment joining the mid-points M and N of parallel sides AB and DC, respectively, of a trapezium $ABCD$ is perpendicular to both the sides AB and DC. Prove that $AD = BC$.

9. ΔABC and ΔDBC are two isosceles triangles on the same base BC, and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
   (i) $\Delta ABD \cong \Delta ACD$
   (ii) $\Delta ABP \cong \Delta ACP$
   (iii) AP bisects $\angle A$ as well as $\angle D$
   (iv) AP is the perpendicular bisector of BC.

10. In ΔPQR, S is any point on the side QR. Show that $PQ + QR + RP > 2PS$.

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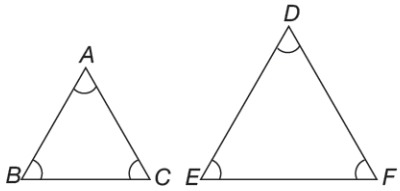
Similar Triangles

Two triangles ABC and DEF are said to be similar if their

(a) Corresponding angles are equal.

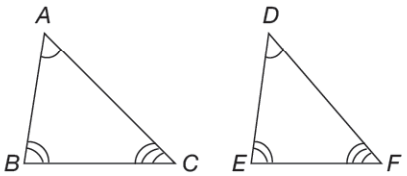
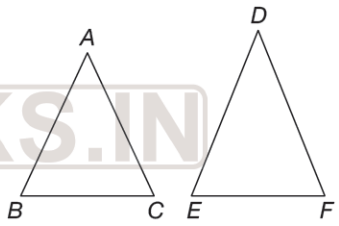
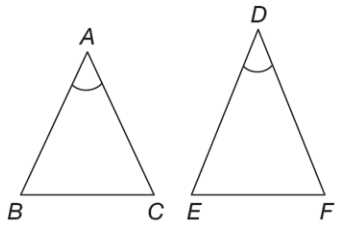


i.e., $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

(b) Corresponding sides are proportional, i.e.,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Characteristic Properties of Similar Triangles

Test	Property	Diagram
A – A – A (similarity)	If in two triangles, corresponding angles are equals, i.e., the two triangles are equiangular, then the triangles are similar. $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ $\Delta ABC \sim \Delta DEF$	
S – S – S (Similarity)	If the corresponding sides of two triangles are proportional, then they are similar. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ $\Delta ABC \sim \Delta DEF$	
S – A – S (Similarity)	If in two triangles, one pair of corresponding sides are proportional and the included angles are equal, then the two triangles arc similar. $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle BAC = \angle EDF$ $\Delta ABC \sim \Delta DEF$	

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Result Based Upon Characteristic Properties of Similar Triangles

(a) If two triangles are equiangular, then the ratio of the corresponding sides is the same as the ratio of the corresponding medians.

(b) If two triangles are equiangular, then the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.

(c) If two triangles are equiangular, then the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.

(d) If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite side in the same ratio, then the triangles are similar.

(e) If two sides and a median bisecting one of these sides of a triangle are, respectively, proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.

(f) If two sides and a median bisecting the third side of a triangle are, respectively, proportional to the corresponding sides and the median of another triangle, then two triangles are similar.

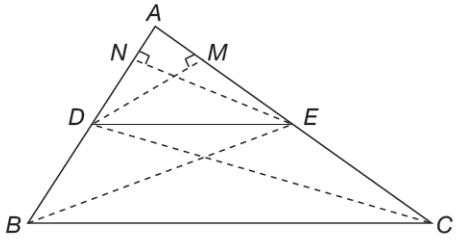
Note

If the ratio of sides of triangle is $a: b: c$, then ratio of their altitudes is $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$.

Thales Theorem (Basic Propotionality Theorem)

Statement: If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.

Given: A triangle ABC in which a line parallel to side BC intersects other two sides AB and AC at D and E, respectively.



To prove: $\frac{AD}{BD} = \frac{AE}{EC}$.

Construction: Join BE and CD and draw $DM \perp AC$ and $EN \perp AB$.

Proof: Area of $\Delta ADE = \frac{1}{2}$ base $\times$ height $= \frac{1}{2} AD \times EN$.

Area of ΔADE is denoted as ar(ADE)

So, $$\text{ar}(ADE) = \frac{1}{2} AD \times \text{EN}$$

And $$\text{ar}(BDE) = \frac{1}{2} DB \times EN,$$

Therefore, $$\frac{\text{ar}(ADE)}{\text{ar}(BDE)} = \frac{\frac{1}{2} AD \times EN}{\frac{1}{2} DB \times EN} = \frac{AD}{DB} \quad \ldots(\text{i})$$

Similarly, $\text{ar}(ADE) = \frac{1}{2} AE \times DM$ and $\text{ar}(DEC) = \frac{1}{2} EC \times DM$.

And $$\frac{\text{ar}(ADE)}{\text{ar}(DEC)} = \frac{\frac{1}{2} AE \times DM}{\frac{1}{2} EC \times DM} = \frac{AE}{EC} \quad \ldots(\text{ii})$$

Note that ΔBDE and ΔDEC are on the same base DE and between the two parallel lines BC and DE.

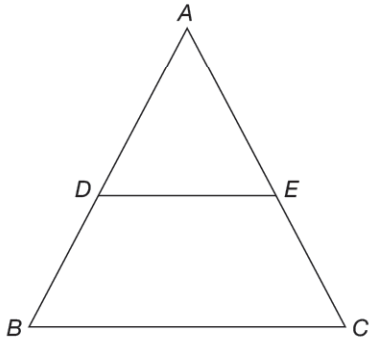
So, $$\text{ar}(BDE) = \text{ar}(DEC) \quad \ldots(\text{iii})$$

Therefore, from (i), (ii), and (iii), we have:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Hence proved.

Corollary: If in a ΔABC, a line $DE \parallel BC$, intersects AB in D and AC in E, then



(i) $\frac{DB}{AD} = \frac{EC}{AE}$ (ii) $\frac{AB}{AD} = \frac{AC}{AE}$

(iii) $\frac{AD}{AB} = \frac{AE}{AC}$ (iv) $\frac{AB}{DB} = \frac{AC}{EC}$

(v) $\frac{DB}{AB} = \frac{EC}{AC}$

Converse of Basic Proportionality Theorem

If a line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

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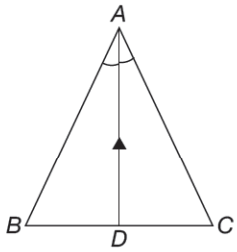
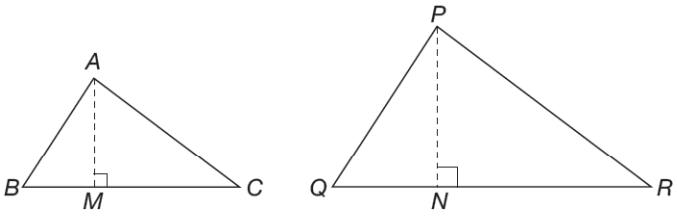
Some Important Results and Theorem

Theorems	Statement/ Explanation	Diagram
Vertical angle bisector	The bisector of the vertical angle of a triangle divides the base in the ratio of other two sides. $\frac{BD}{DC} = \frac{AB}{AC}$	

Areas of Similar Triangle

Theorem: The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Given: Two triangles ABC and PQR, such that $\Delta ABC \sim \Delta PQR$ [Shown in the figure]



To prove: $\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$.

Construction: Draw altitudes AM and PN of the triangle ABC and PQR.

Proof: $\text{ar}(ABC) = \frac{1}{2} BC \times AM$

And $\text{ar}(PQR) = \frac{1}{2} QR \times PN$

So, $\frac{\text{ar}(ABC)}{\text{ar}(PQR)} = \frac{\frac{1}{2} BC \times AM}{\frac{1}{2} QR \times PN} = \frac{BC \times AM}{QR \times PN}$ …(i)

Now, in ΔABM and ΔPQN,

And $\angle B = \angle Q$ [As $\Delta ABC \sim \Delta PQR$]

$\angle M = \angle N$ [90° each]

So, $\Delta ABM \sim \Delta PQN$ [*AA* similarity criterion]

Therefore, $\frac{AM}{PN} = \frac{AB}{PQ}$ …(ii)

Also, $\Delta ABC \sim \Delta PQR$ [Given]

So, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$ …(iii)

Therefore, $\frac{\text{ar}(ABC)}{\text{ar}(PQR)} = \frac{BC}{QR} \times \frac{AB}{PQ}$ [From (i) and (ii)]

$= \frac{AB}{PQ} \times \frac{AB}{PQ}$ [From (iii)]

$= \left(\frac{AB}{PQ}\right)^2$

Now using (iii), we get

$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$

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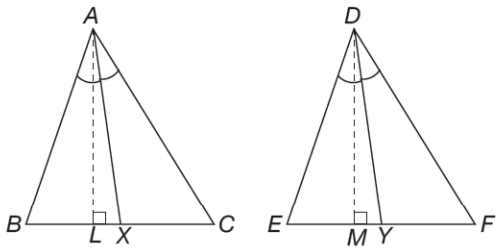
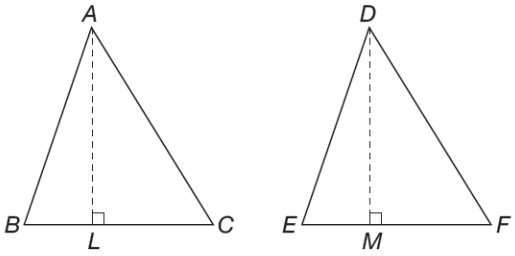
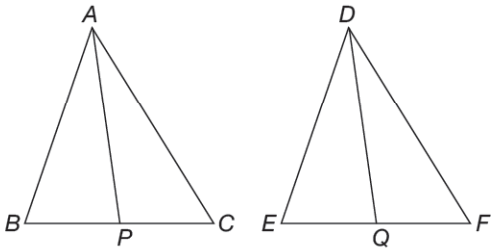
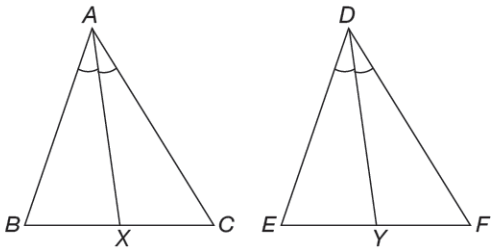
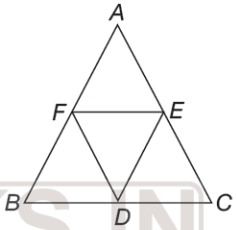
Properties of Areas of Similar Triangle

S. No.	Statement	Diagram
1.	The areas of two similar triangles are proportional to the squares of their corresponding sides. If $\Delta ABC \sim \Delta DEF$, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$	
2.	The areas of two similar triangles are proportional to the squares of their corresponding altitude. If $\Delta ABC \sim \Delta DEF$, $AL \perp BC$ and $DM \perp EF$, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AL^2}{DM^2}$	

(*Continued*)

S. No.	Statement	Diagram
3.	The areas of two similar triangles are proportional to the squares of their corresponding medians. If $ABC \sim DEF$ and AP, DQ are their medians, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AP^2}{DQ^2}$	
4.	The areas of two similar triangles are proportional to the squares of their corresponding angle bisector segments. If $\Delta ABC \sim \Delta DEF$ and AX. DY are their bisectors of $\angle A$ and $\angle D$, respectively, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AX^2}{DY^2}$	
5.	If D, E, and F are, respectively, the mid-points of sides BC, CA, and AB of an equilateral triangle ABC then ΔDEF is also an equilateral triangle.	

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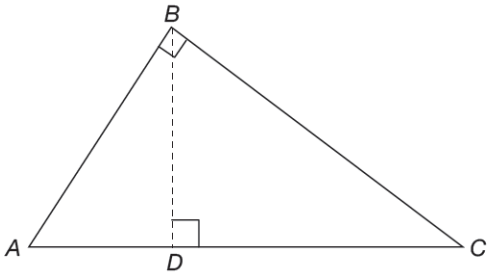
Pythagoras Theorem

Statement: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given: A right triangle ABC, right angled at B.

To prove: $AC^2 = AB^2 + BC^2$

Construction: $BD \perp AC$



Proof: ΔADB and ΔABC

$\angle DAB = \angle CAB$ [Common]

$\angle BDA = \angle CBA$ [90° each]

So, $\Delta ADB \sim \Delta ABC$ [By AA similarity]

$$\frac{AD}{AB} = \frac{AB}{AC}$$ [Sides are proportional]

or, $AD \cdot AC = AB^2$ …(i)

Similarly $\Delta BDC \sim \Delta ABC$

So, $\frac{CD}{BC} = \frac{BC}{AC}$

or $CD \cdot AC = BC^2$ …(ii)

Adding (i) and (ii),

$AD \cdot AC + CD \cdot AC = AB^2 + BC^2$

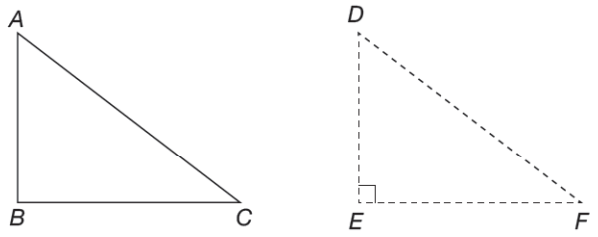
or, $AC(AD + CD) = AB^2 + BC^2$

or, $AC \cdot AC = AB^2 + BC^2$

or, $AC^2 = AB^2 + BC^2$ Hence proved.

Converse of Pythagoras Theorem

Theorem: In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.



Given: A triangle ABC such that $AC^2 = AB^2 + BC^2$

Construction: Construct a triangle DEF such that $DE = AB$, $EF = BC$, and $\angle E = 90°$.

Proof: In order to prove that $\angle B = 90°$, it is sufficient to show $\Delta ABC \sim \Delta DEF$. For this, we proceed as follows.

Since ΔDEF is a right–angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have

$$DF^2 = DE^2 + EF^2$$

$\Rightarrow$ $DF^2 = AB^2 + BC^2$

$[\because DE = AB$ and $EF = BC$ (By construction)$]$

$\Rightarrow$ $DF^2 = AC^2$ $[\because AB^2 + BC^2 = AC^2$ (Given)$]$

$\Rightarrow$ $DF = AC$ …(i)

Thus, in ΔABC and ΔDEF, we have

$AB = DE$, $BC = EF$ [By construction]

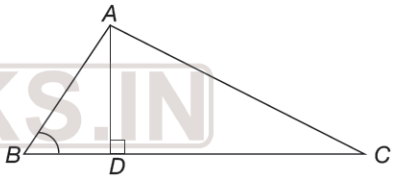
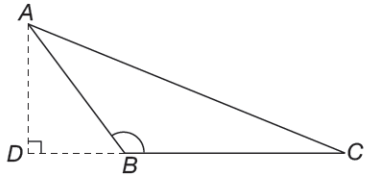
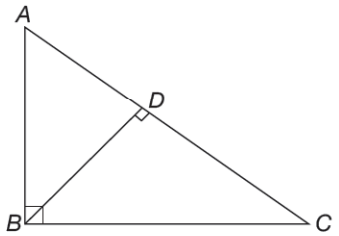
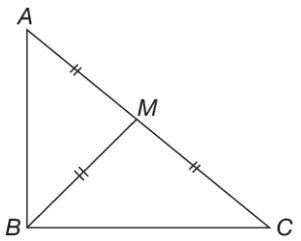
And $AC = DF$ [From equation (i)]

$\therefore$ $\Delta ABC \cong \Delta DEF$ [By *SSS* criteria of congruency]

$\Rightarrow$ $\angle B = \angle E = 90°$

Hence, ΔABC is a right triangle, right angled at B.

Some Results Deduced From Pythagoras Theorem

S.No.	Statement	Diagram
1.	In an acute angle ΔABC, AD is a perpendicular dropped on the opposite side of $\angle A$, then $AC^2 = AB^2 + BC^2 - 2\,BD \cdot BC$ $(\angle B < 90°)$	
2.	In a obtuse angle ΔABC, AD is perpendicular dropped on $BC \cdot BC$ is produced to D to meet AD, then $AC^2 = AB^2 + BC^2 + 2\,BD\,BC$ $(\angle B > 90°)$	
3.	In a right-angled ΔABC, $\angle B = 90°$ and AC is hypotenuse, the perpendicular BD is dropped on hypotenuse AC from right angle vertex B, then (i) $BD = \frac{AB \times BC}{AC}$ (ii) $AD = \frac{AB^2}{AC}$ (iii) $CD = \frac{BC^2}{AC}$ (iv) $\frac{1}{BD^2} = \frac{1}{AB^2} + \frac{1}{BC^2}$	
4.	In a right-angled triangle, the median to the hypotenuse $= \frac{1}{2} \times$ hypotenuse, i.e., $BM = \frac{AC}{2}$	

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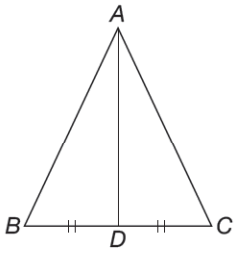
(*Continued*)

5. Apollonius theorem
In a triangle, the sum of the squares of any two sides of a triangle is equal to twice the sum of the square of the median to the third side and square of half the third side.
i.e., $AB^2 + AC^2 = 2(AD^2 + BD^2)$



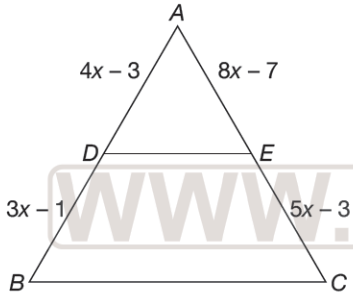
$BD = CD$, AD is the median

Example 16

In a ΔABC, D and E are points on the sides AB and AC, respectively, such that $DE \parallel BC$.

If $AD = 4x - 3$, $AE = 8x - 7$, $BD = 3x - 1$ and $CE = 5x - 3$, find the value of x.

Solution



In ΔABC, we have

$$DE \parallel BC$$

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$$

[By Basic Proportionality Theorem]

$$\Rightarrow \quad \frac{4x-3}{3x-1} = \frac{8x-7}{5x-3}$$

$$\Rightarrow \quad 20x^2 - 15x - 12x + 9 = 24x^2 - 21x - 8x + 7$$

$$\Rightarrow \quad 20x^2 - 27x + 9 = 24x^2 - 29x + 7$$

$$\Rightarrow \quad 4x^2 - 2x - 2 = 0$$

$$\Rightarrow \quad 2x^2 - x - 1 = 0$$

$$\Rightarrow \quad (2x + 1)(x - 1) = 0$$

$$\Rightarrow \quad x = 1 \text{ or } x = -\frac{1}{2}$$

So, the required value of x is 1 [$x = -\frac{1}{2}$ is neglected as length can not be negative].

Example 17

D and E are, respectively, the points on the sides AB and AC of a ΔABC such that $AB = 12$ cm, $AD = 8$ cm, $AE = 12$ cm, and $AC = 18$ cm, show that $DE \parallel BC$.

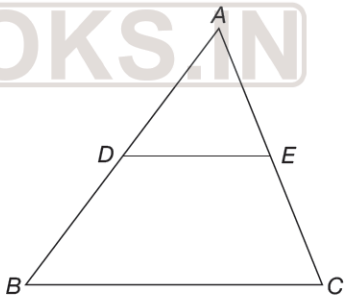
Solution

We have,

$AB = 12$ cm, $AC = 18$ cm, $AD = 8$ cm, and $AE = 12$ cm.

$\therefore \quad BD = AB - AD = (12 - 8)\text{cm} = 4 \text{ cm}$

$CE = AC - AE = (18 - 12)\text{cm} = 6 \text{ cm}$



Now, $\frac{AD}{BD} = \frac{8}{4} = \frac{2}{1}$

And, $\frac{AE}{CE} = \frac{12}{6} = \frac{2}{1} \Rightarrow \frac{AD}{BD} = \frac{AE}{CE}$

Thus, DE divides sides AB and AC of ΔABC in the same ratio. Therefore, by the converse of basic proportionality theorem, we have $DE \parallel BC$.

Example 18

In a trapezium $ABCD$, $AB \parallel DC$ and $DC = 2AB \cdot EF$ drawn parallel to AB cuts AD in F and BC in E such that $\frac{BE}{EC} = \frac{3}{4}$. Diagonal DB intersects EF at G. Prove that $7FE = 10AB$.

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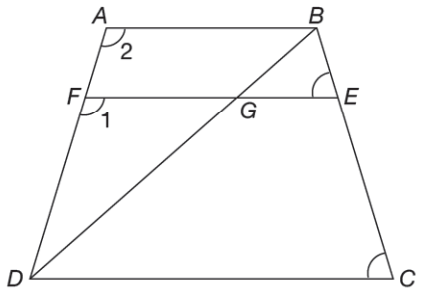
Solution



In ΔDFG and ΔDAB,

$\angle 1 = \angle 2$

[Corresponding $\angle s$ $\because AB \,||\, FG$]

$\angle FDG = \angle ADB$ [Common]

$\therefore$ $\Delta DFG \sim \Delta DAB$ [By AA rule of similarity]

$\therefore$ $\dfrac{DF}{DA} = \dfrac{FG}{AB}$ …(i)

$\angle FDG = \angle ADB$ [Common]

$\therefore$ $\Delta DFG \sim \Delta DAB$ [By AA rule of similarity]

$\therefore$ $\dfrac{DF}{DA} = \dfrac{FG}{AB}$ …(i)

Again in trapezium $ABCD$

$EF \,||\, AB \,||\, DC$

$\therefore$ $\dfrac{AF}{DF} = \dfrac{BE}{EC}$

$\Rightarrow$ $\dfrac{AF}{DF} = \dfrac{3}{4}$ $\left[\dfrac{BE}{EC} = \dfrac{3}{4} \text{ (given)}\right]$

$\Rightarrow$ $\dfrac{AF}{DF} + 1 = \dfrac{3}{4} + 1$

$\Rightarrow$ $\dfrac{AF + DF}{DF} = \dfrac{7}{4}$

$\Rightarrow$ $\dfrac{AD}{DF} = \dfrac{7}{4} \Rightarrow \dfrac{DF}{AD} = \dfrac{4}{7}$ …(ii)

From (i) and (ii), we get

$\dfrac{FG}{AB} = \dfrac{4}{7}$ i.e., $FG = \dfrac{4}{7}\ AB$ …(iii)

In ΔBEG and ΔBCD, we have

$\angle BEG = \angle BCD$

[Corresponding angle $\because$ $EG \,||\, CD$]

$\angle GBE = \angle DBC$ [Common]

$\therefore$ $\Delta BEG \sim \Delta BCD$

[By AA rule of similarity]

$\therefore$ $\dfrac{BE}{BC} = \dfrac{EG}{CD}$

$\therefore$ $\dfrac{3}{7} = \dfrac{EG}{CD}$

$\left[\dfrac{BE}{EG} = \dfrac{3}{7} \text{ i.e., } \dfrac{EC}{BE} = \dfrac{4}{3} \Rightarrow \dfrac{EC + BE}{BE} = \dfrac{4+3}{3} \Rightarrow \dfrac{BC}{BE} = \dfrac{7}{3}\right]$

$\therefore$ $EG = \dfrac{3}{7}\ CD = \dfrac{3}{7}\ (2\ AB)$

[$\because$ $CD = 2AB$ (given)]

$\therefore$ $EG = \dfrac{6}{7}\ AB$ …(iv)

Adding (iii) and (iv), we get

$$FG + EG = \frac{4}{7}\ AB + \frac{6}{7}\ AB = \frac{10}{7} AB$$

$\Rightarrow$ $EF = \dfrac{10}{7} AB$ i.e., $7EF = 10AB$.

Hence proved.

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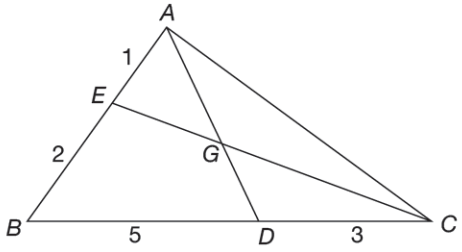
Example 19

In the figure $AE : EB = 1 : 2$ and $BD : DC = 5 : 3$. Find $EG : GC$.



Solution

Through E, we draw a line parallel to AD, intersecting BC at M.

$AE: EB = 1: 2$

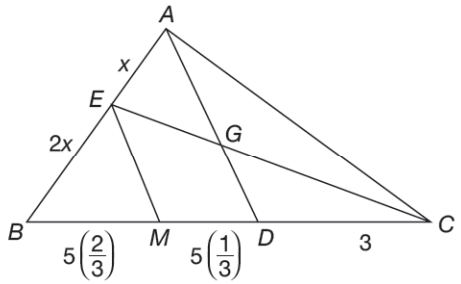
Let $AE = x$, $EB = 2x$,

ΔABD $BD : DC = 5 : 3$ Let $BD = 5\gamma$ $CD = 3\gamma$

$EM \,||\, AD$

By BDT

Let $Bx = 5x$



$$\frac{BM}{MD} = \frac{BE}{EA} = \frac{2x}{x} = \frac{2}{1}$$

$$\therefore \quad BM = \frac{2}{3} \times BD = \frac{2}{3} \times 5y$$

$$MD = \frac{1}{3} \times 5y$$

$\therefore \Delta CEM, \quad GB \,||\, EM$

By BPT

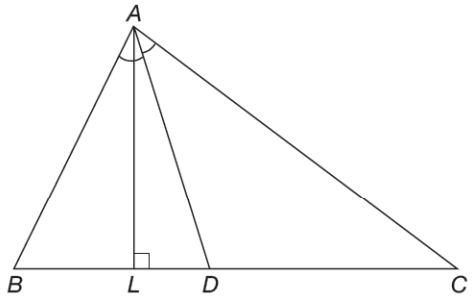
$$\frac{EG}{GC} = \frac{MD}{DC} = \frac{5y/3}{3y} = \frac{5}{9}$$

Example 20

In ΔABC, if AD is the bisector of $\angle A$, prove that $\frac{\text{Area}\,(\Delta ABD)}{\text{Area}\,(\Delta ACD)} = \frac{AB}{AC}$.

Solution

In ΔABC, AD is the bisector of $\angle A$.



$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC} \qquad \ldots(i)$$

[By internal bisector theorem]

From A draw $AL \perp BC$

$$\therefore \quad \frac{\text{Area}\,(\Delta ABD)}{\text{Area}\,(\Delta ACD)} = \frac{\frac{1}{2} BD \cdot AL}{\frac{1}{2} DC \cdot AL} = \frac{BD}{DC} = \frac{AB}{AC}$$

[From (i)]

Hence proved.

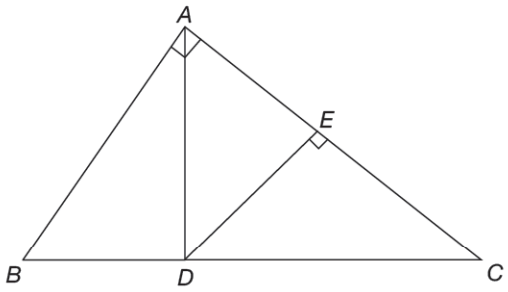
Example 21

In triangle ABC, $\angle BAC = 90°$, AD is its bisector. If $DE \perp AC$, prove that $DE \times (AB + AC) = AB \times AC$.

Solution

It is given that AD is the bisector of $\angle A$ of ΔABC.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$



$$\Rightarrow \quad \frac{AB}{AC} + 1 = \frac{BD}{DC} + 1 \quad \text{[Adding 1 on both sides]}$$

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{BD + DC}{DC}$$

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{BC}{DC} \qquad \ldots(i)$$

In Δ's CDE and CBA, we have

$$\angle DCE = \angle BCA \quad \text{[Common]}$$

$$\angle DEC = \angle BAC \quad \text{[Each equal to 90°]}$$

So, by AA–criterion of similarity

$$\Delta CDE \sim \Delta CBA$$

$$\Rightarrow \quad \frac{CD}{CB} = \frac{DE}{BA}$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{DC} \qquad \ldots(ii)$$

From (i) and (ii), we have

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{AB}{DE}$$

$$\Rightarrow \quad DE \times (AB + AC) = AB \times AC.$$

Example 22

Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.

Solution

Given: A square $ABCD$. Equilateral triangles ΔBCE and ΔACF have been described on side BC and diagonal AC, respectively.

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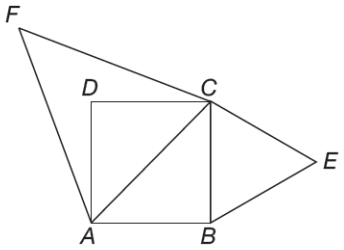
To prove: Area $(\Delta BCE) = \frac{1}{2} \cdot \text{Area}(\Delta ACF)$

Proof: Since ΔBCE and ΔACF are equilateral. Therefore, they are equiangular (each angle being equal to 60°) and hence $\Delta BCE \sim \Delta ACF$.



$\because$ ABCD is a square
$\therefore$ Diagonal = $\sqrt{2}$(side)
$\Rightarrow AC = \sqrt{2}BC$

$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{AC^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{\left(\sqrt{2}BC\right)^2} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{1}{2}.$$

Hence proved.

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Example 23

BL and CM are medians of ΔABC right angled at A. Prove that $4\,(BL^2 + CM^2) = 5\,BC^2$.

Solution

In ΔBAL

$$\Rightarrow \quad BL^2 = AL^2 + AB^2 \qquad \ldots(i)$$

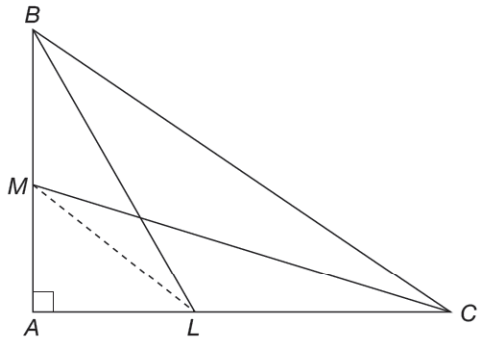
[Using Pythagoras theorem]

and, In ΔCAM

$$CM^2 = AM^2 + AC^2 \qquad \ldots(ii)$$

[Using Pythagoras theorem]

Adding (1) and (2) and then multiplying by 4, we get



$$4(BL^2 + CM^2) = 4(AL^2 + AB^2 + AM^2 + AC^2)$$

$$= 4\{AL^2 + AM^2 + (AB^2 + AC^2)\}$$

[$\therefore \Delta ABC$ is a right triangle]

$$= 4(AL^2 + AM^2 + BC^2) = 4(ML^2 + BC^2)$$

[$\therefore \Delta LAM$ is a right triangle]

$$= 4ML^2 + 4\,BC^2$$

[A line joining mid–points of two sides is parallel to third side and is equal to half of it, $ML = BC/2$]

$$= BC^2 + 4BC^2 = 5BC^2.$$

Hence proved.

Example 24

In the given figure, $BC \perp AB$, $AE \perp AB$ and $DE \perp AC$. Prove that $DE \cdot BC = AD \cdot AB$.

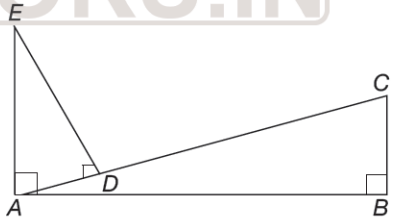
Solution

In ΔABC and ΔEDA,

We have

$\angle ABC = \angle ADE$ [Each equal to 90°]

$\angle ACB = \angle EAD$ [Alternate angles]



$\therefore$ By AA Similarity

$$\Delta ABC \sim \Delta EDA$$

$$\Rightarrow \quad \frac{BC}{AB} = \frac{AD}{DE}$$

$$\Rightarrow \quad DE \cdot BC = AD \cdot AB.$$

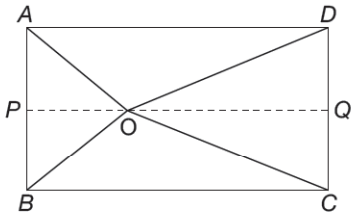
Hence proved.

Example 25

O is any point inside a rectangle $ABCD$ (shown in the figure). Prove that $OB^2 + OD^2 = OA^2 + OC^2$.

Solution

Through O, draw $PQ \,||\, BC$ so that P lies on AB and Q lies on DC.



Now, $PQ \parallel BC$

Therefore,

$$PQ \perp AB \text{ and } PQ \perp DC$$

$$[\angle B = 90^\circ \text{ and } \angle C = 90^\circ]$$

So, $\angle BPQ = 90^\circ$ and $\angle CQP = 90^\circ$

Therefore, $BPQC$ and $APQD$ are both rectangles.

Now, from ΔOPB,

$$OB^2 = BP^2 + OP^2 \quad \ldots(i)$$

Similarly, from ΔODQ,

$$OD^2 = OQ^2 + DQ^2 \quad \ldots(ii)$$

From ΔOQC, we have

$$OC^2 = OQ^2 + CQ^2 \quad \ldots(iii)$$

And from ΔOAP, we have

$$OA^2 = AP^2 + OP^2 \quad \ldots(iv)$$

Adding (i) and (ii)

$$OB^2 + OD^2 = BP^2 + OP^2 + OQ^2 + DQ^2$$

$$= CQ^2 + OP^2 + OQ^2 + AP^2$$

[As $BP = CQ$ and $DQ = AP$]

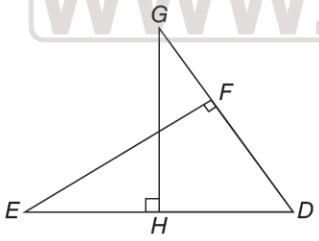
$$= CQ^2 + OQ^2 + OP^2 + AP^2 = OC^2 + OA^2$$

[From (iii) and (iv)]

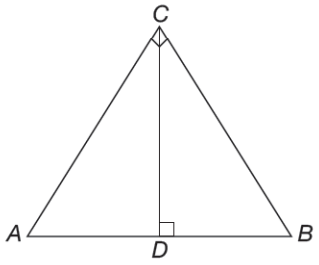
Hence proved.

Know Your Progress - 3

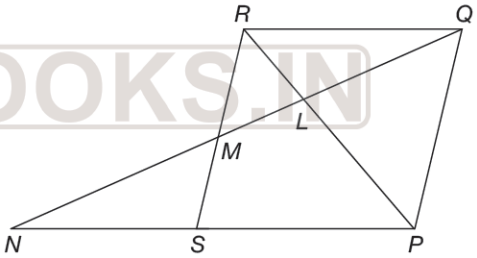
1. In figure, $\angle GHE = \angle DFE = 90^\circ$, $DH = 8$, $DF = 12$, $DG = 3x - 1$ and $DE = 4x + 2$. Find DE.



2. In figure, $\angle ACB = 90^\circ$, $CD \perp AB$, prove that $CD^2 = BD \cdot AD$.



3. In the figure, $PQRS$ is a parallelogram with $PQ = 16$ cm and $QR = 10$ cm. L is a point on PR such that $RL : LP = 2 : 3$. QL produced meets RS at M and PS produced at N.



Find the lengths of PN and RM.

4. ABC is a right triangle, right–angled at C. Let $BC = a$, $CA = b$, $AB = c$, and let p be the length of perpendicular from C on AB. Prove that:

(i) $cp = ab$ (ii) $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$

5. In an equilateral triangle ABC, the side BC is trisected at D. Prove that $9\,AD^2 = 7AB^2$.

6. In ΔABC, $\angle ABC = 135^\circ$. Prove that $AC^2 = AB^2 + BC^2 + 4ar(\Delta ABC)$.

7. In the figure $DF \parallel AG$, $DE \parallel AB$, $AB = 15$, $CD = 8$, $AD = x$, $DE = 10$, $FG = y$, and $CG = 6$. The ratio $x : y$ equal to:

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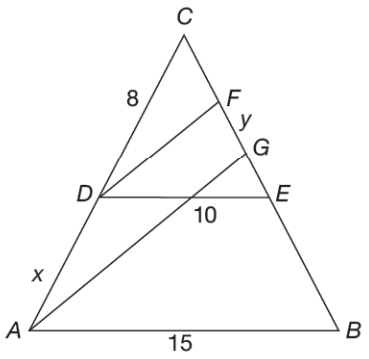
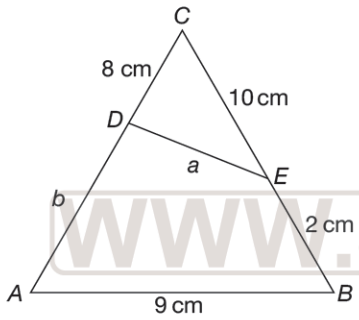


(A) 1 : 2 **(B)** 1 : 3
(C) 2 : 1 **(D)** 3 : 2

8. In the figure $\angle A = \angle CED$, $CD = 8$ cm $CE = 10$ cm, $BE = 2$ cm, $AB = 9$ cm, $AD = b$ and $DE = a$. The value of $a + b$ is:



(A) 13 cm **(B)** 15 cm
(C) 12 cm **(D)** 9 cm

9. In ΔABC, D is a point on BC such that $3BD = BC$. If each side of the triangle is 12 cm, then AD equals

(A) $4\sqrt{5}$ **(B)** $4\sqrt{6}$
(C) $4\sqrt{7}$ **(D)** $4\sqrt{11}$

10. In ΔABC, $\overline{XY}$ is parallel to $\overline{AC}$ and divides the triangle into the two parts of equal area. Then, $\frac{AX}{AB}$ equals

(A) $\frac{\sqrt{2}+1}{2}$ **(B)** $\frac{2-\sqrt{2}}{2}$
(C) $\frac{2+\sqrt{2}}{2}$ **(D)** $\frac{\sqrt{2}-1}{2}$

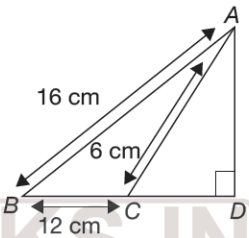
11. $PQRS$ is a smallest square whose vertices are the opposite sides of the square $ABCD$. The ratio of the area of $PQRS$ to $ABCD$ is

(A) 1: 2 **(B)** 1: $\sqrt{2}$
(C) 1: 3 **(D)** 2: 3

12. The length of a ladder is exactly equal to the height of the wall it is leaning against. If the lower end of the ladder is kept on a bench of height 3 m and the bench is kept 9 m away from the wall, the upper end of the ladder coincides with the top of the wall. The height of the wall is:

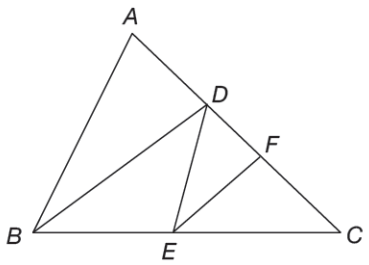
(A) 11 m **(B)** 12 m
(C) 15 m **(D)** 18 m

13. In the figure $\angle D = 90°$, $AB = 16$ cm, $BC = 12$ cm, and $CA = 6$ cm, then CD is:



(A) $\frac{13}{6}$ cm **(B)** $\frac{17}{6}$ cm
(C) $\frac{19}{6}$ cm **(D)** $\frac{18}{5}$ cm

14. In the given figure, ΔABC has points D and F in $\overline{AC}$ and point E in $\overline{BC}$ such that $\overline{DE} \parallel \overline{AB}$ and $\overline{EF} \parallel \overline{BD}$. If $CF = 4$ cm and $AC = 9$ cm, what is the length of $\overline{DC}$?



(A) 7 cm **(B)** 6 cm
(C) 5 cm **(D)** 4 cm

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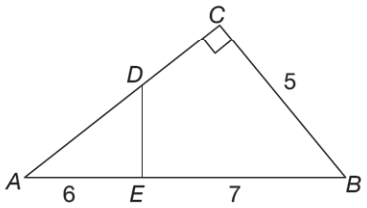
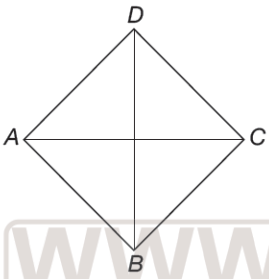
15. In the figure, C is a right angle, $DE \perp AB$, $AE = 6$, $EB = 7$, and $BC = 5$. The area of the quadrilateral $EBCD$ is:



(A) 27.5 **(B)** 25
(C) 22.5 **(D)** 20

Quadrilateral

A **quadrilateral** is a four-sided closed figure.

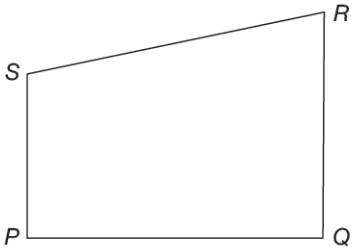


Let A, B, C, and D be four points in a plane such that:

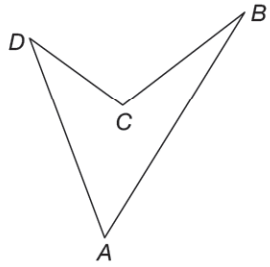
(i) No three of them are collinear.
(ii) The line segments AB, BC, CD, and DA do not intersect except at their end points, then figure obtained by joining A, B, C, and D is called a quadrilateral.

Convex and Concave Quadrilaterals

(i) A quadrilateral in which the measure of each interior angle is less than 180° is called a **convex quadrilateral.**
In figure, $PQRS$ is convex quadrilateral.

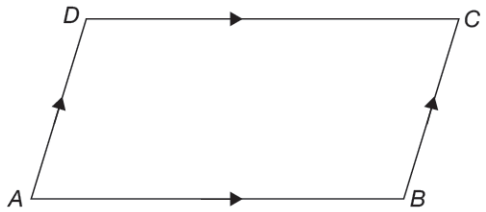


(ii) A quadrilateral in which the measure of one of the interior angles is more than 180° is called a **concave quadrilateral.** In figure, $ABCD$ is concave quadrilateral.



Special Quadrilaterals

(i) Parallelogram: A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.
In figure, $AB \parallel DC$, $AD \parallel BC$. Therefore, $ABCD$ is a parallelogram.



Properties:
(a) A diagonal of a parallelogram divides it into two congruent triangles.
(b) In a parallelogram, opposite sides are equal.
(c) The opposite angles of a parallelogram are equal.
(d) The diagonals of a parallelogram bisect each other.

(ii) Rectangle: A rectangle is a parallelogram, in which each of its angle is a right angle. If $ABCD$ is a rectangle, then $\angle A = \angle B = \angle C = \angle D = 90°$, $AB = CD$, $BC = AD$, and diagonals $AC = BD$.

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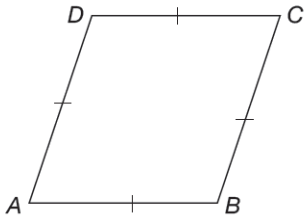
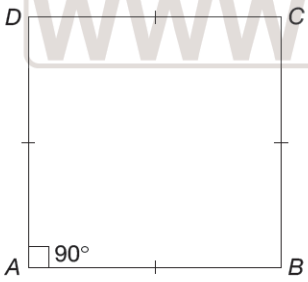
(iii) Rhombus: A rhombus is a parallelogram in which all its sides are equal in length. If $ABCD$ is a rhombus, then $AB = BC = CD = DA$.



The diagonals of a rhombus are perpendicular to each other.

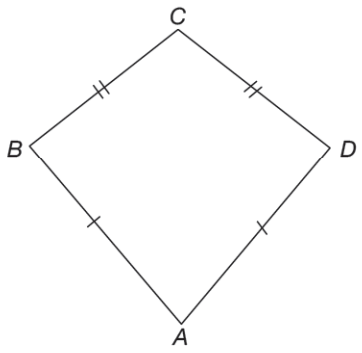
(iv) Square: A square is a parallelogram having all sides equal and each angle equal to right angle. If $ABCD$ is a square, then $AB = BC = CD = DA$, diagonal $AC = BD$, and $\angle A = \angle B = \angle C = \angle D = 90°$.



The diagonals of a square are perpendicular to each other.

(v) Trapezium: A trapezium is a quadrilateral with only one pair of opposite sides parallel. In figure, $ABCD$ is a trapezium with $AB \parallel DC$.

(vi) Kite: A kite is a quadrilateral in which two pairs of adjacent sides are equal. If $ABCD$ is a kite then $AB = AD$ and $BC = CD$.



(vii) Isosceles trapezium: A trapezium is said to be an isosceles trapezium, if its non–parallel sides are equal. Thus, a quadrilateral $ABCD$ is an isosceles trapezium, if $AB \parallel DC$ and $AD = BC$.

In isosceles trapezium $\angle A = \angle B$ and $\angle C = \angle D$.

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Properties

Theorem 1: The sum of the four angles of a quadrilateral is 360°.

Theorem 2: A diagonal of a parallelogram divides it into two congruent triangles.

Theorem 3: In a parallelogram, opposite sides are equal.

Theorem 4: The opposite angles of a parallelogram are equal.

Theorem 5: The diagonals of a parallelogram bisect each other.

Theorem 6: Each of the four angles of a rectangle is a right angle.

Theorem 7: Each of the four sides of a rhombus is of the same length.

Theorem 8: Each of the angles of a square is a right angle and each of the four sides is of the same length.

Theorem 9: The diagonals of a rectangle are of equal length.

Theorem 10: The diagonals of a rhombus are perpendicular to each other.

Theorem 11: The diagonals of a square are equal and perpendicular to each other.

A quadrilateral become a parallelogram when:

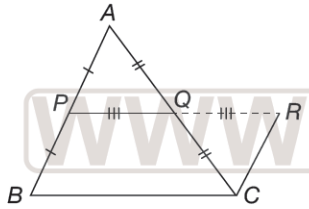
(i) Opposite angles are equal.
(ii) Both the pair of opposite sides are equal
(iii) A pair of opposite side is equal as well as parallel
(iv) Diagonals of quadrilateral bisect each other.

Note

(i) Square, rectangle, and rhombus are all parallelograms.
(ii) Kite and trapezium are not parallelograms.
(iii) A square is a rectangle.
(iv) A square is a rhombus.
(v) A parallelogram is a trapezium.

Mid Point Theorem

In a triangle, the line segment joining the mid-points of any two sides is parallel to the third side and is half of it.



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Given: A triangle ABC in which P is the mid–point of side AB and Q is the mid-point of side AC.

To prove: PQ is parallel to BC and is half of it

i.e., $PQ \parallel BC$ and $PQ = \frac{1}{2} BC$.

Construction: Produce PQ upto point R such that $PQ = QR$. Join R and C.

Proof: In ΔAPQ and ΔCRQ

$PQ = QR$ [By construction]

$AQ = QC$ [Given]

And, $\angle AQP = \angle CQR$ [Vertically opposite angles]

So, $\Delta APQ \cong \Delta CRQ$ [By SAS]

$\therefore$ $AP = CR$ [By CPCT]

And, $\angle APQ = \angle CRQ$ [By CPCT]

But, $\angle APQ$ and $\angle CRQ$ are alternate angles and whenever the alternate angles are equal, the lines are parallel.

$\Rightarrow$ $AP \parallel CR \Rightarrow AB \parallel CR \Rightarrow BP \parallel CR$

$AP = BP$ [Given, P is mid–point of AB]

$\Rightarrow$ $CR = BP$ [As, $AP = CR$]

Now, $BP = CR$ and $BP \parallel CR \Rightarrow BCRP$ is a parallelogram.

[When any pair of opposite sides are equal and parallel, the quadrilateral is a parallelogram]

$BCRP$ is a parallelogram and opposite sides of a parallelogram are equal and parallel.

$\therefore$ $PR = BC$ and $PR \parallel BC$

Since,

$$PQ = QR \Rightarrow PQ = \frac{1}{2} PR = \frac{1}{2} BC \quad [\text{As, } PR = BC]$$

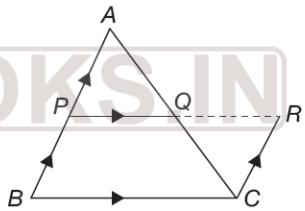
Also, $PQ \parallel BC$ [As, $PR \parallel BC$]

$\therefore$ $PQ \parallel BC$ and $PQ = \frac{1}{2} BC$

Hence proved.

Converse of the Mid–Point Theorem

The line drawn through the mid–point of one side of a triangle parallel to the another side; bisects the third side.



Given: A triangle ABC in which P is the mid–point of side AB and PQ is parallel to BC.

To prove: PQ bisects the third side AC i.e., $AQ = QC$.

Construction: Through C, draw CR parallel to BA, which meets PQ produced at point R.

Proof: Since, $PQ \parallel BC$ i.e., $PR \parallel BC$ [Given]

and $CR \parallel BA$ i.e., $CR \parallel BP$. [By construction]

$\therefore$ Opposite sides of quadrilateral $PBCR$ are parallel.

$\Rightarrow$ $PBCR$ is a parallelogram

$\Rightarrow$ $BP = CR$

Also, $BP = AP$ [As, P is mid–point of AB]

$\therefore$ $CR = AP$

$\because$ $AB \parallel CR$ and AC is transversal, $\angle PAQ = \angle RCQ$ [Alternate angles]

$\therefore$ $AB \parallel CR$ and PR is transversal, $\angle APQ = \angle CRQ$ [Alternate angles]

In ΔAPQ and ΔCRQ

$CR = AP$, $\angle PAQ = \angle RCQ$ and $\angle APQ = \angle CRQ$

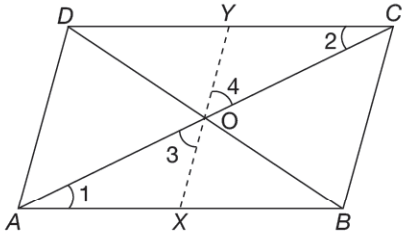
$\Rightarrow$ $\Delta APQ \cong \Delta CRQ$ [By ASA]

$\Rightarrow$ $AQ = QC$

Hence proved.

Example 26

The diagonals of a parallelogram $ABCD$ intersect at O. A line through O intersects AB at X and DC at Y. Prove that $OX = OY$.



Solution

Since $AB \parallel CD$.

In ΔOAX and ΔOCY

$\angle 1 = \angle 2$ [Alternate angles]

$OA = OC$ and, $\angle 3 = \angle 4$ [Vertically opposite angles]

So, by ASA criterion of congruence,

$\Delta OAX \cong \Delta OCY$

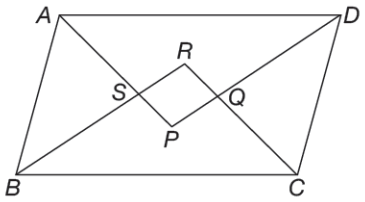
$\Rightarrow$ $OX = OY$.

Example 27

The angle bisectors of a parallelogram form a rectangle.

Solution

A parallelogram $ABCD$ in which bisectors of angles A, B, C, D intersect at P, Q, R, S to form a quadrilateral $PQRS$.



Since, $ABCD$ is a parallelogram, so $AD \parallel BC$ and transversal AB intersects them at A and B, respectively.

$$\angle A + \angle B = 180^\circ \Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^\circ$$

$\Rightarrow$ $\angle BAS + \angle ABS = 90^\circ$ ...(i)

[AS and BS are bisectors of $\angle A$ and $\angle B$, respectively]

$\angle BAS + \angle ABS + \angle ASB = 180^\circ \Rightarrow 90^\circ + \angle ASB = 180^\circ$

$\Rightarrow$ $\angle ASB = 90^\circ \Rightarrow \angle RSP = 90^\circ$

[$\angle ASB$ and $\angle RSP$ are vertically opposite angles]

Similarly, $\angle SRQ = 90^\circ$, $\angle RQP = 90^\circ$ and $\angle SPQ = 90^\circ$.

Hence, $PQRS$ is rectangle.

Example 28

$ABCD$ is a trapezium in which $AB \parallel CD$ and $AD = BC$. Show that:

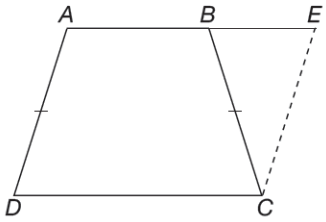
(i) $\angle A = \angle B$

(ii) $\angle C = \angle D$

(iii) $\Delta ABC \cong \Delta BAD$

(iv) Diagonal AC = diagonal BD.

Solution



(i) Extend AB and draw a line through C parallel to DA intersecting AB produced at E.

Since, $AD \parallel CE$ and transversal AE cuts them at A and E, respectively.

$\therefore$ $\angle A + \angle E = 180^\circ \Rightarrow 180^\circ - \angle E = \angle A$

Since, $AB \parallel CD$ and $AD \parallel CE$

$\therefore$ $AECD$ is a parallelogram.

$\Rightarrow$ $AD = CE \Rightarrow BC = CE$

Thus, in ΔBCE

$BC = CE \Rightarrow \angle CBE = \angle CEB \Rightarrow 180^\circ - \angle B = \angle E$

$\Rightarrow$ $180^\circ - \angle E = \angle B \Rightarrow \angle A = \angle B$.

(ii) Consecutive interior angles on the same side of a transversal are supplementary.

$\therefore$ $\angle A + \angle D = 180^\circ$ and $\angle E + \angle C = 180^\circ$

$\Rightarrow$ $\angle A + \angle D = \angle E + \angle C$

$\Rightarrow$ $\angle B + \angle D = \angle E + \angle C \Rightarrow \angle D = \angle C$

[$\angle B = \angle E$]

(iii) In ΔABC and ΔBAD

$AB = BA$

$\angle B = \angle A$ and $BC = AD$

So, by SAS congruence criterion

$\Delta ABC \cong \Delta BAD$

(iv) Since, $\Delta ABC \cong \Delta BAD$

$AC = BD$

Hence, diagonal AC = diagonal BD.

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Example 29

If the medians of a ΔABC intersect at G, show that ar(ΔAGB) = ar(ΔAGC) = ar(ΔBGC) = $\frac{1}{3}$ ar(ΔABC).

Solution

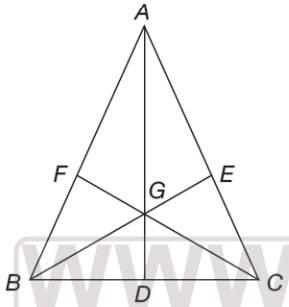
Given: A ΔABC and its medians AD, BE and CF intersect at G.

To prove: ar(ΔAGB) = ar(ΔAGC) = ar(ΔBGC) = $\frac{1}{3}$ ar(ΔABC).

Proof: A median of a triangle divides it into two triangles of equal area.

In ΔABC, AD is the median.

$\therefore$ ar(ΔABD) = ar(ΔACD) ...(i)



In ΔGBC, GD is the median.

$\therefore$ ar(ΔGBD) = ar(ΔGCD) ...(ii)

Subtract equation (ii) from (i), we get

ar(ΔABD) – ar(ΔGBD) = ar(ΔACD) – ar(ΔGCD)

$\therefore$ ar(ΔAGB) = ar(ΔAGC) ...(iii)

Similarly, ar(ΔAGB) = ar(ΔBGC) ...(iv)

From (iii) and (iv)

ar(ΔAGB) = ar(ΔAGC) = ar(ΔBGC)

But, ar(ABC) = ar(ΔAGB) + ar(ΔAGC) + ar(ΔBGC)

= 3 ar (ΔAGB)

$\therefore$ ar(ΔAGB) = $\frac{1}{3}$ ar(ΔABC).

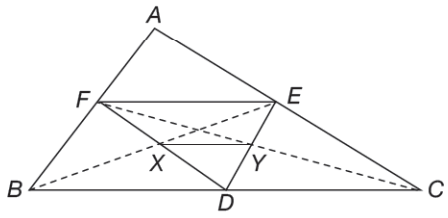
Hence, ar(ΔAGB) = ar(ΔAGC) = ar(ΔBGC) = $\frac{1}{3}$ ar(ΔABC).

Example 30

In the adjoining figure, D, E, F are the mid-points of the sides BC, CA and AB of ΔABC. If BE and DF intersect at X while CF and DE intersect at Y, prove that $XY = \frac{1}{4}$ BC.

Solution

In ΔABC, F and E are the mid-points of AB and AC, respectively.



$\therefore$ $FE \parallel BC$ and $FE = \frac{1}{2}$ $BC = BD$.

$\therefore$ $FE \parallel BD$ and $FE = BD$.

So, $BDEF$ is a parallelogram whose diagonals BE and DF intersect each other at X.

$\therefore$ X is the mid-point of DF.

Similarly, Y is the mid-point of DE.

Thus, in ΔDEF, X and Y are the mid-points of DF and DE, respectively.

So, $XY \parallel FE$ and $XY = \frac{1}{2}$ $FE = \frac{1}{2} \times \frac{1}{2}$ $BC = \frac{1}{4} BC$.

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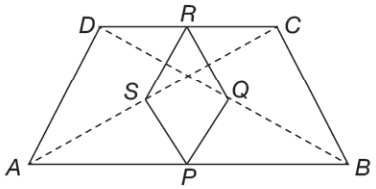
Example 31

In the adjoining figure, $ABCD$ is a trapezium in which $AB \parallel DC$ and $AD = BC$. If P, Q, R, S be, respectively, the mid-points of BA, BD and CD, CA, then show that $PQRS$ is a rhombus.

Solution

In ΔBDC, Q and R are the mid-points of BD and CD, respectively.



$\therefore$ $QR \parallel BC$ and $QR = \frac{1}{2}$ BC.

Similarly, $PS \parallel BC$ and $PS = \frac{1}{2}$ BC.

$\therefore$ $PS \parallel QR$ and $PS = QR$ [Each equal to $\frac{1}{2} BC$.]

$\therefore$ $PQRS$ is a parallelogram.

In ΔACD, S and R are the mid-points of AC and CD

respectively.

$\therefore \quad SR \,||\, AD$ and $SR = \frac{1}{2} AD = \frac{1}{2} BC \quad [\because AD = BC]$

$\therefore \quad PS = QR = SR = PQ.$

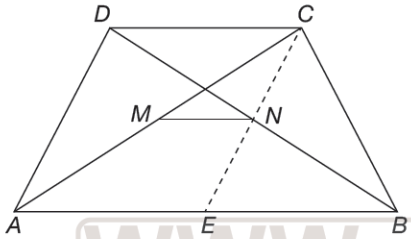
Hence, $PQRS$ is a rhombus.

Example 32

Prove that the line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half of their difference.

Solution

Let $ABCD$ be a trapezium in which $AB \,||\, DC$, and let M and N be the mid-points of the diagonal AC and BD, respectively.



Join CN and produce it to meet AB at E.

In ΔCDN and ΔEBN, we have

$DN = BN \quad [\because$ N is mid-point of BD$]$

$\angle DCN = \angle BEN \quad$ [Alt. Int. $\angle$]

$\angle CDN = \angle EBN \quad$ [Alt. Int. $\angle$]

$\therefore \quad \Delta CDN \cong \Delta EBN \quad$ [AAS–criteria]

$\therefore \quad DC = EB$ and $CN = NE \quad$ [CPCT]

Thus; in ΔCAE, the points M and N are the mid-points of AC and CE respectively.

$\therefore MN \,||\, AE$ and $MN = \frac{1}{2} AE \Rightarrow MN \,||\, AB \,||\, DC$

and

$$MN = \frac{1}{2} AE = \frac{1}{2} (AB - EB) = \frac{1}{2} (AB - DC)$$

$[\because EB = DC].$

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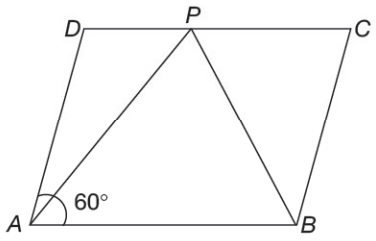
Know Your Progress - 4

1. In the following figure, $ABCD$ is a parallelogram in which $\angle A = 60°$. If the bisectors of $\angle A$ and $\angle B$ meet at P, prove that $\angle APB = 90°$. Also, prove that $AD = DP$, $PC = BC$, and $DC = 2AD$.

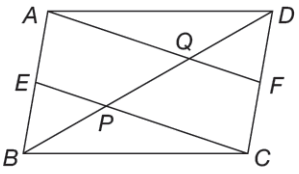


2. In ΔABC, AD is the median through A and E is the mid–point of AD. BE produced meets AC in F. Prove that $AF = \frac{1}{3} AC$.

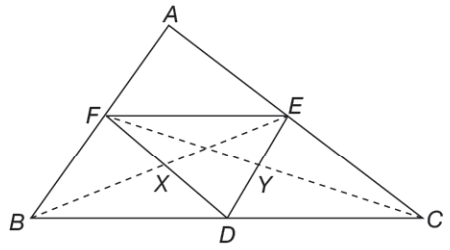
3. Let $ABCD$ be a trapezium in which $AB \,||\, DC$, and let M and N be the mid-points of the diagonals AC and BD, respectively. If $AB = 15$ cm and $CD = 5$ cm, then find MN.

4. Prove that quadrilateral formed by line segments joining the mid-points of the sides of an isosceles trapezium is a rhombus.

5. In figure, $ABCD$ is a parallelogram. E and F are the mid-point of the sides AB and CD, respectively. Prove that the line segments AF and CE trisect (divide into three equal parts) the diagonal BD.

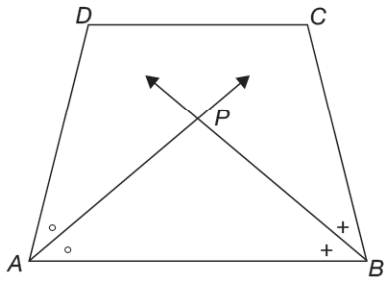


6. In the adjoining figure, D, E, F are the mid-points of the sides BC, CA and AB of ΔABC. If BE and DF intersect at X while CF and DE intersect at Y, prove that $XY = \frac{1}{4} BC$.



7. If the angle bisectors of $\angle DAB$ and $\angle CBA$ of any quadrilateral $ABCD$ intersects at the Point P, then prove that

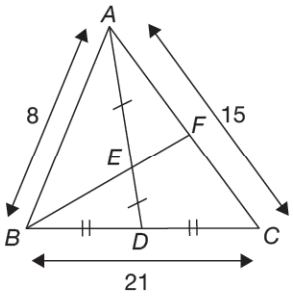
$$2\angle APB = \angle BCD + \angle BDC$$



8. If the line segment joining the mid-point of the consecutive side of quadrilateral $ABCD$ form a rectangle, then $ABCD$ must be

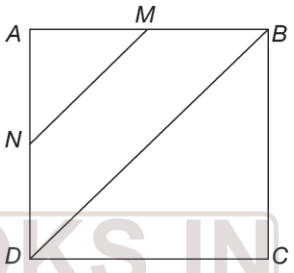
(A) rhombus **(B)** square
(C) kite **(D)** all of the above

9. In ΔABC, AD is median and E is the mid-point of AD. If BE is extended, it meets AC in F. $AB = 8$ cm, $BC = 21$ cm, and $AC = 15$ cm, then $AF = ?$



(A) 7 cm **(B)** 3 cm
(C) 12 cm **(D)** 5 cm

10. In the given figure, $ABCD$ is a square in which M and N are the mid-point of the sides $\overline{AB}$ and $\overline{AD}$ respectively. How many square units is the area of $MNDB$ quadrilateral, if each side of the square is 2 units long?



(A) 1.0 **(B)** 1.5
(C) 2.0 **(D)** 2.5

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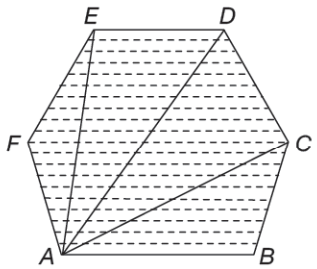
Area of Parallelograms and Triangles

Polygon Region

Polygon region can be expressed as the union of a finite number of triangular regions in a plane such that if two of these intersect, their intersection is either a point or a line segment. It is the shaded portion including its sides as shown in the figure.



Area Axioms

Every polygonal region R has an area, measured in square units and denoted by ar(R).

(i) Congruent area axiom: If R_1 and R_2 be two regions such that $R_1 \cong R_2$ then ar(R_1) = ar (R_2).

(ii) Area addition axiom: If R_1 and R_2 are two polygonal regions, whose intersection is a finite number of points and line segments such that $R = R_1 \cup R_2$, then ar(R) = ar(R_1) + ar(R_2).

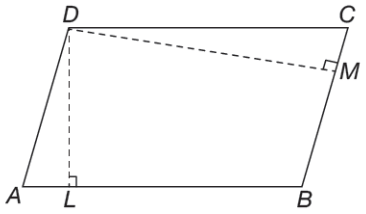
(iii) Rectangular area axiom: If AB = a metre and $AD = b$ metre then, ar (rectangular region $ABCD$) = ab sq.m.

Area of a Parallelogram

Base and altitude of a Parallelogram:

(i) Base: Any side of a parallelogram can be called its base.

(ii) Altitude: The length of the line segment that is perpendicular to the base from the opposite side is called the altitude or height of the parallelogram corresponding to the given base.



(a) DL is the altitude of $||^{gm}$ $ABCD$, corresponding to the base AB.

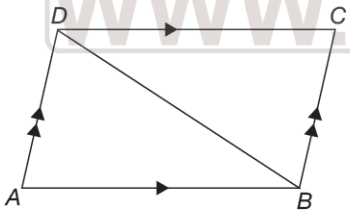
(b) DM is the altitude of $||^{gm}$ $ABCD$, corresponding to the base BC.

Theorem: A diagonal of a parallelogram divides it into two triangles of equal area.

Given: A parallelogram $ABCD$ whose one of the diagonals is BD.

To prove: ar (ΔABD) = ar(ΔCDB).

Proof:



In ΔABD and ΔCDB.

$AB = DC$ [Opposite sides of a $||$gm]

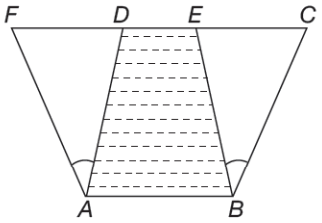
$AD = BC$ [Opposite sides of a $||$gm]

$BD = BD$ [Common side]

$\therefore$ $\Delta ABD \cong \Delta CDB$ [By *SSS* congruency]

$\therefore$ ar(ΔABD) = ar(ΔCDB) Hence proved.

Theorem: Parallelograms on the same base and between the same parallels are equal in area.



Given: Two $||^{gms}$ $ABCD$ and $ABEF$ on the same base AB and between the same parallels AB and FC.

To prove: ar$(||^{gm} ABCD)$ = ar$(||^{gm} ABEF)$

Proof: In ΔADF and ΔBCE, we have

$AD = BC$ [Opposite sides of a $||^{gm}$]

$AF = BE$ [Opposite sides of a $||^{gm}$]

$\angle DAF = \angle CBE$ [$\because$ $AD \,||\, BC$ and $AF \,||\, BE$]

[Angle between AD and AF = Angle between BC and BE]

$\therefore$ $\Delta ADF \cong \Delta BCE$ [By *SAS* congruency]

$\therefore$ ar(ΔADF) = ar(ΔBCE) …(i)

$\therefore$ ar$(||^{gm} ABCD)$ = ar$(ABED)$ + ar(ΔBCE)

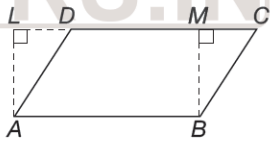
= ar$(ABED)$ + ar(ΔADF) [Using (i)]

= ar$(||^{gm} ABEF)$.

Hence, ar$(||^{gm} ABCD)$ = ar$(||^{gm} ABEF)$.

Hence proved.

Theorem: The area of parallelogram is the product of its base and the corresponding altitude.



Given: A $||^{gm}$ $ABCD$ in which AB is the base and AL is the corresponding height.

To prove: Area $(||^{gm} ABCD) = AB \times AL$.

Construction: Draw $BM \perp DC$, so that rectangle $ABML$ is formed.

Proof: $||^{gm}$ $ABCD$ and rectangle $ABML$ are on the same base AB and between the same parallel lines AB and LC.

$\therefore$ ar$(||^{gm} ABCD)$ = ar(rectangle $ABML$) = $AB \times AL$.

$\therefore$ area of a $||^{gm}$ = base $\times$ height. Hence proved.

Area of a Triangle

Theorem: Two triangles on the same base (or equal bases) and between the same parallels are equal in area.

Given: Two triangles ABC and PBC on the same base BC and between the same parallel lines BC and AP.

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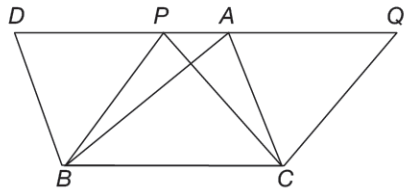
To prove: ar(ΔABC) = ar(ΔPBC).

Construction: Through B, draw $BD \,||\, CA$ intersecting AP produced in D and through C, draw $CQ \,||\, BP$, intersecting PA produced in Q.



Proof: $BD \,||\, CA$ [By construction]

And, $BC \,||\, DA$ [Given]

$\therefore$ Quadrilateral $BCAD$ is a parallelogram.

Similarly, Quadrilateral $BCQP$ is a parallelogram.

Now, parallelogram $BCQP$ and $BCAD$ are on the same base BC, and between the same parallels.

$\therefore$ ar ($||^{gm}$ $BCQP$) = ar ($||^{gm}$ $BCAD$) ...(i)

Diagonals of a parallelogram divides it into two triangles of equal area.

$\therefore$ ar (ΔPBC) = $\frac{1}{2}$ ar ($||^{gm}$ $BCQP$) ...(ii)

And ar (ΔABC) = $\frac{1}{2}$ ar ($||^{gm}$ $BCAD$) ...(iii)

Now, ar ($||^{gm}$ $BCQP$) = ar ($||^{gm}$ $BCAD$) [From (i)]

$\Rightarrow$ $\frac{1}{2}$ ar ($||^{gm}$ $BCAD$) = $\frac{1}{2}$ ar ($||^{gm}$ $BCQP$)

Hence, ar (ΔABC) = ar (ΔPBC) [Using (ii) and (iii)]

Hence proved.

Theorem: If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to half of the parallelogram.

Given: A ΔABC and a paralellogram $BCDE$ on the same base BC and between the same parallels BC and AD.

To prove: ar (ΔABC) = $\frac{1}{2}$ ar (parallelogram $BCDE$)

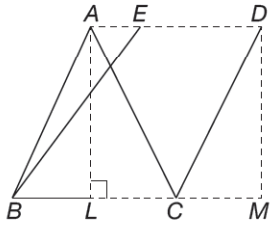
Construction: Draw $AL \perp BC$ and $DM \perp BC$, meeting BC producted in M.

Proof: Since E and D are colinear and $BC \,||\, AD$

$\therefore$ $AL = DM$ ...(i)

[$\because$ distance between parallel lines is always same]

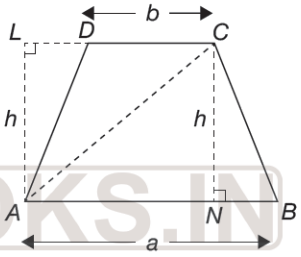
Now, ar(ΔABC) = $\frac{1}{2}$ ($BC \times AL$)



$\Rightarrow$ ar (ΔABC) = $\frac{1}{2}$ ($BC \times DM$) [$\because$ $AL = DM$ (from (i)]

$\Rightarrow$ ar (ΔABC) = $\frac{1}{2}$ ar(parallelogam $BCDE$).

Theorem: The area of a trapezium is half the product of its height and the sum of the parallel sides.



Given: Trapezium $ABCD$ in which $AB \,||\, DC$, $AL \perp DC$, $CN \perp AB$ and $AL = CN = h$ (say), $AB = a$, $DC = b$.

To prove: ar(trapezium $ABCD$) = $\frac{1}{2}$ $h \times (a + b)$.

Construction: Join AC.

Proof: $\because$ AC is a diagonal of quad. $ABCD$.

$\therefore$ ar(trapezium $ABCD$) = ar(ΔABC) + ar(ΔACD)

$= \frac{1}{2} h \times a + \frac{1}{2} h \times b = \frac{1}{2} h(a + b)$.

Hence proved.

Theorem: Median of a triangle divides it into two triangles of equal area.

Given: A ΔABC in which AD is the median.

To prove: ar (ΔABD) = ar(ΔADC).

Construction: Draw $AL \perp BC$.

Proof: Since AD is the median of ΔABC. Therefore, D is the mid-point of BC.

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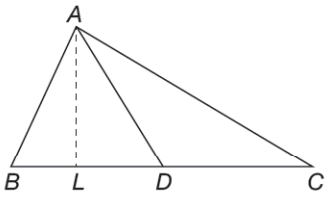


$$\Rightarrow \quad \text{ar}(\Delta ABD) = \frac{1}{2}(BD \times AL) \qquad \ldots(i)$$

$$\Rightarrow \quad \text{ar}(\Delta ADC) = \frac{1}{2}(CD \times AL)$$

$$\Rightarrow \quad \text{ar}(\Delta ADC) = \frac{1}{2}(BD \times AL) \qquad \ldots(ii)$$

[$\because$ $BD = CD$, AD is the median of ΔABC]

From (i) and (ii)

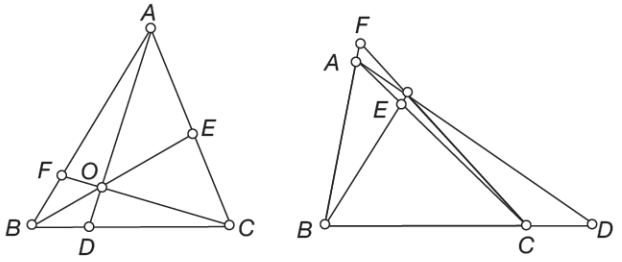
$$\text{ar}(\Delta ABD) = \text{ar}(\Delta ADC).$$

Hence proved.

Some Important Theorem

Theorem (Ceva) If points D, E, F are taken on the sides BC, CA, AB of ΔABC so that the lines AD, BE, CF are concurrent at a point O, then

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \qquad \ldots(1)$$



(i) (ii)

Proof: There are two possible cases depending on whether O lies inside or outside ΔABC. First note that in figure (i) all the ratios $\frac{BD}{DC}$ etc. are positive, while in figure (ii) exactly two are negative and so the product in (1) is positive in all cases. Hence we now ignore the signs of the segments and prove that numerical value of the product in (1) is 1.

In figure (i) and (ii), since the triangle ABD, ADC have the same altitude and also the triangle OBD, ODC have the same altitude, we get

$$\frac{BD}{DC} = \frac{\Delta ABD}{\Delta ADC} = \frac{\Delta OBD}{\Delta ODC} = \frac{\Delta ABD - \Delta OBD}{\Delta ADC - \Delta ODC} = \frac{\Delta ABO}{\Delta CAO}$$

Similarly,

$$\frac{CE}{EA} = \frac{\Delta BCO}{\Delta ABO}, \quad \frac{AF}{FB} = \frac{\Delta CAO}{\Delta BCO}$$

Multiplying these three equations, we get (1).

Theorem (Converse of Ceva's theorem). If three points D, E, F taken on the sides BC, CA, AB of ΔABC are such that equation (1) holds, then AD, BE, CF are concurrent.

Proof. Suppose AD and BE meet in O and meets AB in 'F'. Then since AD, BE, CF' are concurrent, we get by Ceva's Theorem,

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF'}{F'B} = 1.$$

Hence by (1) we get $\frac{AF}{FB} = \frac{AF'}{F'B}$

Hence $\frac{AF}{AF + FB} = \frac{AF'}{AF' + F'B}$ or $\frac{AF}{AB} = \frac{AF'}{AB}$

Thus $AF = AF'$ and so the points F and F' coincide. Hence CF also passes through O

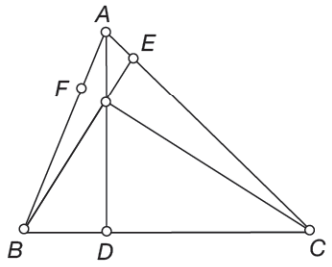
Trigonometric form of Ceva's Theorem: Let points D, E, F be taken on the sides BC, CA, AB of ΔABC. Then the lines AD, BE, CF are concurrent if and only if

$$\frac{\sin\angle BAD}{\sin\angle DAC} \cdot \frac{\sin\angle CBE}{\sin\angle EBA} \cdot \frac{\sin\angle ACF}{\sin\angle FCB} = 1.$$

Corollary 1. The medians of a triangle are concurrent.

[For in this case, figure (i), $BD/DC = 1$ etc. and so (1) holds.]

Corollary 2. The altitudes of a triangle are concurrent.



(iii)

[For acute-angled triangle ABC, figure (iii) $BE \perp CA$ and $CF \perp AB$ and so $\Delta AEB \sim \Delta AFC$ as $\angle A$ is common to them. Hence $\frac{AE}{AF} = \frac{AB}{AC}$ Similarly, if $AD \perp BC$, then $\frac{BF}{BD} = \frac{BC}{BA}$ and $\frac{CD}{CE} = \frac{CA}{CB}$ so $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{AB}{BC} \cdot \frac{BC}{CA} \cdot \frac{CA}{AB} = 1$. And so (1) holds. A similar proof holds for obtuse angled triangles]

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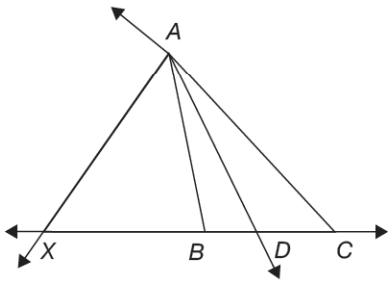
Corollary 3. The internal bisectors of the angle of a triangle are concurrent.



(iv)

In ΔABC, the internal bisector AD of $\angle A$ divides the opposite side BC internally in the ratio $\frac{BD}{DC} = \frac{AB}{AC}$; the external bisector AX of $\angle A$ divides BC externally in the ratio $\frac{BX}{XC} = -\frac{AB}{AC}$. (see figure iv)

[For in this case, figure $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$, etc., where a, b, c, respectively, denotes the lengths of the sides BC, CA, AB of ΔABC. Hence

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = \frac{c}{b}\cdot\frac{a}{c}\cdot\frac{b}{a} = 1 \text{ and so (1) holds]}$$

Corollary 4. The external bisectors of any two angles of a triangle and the internal bisector of the third angles are concurrent.

[For in this case, (Figure iv)

$$\frac{BD}{DC} = \frac{c}{b},\ \frac{CE}{EA} = \frac{-a}{c},\ \frac{AF}{FB} = \frac{-b}{a} \text{ and so (1) holds.]}$$

Corollary 5. If the incircle of ΔABC touches BC, CA, AB at X, Y, Z, respectively, then AX, BY, CZ are concurrent (the point of concurrence is called the Gergonne point of ΔABC)

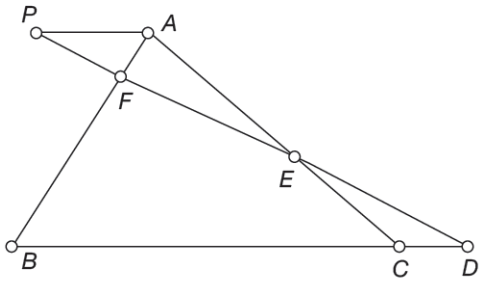
[For in this case, (figure) $BX = BZ$, etc. and so

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = \frac{BX}{CY}\cdot\frac{CY}{AZ}\cdot\frac{AZ}{BX} = 1, \text{ and so (1) holds]}$$

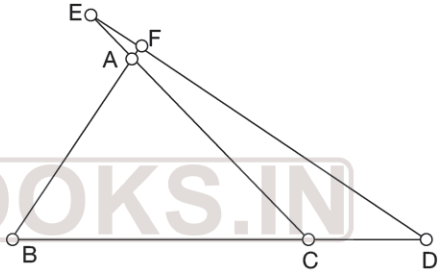
Theorem (Menelaus) If a transversal cut the sides BC, CA, AB (Suitably extended) of ΔABC in point D, E, F, respectively, then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1 \qquad \ldots(2)$$

Proof. There are two cases depending on whether the transversal cuts one side externally or all three sides externally. Note that only one ratio (as in Figure (v)) or else all the three ratios (as in figure (vi)) in (2) are negative. Thus, the product in (2) is always negative. Hence, we now ignore the signs of the segments and prove that the numerical value of the product in (2) is 1.



(v)



(vi)

With the notation of figure (v) draw AP parallel to DB meeting DF in P. Then, since $\Delta APF \sim \Delta BDF$,

$\frac{AF}{FB} = \frac{PA}{BD}$ and since $\Delta CDE - \Delta APE$, $\frac{DC}{AP} = \frac{CE}{EA}$.

Hence

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = \frac{BD}{DC}\cdot\frac{DC}{AP}\cdot\frac{PA}{BD} = 1, \text{ numerically.}$$

Theorem 17 (Converse of Menelaus' Theorem): If points D, E, F are taken on the sides of ΔABC such that equation (2) holds, then D, E, F are collinear.

Proof. Let DE meet AB in F'. Then by Menelaus' theorem, we get

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF'}{F'B} = -1.$$

Hence, by (2) we get $\frac{AF}{FB} = \frac{AF'}{F'B}$ and so $F = F'$ Hence F lies on DE, as required.

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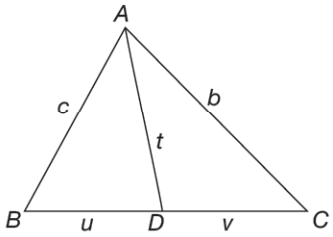
Stewart's Theorem



(i) If u, v are the actual length of BD, DC respectively then

$$t^2 = \frac{ub^2 + vc^2}{u+v} - uv$$

(ii) If $BD : DC = u : v$ then, $ub^2 + vc^2 = (u + v)t^2 + u \cdot DC^2 + v \cdot BD^2$

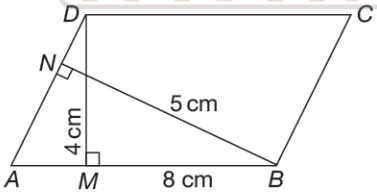
Note: Apollonius thoerem is special case of Stewart theorem.

Example 33

In a parallelogram $ABCD$, $AB = 8$ cm. The altitudes corresponding to sides AB and AD are, respectively, 4 cm and 5 cm. Find AD.

Solution

Area of a $||^{gm}$ = Base × corresponding altitude

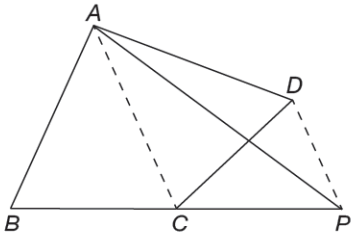


$\therefore$ Area of parallelogram $ABCD = AD \times BN = AB \times DM$

$$\Rightarrow \quad AD \times 5 = 8 \times 4 \Rightarrow AD = \frac{8 \times 4}{5} = 6.4 \text{ cm.}$$

Example 34

$ABCD$ is a quadrilateral. A line through D, parallel to AC, meets BC produced in P as shown in figure. Prove that ar (ΔABP) = ar(quad. $ABCD$).



Solution

Since Δs ACP and ACD are on the base AC and between the same parallel's AC and DP.

$$\therefore \quad \text{ar}(\Delta ACP) = \text{ar}(\Delta ACD)$$

$$\Rightarrow \quad \text{ar}(\Delta ACP) + \text{ar}(\Delta ABC) = \text{ar}(\Delta ACD) + \text{ar}(\Delta ABC)$$

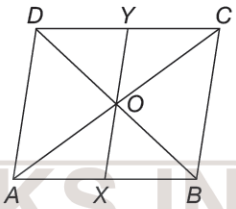
$$\Rightarrow \quad \text{ar}(\Delta ABP) = \text{ar}(\text{quad. } ABCD).$$

Example 35

The diagonals of a parallelogram $ABCD$ intersect in O. A line through O meets AB in X and the opposite side CD in Y. Show that ar (quadrilateral $AXYD$) = $\frac{1}{2}$ ar(parallelogram $ABCD$).

Solution

$\because$ AC is a diagonal of the parallelogram $ABCD$.



$$\text{ar}(\Delta ACD) = \frac{1}{2}\text{ar}(||^{gm} ABCD) \qquad \ldots(i)$$

Now, in Δs AOX and COY,

$$AO = CO$$

[$\because$ Diagonals of a parallelogram bisect each other]

$$\angle AOX = \angle COY \quad \text{[Vertically opposite } \angle s\text{]}$$

$$\angle OAX = \angle OCY \quad \text{[Alternate interior } \angle s\text{]}$$

[$\because$ $AB \,||\, DC$ and transversal AC intersects them]

$$\therefore \quad \Delta AOX \cong \Delta COY \quad \text{[By } ASA \text{ congruency]}$$

$$\therefore \quad \text{ar}(\Delta AOX) = \text{ar}(\Delta COY) \qquad \ldots(ii)$$

Adding ar(quad. $AOYD$) to both sides of (ii), we get

$$\text{ar}(\text{quad. } AOYD) + \text{ar}(\Delta AOX) = \text{ar}(\text{quad. } AOYD) + \text{ar}(\Delta COY)$$

$$\Rightarrow \quad \text{ar}(\text{quad. } AXYD) = \text{ar}(\Delta ACD) = \frac{1}{2}\text{ar}(||^{gm} ABCD)$$

Hence proved.

Example 36

$ABCD$ is a trapezium with $AB \,||\, DC$. A line parallel to AC intersects AB at X and BC at Y. Prove that ar(ΔADX) = ar(ΔACY).

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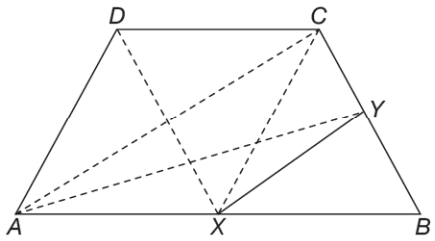
Solution



Join CX, DX, and AY.

Clearly, triangles ADX and ACX are on the same base AX and between the parallels AB and DC.

$\therefore$ $\quad ar(\Delta ADX) = ar(\Delta ACX)$ $\quad \ldots(i)$

Also, ΔACX and ΔACY are on the same base AC and between the parallels AC and XY.

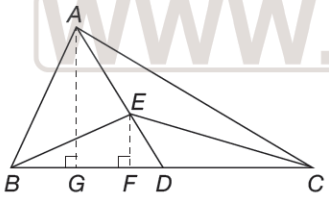
$\therefore$ $\quad ar(\Delta ACX) = ar(\Delta ACY)$ $\quad \ldots(ii)$

From (i) and (ii), we get

$$ar(\Delta ADX) = ar(\Delta ACY).$$

Example 37

In figure, E is any point on median AD of a ΔABC. Show that $ar(\Delta ABE) = ar(\Delta ACE)$.



Solution

Construction: From A, draw $AG \perp BC$ and from E draw $EF \perp BC$.

Proof: $ar(\Delta ABD) = \dfrac{BD \times AG}{2}$ and $ar(\Delta ADC) = \dfrac{DC \times AG}{2}$

But, $\quad BD = DC$

[$\because$ D is the mid-point of BC, AD being the median]

$$ar(\Delta ABD) = ar(\Delta ADC) \quad \ldots(i)$$

Again, $ar(\Delta EBD) = \dfrac{BD \times EF}{2}$ and $ar(\Delta EDC) = \dfrac{DC \times EF}{2}$

But, $\quad BD = DC$

$\therefore$ $\quad ar(\Delta EBD) = ar(\Delta EDC)$ $\quad \ldots(ii)$

Subtracting (ii) from (i), we get

$$ar(\Delta ABD) - ar(\Delta EBD) = ar(\Delta ADC) - ar(\Delta EDC)$$

$\Rightarrow$ $\quad ar(\Delta ABE) = ar(\Delta ACE)$.

Hence proved.

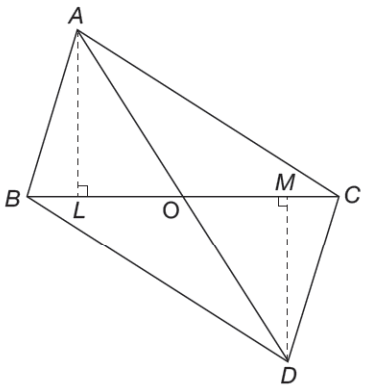
Example 38

Triangles ABC and DBC are on the same base BC; with A, D on opposite sides of the line BC, such that $ar(\Delta ABC) = ar(\Delta DBC)$. Show that BC bisects AD.

Solution

Construction: Draw $AL \perp BC$ and $DM \perp BC$.

Proof: $ar(\Delta ABC) = ar(\Delta DBC)$ [Given]



$\Rightarrow$ $\quad \dfrac{BC \times AL}{2} = \dfrac{BC \times DM}{2}$

$\Rightarrow$ $\quad AL = DM$ $\quad \ldots(i)$

Now in Δs OAL and OMD

$AL = DM$ [From (i)]

$\Rightarrow$ $\quad \angle ALO = \angle DMO$ [Each = 90°]

$\Rightarrow$ $\quad \angle AOL = \angle MOD$ [Vertically opposite $\angle s$]

$\therefore$ $\quad \Delta OLA \cong \Delta OMD$ [By AAS congruency]

$\therefore$ $\quad OA = OD$ [By CPCT]

i.e., BC bisects AD.

Hence proved.

Example 39

ABC is a triangle in which D is the mid-point of BC and E is the mid-point of AD. Prove that the area of $\Delta BED = \dfrac{1}{4}$ area of ΔABC.

Solution

Given: A ΔABC in which D is the mid-point of BC and E is the mid-point of AD.

To prove: $ar(\Delta BED) = \dfrac{1}{4} ar(\Delta ABC)$.

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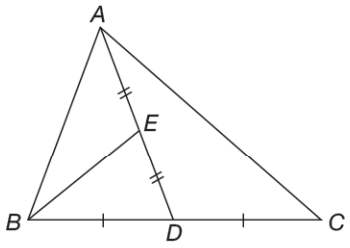


Proof: ∵ AD is a median of ΔABC.

$$\therefore \quad \text{ar}(\Delta ABD) = \text{ar}(\Delta ADC) = \frac{1}{2}\text{ar}(\Delta ABC) \qquad \ldots(i)$$

[∵ Median of a triangle divides it into two triangles of equal area]

Again,

∵ BE is a median of ΛABD.

$$\therefore \quad \text{ar}(\Delta BEA) = \text{ar}(\Delta BED) = \frac{1}{2}\text{ar}(\Delta ABD)$$

[∵ Median of a triangle divides it into two triangles of equal area]

$$\text{And} \quad \text{ar}(\Delta BED) = \frac{1}{2}\text{ar}(\Delta ABD) = \frac{1}{2} \times \frac{1}{2}\text{ar}(\Delta ABC)$$

[From (i)]

$$\therefore \quad \text{ar}(\Delta BED) = \frac{1}{4}\text{ar}(\Delta ABC).$$

Hence proved.

Example 40

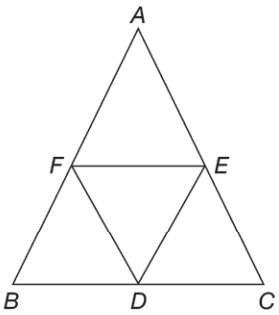
D, E and F are respectively the mid points of the sides BC, CA and AB of a ΔABC. Show that:

(i) $BDEF$ is a parallelogram

(ii) $\text{ar}(||^{gm} BDEF) = \frac{1}{2}\text{ar}(\Delta ABC)$

(iii) $\text{ar}(\Delta DEF) = \frac{1}{4}\text{ar}(\Delta ABC)$

Solution



(i) In ΔABC,

∵ F is the mid-point of side AB and E is the mid-point of side AC.

$\therefore$ $EF \,||\, BD$ [∵ Line joining the mid-points of any two sides of a Δ is parallel to the third side.]

Similarly, $ED \,||\, FB$.

Hence, $BDEF$ is a parallelogram.

Hence proved.

(ii) Similarly, we can prove that $AFDE$ and $FDCE$ are parallelograms.

∵ FD is a diagonal of parallelogram $BDEF$.

$$\therefore \quad \text{ar}(\Delta FBD) = \text{ar}(\Delta DEF) \qquad \ldots(i)$$

$$\text{Similarly,} \quad \text{ar}(\Delta FAE) = \text{ar}(\Delta DEF) \qquad \ldots(ii)$$

$$\text{And,} \quad \text{ar}(\Delta DCE) = \text{ar}(\Delta DEF) \qquad \ldots(iii)$$

From above equations, we have

$$\text{ar}(\Delta FBD) = \text{ar}(\Delta FAE) = \text{ar}(\Delta DCE) = \text{ar}(\Delta DEF)$$

$$\text{and ar}(\Delta FBD) + \text{ar}(\Delta DCE) + \text{ar}(\Delta DEF) + \text{ar}(\Delta FAE) = \text{ar}(\Delta ABC)$$

$$\Rightarrow 2\,[\text{ar}(\Delta FBD) + \text{ar}(\Delta DEF)] = \text{ar}(\Delta ABC)$$

[By using (ii) and (iii)]

$$\Rightarrow \quad 2\,[\text{ar}.\,(||^{gm} BDEF)] = \text{ar}(\Delta ABC)$$

$$\Rightarrow \quad \text{ar}\,(||^{gm} BDEF) = \frac{1}{2}\text{ar}(\Delta ABC).$$

(iii) Since, ΔABC is divided into four non-overlapping triangles FBD, FAE, DCE, and DEF.

$$\therefore \quad \text{ar}(\Delta ABC) = \text{ar}(\Delta FBD) + \text{ar}(\Delta FAE) + \text{ar}(\Delta DCE) + \text{ar}(\Delta DEF)$$

$$\Rightarrow \quad \text{ar}\,(\Delta ABC) = 4\,\text{ar}\,(\Delta ABC)$$

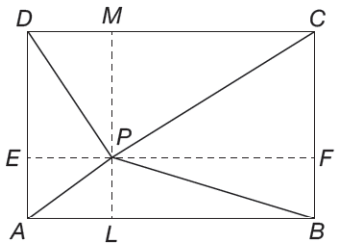
[Using (i), (ii) and (iii)]

$$\Rightarrow \quad \text{ar}(\Delta DEF) = \frac{1}{4}\,\text{ar}(\Delta ABC).$$

Hence proved.

Example 41

In figure, P is a point in the interior of a rectangle $ABCD$. Show that



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(i) $\text{ar}(\Delta APD) + \text{ar}(\Delta PBC) = \frac{1}{2}\text{ar}(\text{rectangle } ABCD)$

(ii) $\text{ar}(\Delta APD) + \text{ar}(\Delta PBC) = \text{ar}(\Delta APB) + \text{ar}(\Delta PCD)$

Solution

Construction: Draw $EPF \,||\, AB \,||\, CD$ and $LPM \,||\, AD \,||\, BC$.

Proof:

(i) $EPF \,||\, AB$ and DA cuts them.

$\therefore \quad \angle DEP = \angle EAB = 90°$ [Corresponding angles]

$\therefore \quad PE \perp AD.$

Similarly, $PF \perp BC$; $PL \perp AB$ and $PM \perp DC$.

$\therefore \quad \text{ar}(\Delta APD) + \text{ar}(\Delta BPC)$

$$= \left(\frac{1}{2} \times AD \times PE\right) + \left(\frac{1}{2} \times BC \times PF\right)$$

$$= \frac{1}{2} AD \times (PE + PF) \qquad [\because BC = AD]$$

$$= \frac{1}{2} \times AD \times EF = \frac{1}{2} \times AD \times AB \quad [\because EF = AB]$$

$$= \frac{1}{2} \times \text{ar}(\text{rectangle } ABCD).$$

(ii) $\text{ar}(\Delta APB) + \text{ar}(PCD)$

$$= \left(\frac{1}{2} \times AB \times PL\right) + \left(\frac{1}{2} \times DC \times PM\right)$$

$$= \frac{1}{2} \times AB \times (PL + PM) \qquad [\because DC = AB]$$

$$= \frac{1}{2} \times AB \times LM = \frac{1}{2} \times AB \times AD \quad [\because LM = AD]$$

$$= \frac{1}{2} \times \text{ar}(\text{rect. } ABCD).$$

$\therefore \quad \text{ar}(\Delta APD) + \text{ar}(PBC) = \text{ar}(\Delta APB) + \text{ar}(PCD)$

Hence proved.

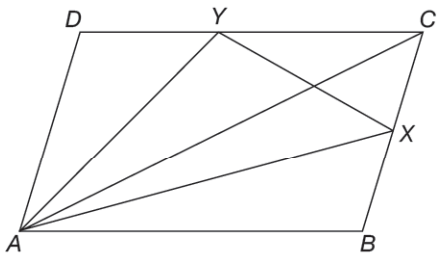
Example 42

$ABCD$ is a parallelogram. X and Y are mid-points of BC and CD respectively.

Prove that $\text{area}(\Delta AXY) = \frac{3}{8}\text{area}(\text{parallelogram } ABCD)$.

Solution

Since X and Y are the mid–points of sides BC and CD, respectively, in ΔBCD.



$\therefore \quad XY \,||\, BD$ and $XY = \frac{1}{2} BD.$

$$\text{area } (\Delta CYX) = \frac{1}{4}\text{area}(\Delta DBC)$$

$$= \frac{1}{8}\text{area}(\text{parallelogram } ABCD) \qquad \ldots(\text{i})$$

$$[\text{Since ar}(\Delta DBC) = \frac{1}{2}\text{area (parallelogram } ABCD)]$$

Parallelogram $ABCD$ and ΔABX are between the same parallels AD and BC and $BX = \frac{1}{2} BC$.

$\therefore \quad \text{area } (\Delta ABX) = \frac{1}{4}\text{area (parallelogram } ABCD) \quad \ldots(\text{ii})$

Similarly, $\text{area } (\Delta AYD) = \frac{1}{4}\text{ area (parallelogram } ABCD)$

Now, $\text{ar}(\Delta AXY) = \text{ar}(\text{parallelogram } ABCD) - [\text{ar}(\Delta ABX) + \text{ar}(\Delta AYD) + \text{ar}(\Delta CYX)]$

$$\text{ar}(\Delta AXY) = \text{ar (parallelogram } ABCD) - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{8}\right)\text{ar}(ABCD)$$

$$\text{ar}(\Delta AXY) = \left(1 - \frac{5}{8}\right)\text{ar}(\text{parallelogram } ABCD)$$

$$= \frac{3}{8}\text{ ar}(\text{parallelogram } ABCD).$$

Example 43

As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.

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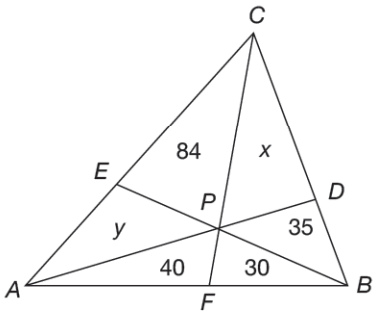
Solution



$$\frac{[CAP]}{[FAP]} = \frac{CP}{FP} = \frac{[CBP]}{[FBP]} \text{ yields}$$

$$\frac{84+y}{40} = \frac{x+35}{30}, \qquad \ldots(1)$$

and $$\frac{[CAP]}{[CDP]} = \frac{AP}{DP} = \frac{[BAP]}{[BDP]} \text{ yields}$$

$$\frac{84+y}{x} = \frac{70}{35} = 2 \qquad \ldots(2)$$

But $\frac{(2)}{(1)}$, it follows that $\frac{x}{40} = \frac{x+35}{60}$,

$\therefore$ $3x = 2x + 70$, i.e., $x = 70$.

Then by (2), $y = 140 - 84 = 56$.

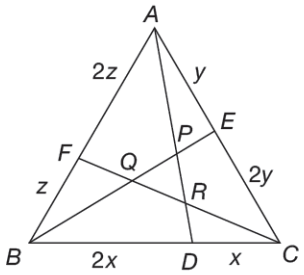
Thus, $[ABC] = 84 + 56 + 40 + 30 + 35 + 70 = 315$.

Example 44

On the sides BC, CA, and AB of ΔABC point D, E, F area takes in such a way that $\frac{DB}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2$. If the area of Δ determined by the line AD, BE, CF in $\frac{1}{K}$ times of area ΔABC find K.

Solution

In ΔADB, CF in cevian.



$\therefore$ $$\frac{AR}{RD} \times \frac{DC}{CB} \times \frac{BF}{AF} = 1$$

$$\frac{AR}{RD} = \frac{6}{1}$$

area $\Delta ABC = a$

$$\text{area } \Delta ADC = \frac{a}{3}$$

$$\text{area } \Delta ARC = \frac{6}{7}\left(\frac{a}{3}\right) = \frac{2}{7}a$$

Similarly, $$\text{area } \Delta BQC = \text{area } \Delta ABP = \frac{2a}{7}$$

Adding area ΔARC + area ΔBQC + area ΔABP

$$= \frac{2a}{7} + \frac{2a}{7} + \frac{2a}{7} = \frac{6a}{7}$$

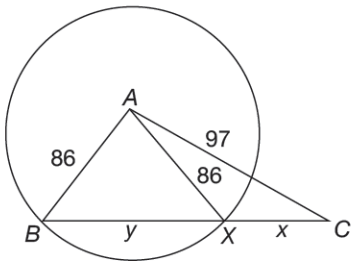
$\therefore$ $$\text{area } \Delta PQR = a - \frac{6a}{7} = \frac{a}{7}.$$

$\therefore$ $$K = 7.$$

Example 45

In ΔABC, $AB = 86$, $AC = 97$ a circle with centre A and Radius AB interest BC at B and X. BX and CX have integer length. What is the length of BC?

Solution



By Stewart Theorem

$$86^2 = \frac{x \cdot 86^2 + y \cdot 97^2}{x+y} - xy$$

$$x(x+y) = 33 \times 61 = 3 \times 11 \times 61$$

Note $x < x + y$

$$x + y < 86 + 97 = 183$$

So possible solution $x = 33$, $x + y = 61$

Example 46

ABC is a triangle, D and E are interior points of sides. AB and BC, respectively, such that $\frac{AD}{DB} = \frac{1}{3}, \frac{DE}{EB} = \frac{1}{3}$, if AE

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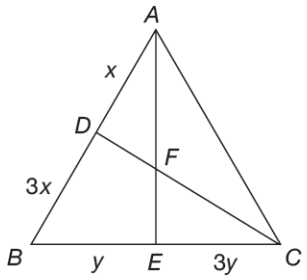
and CD intersect at F. Find $\frac{CF}{FD}$.

Solution



In ΔBCD by Menelaus theorem,

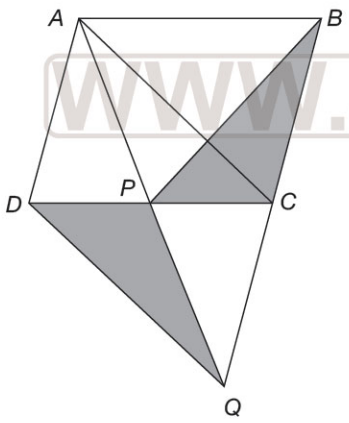
$$\frac{BE}{EC}\times\frac{CF}{FD}\times\frac{AD}{AB}=1$$

$$\frac{y}{3y}\times\frac{CF}{FD}\times\frac{x}{4x}=1$$

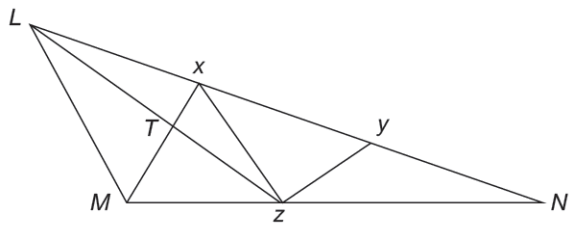
$$\frac{CF}{FD}=\frac{12}{1}$$

Know Your Progress - 5

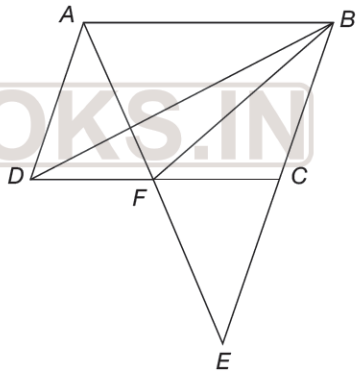
1. $ABCD$ is a parallelogram and BC is produced to a point Q such that $AD = CQ$ (Figure). If AQ intersects DC at P, show that ar(BPC) = ar(DPQ).



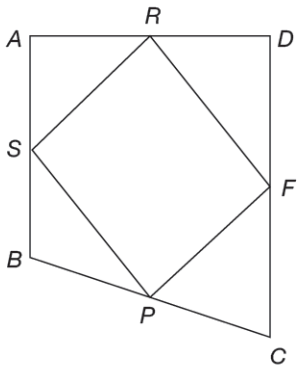
2. X and Y are points on the side LN of the triangle LMN such that $LX = XY = YN$. Through X, a line is drawn parallel to LM to meet MN at Z (See Figure). Prove that ar(LZY) = ar($MZYX$)



3. $ABCD$ is a parallelogram in which BC is produced to E such that $CE = BC$ (Figure). AE intersects CD at F. If ar(DFB) = 3 cm^2, find the area of the parallelogram $ABCD$.



4. If the mid-points of the sides of a quadrilateral are joined in order, prove that the area of the parallelogram so formed will be half of the area of the given quadrilateral (Figure).



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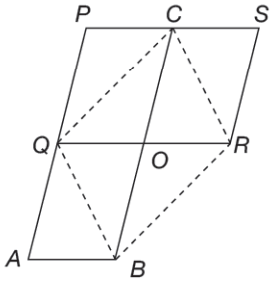
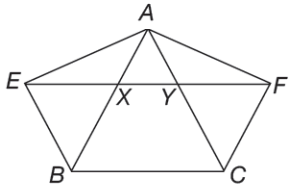
5. In the adjoining figure, $PQRS$ and $PABC$ are two parallelograms of equal area. Prove that $QC \parallel BR$.



6. In the given figure, XY is a line parallel to side BC of a ΔABC. $BE \parallel AC$ and $CF \parallel AB$ meet XY in E and F, respectively. Show that $\text{ar}(\Delta ABE) = \text{ar}(\Delta ACF)$.

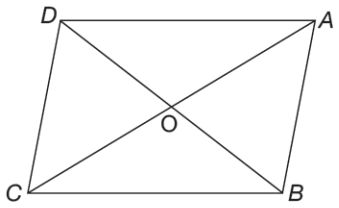


7. Diagonals AC and BD of a quadrilateral $ABCD$ intersect at O, such that $OB = OD$. If $AB = CD$, then show that:

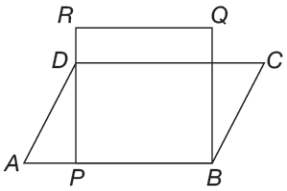
(i) ar ΔDOC = ar ΔAOB

(ii) ar ΔDCB = ar ΔACB

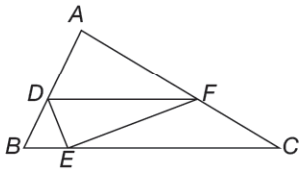
(iii) DA parallel to CB or $ABCD$ is a parallelogram



8. In the figure, $ABCD$ is a parallelogram and $PBQR$ is a rectangle. If $AP : PB = 1 : 2 = PD : DR$, what is the ratio of the **area of** $ABCD$ to the area of $PBQR$?



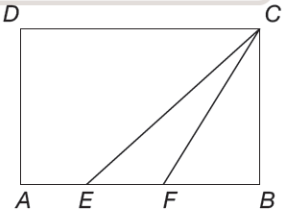
9. In the figure $AD = \frac{1}{2} DB$, $BE = EC$ and $CF = \frac{1}{3} AF$. If the area of ΔABC = 120 cm^2, the area (in cm^2) of ΔDEF is



(A) 21 **(B)** 35

(C) 40 **(D)** 45

10. In the below diagram, $ABCD$ is a rectangle with $AE = EF = FB$. What is the ratio of the areas of ΔCEF and that of the rectangle?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{8}$

(C) $\frac{1}{9}$ **(D)** None of these

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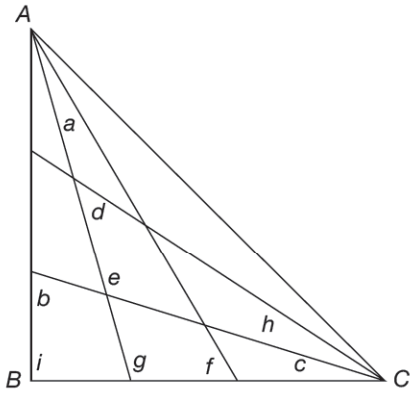
Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. In ΔABC, what is sum of the angles $a + b + c + d + e + f + g + h + i$?



(A) 360°
(B) 540°
(C) 600°
(D) Cannot be determined

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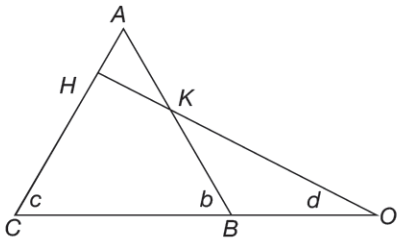
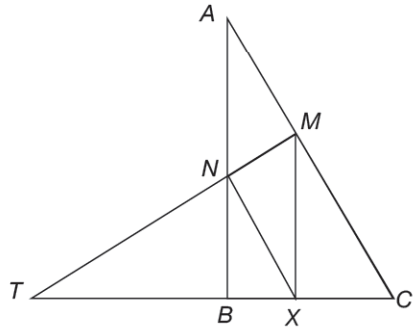
2. In the given figure OBC and OKH are straight lines. If $AH = AK$, $b = 80°$ and $c = 30°$ then the value of d is:



(A) 20° **(B)** 25°
(C) 30° **(D)** 45°

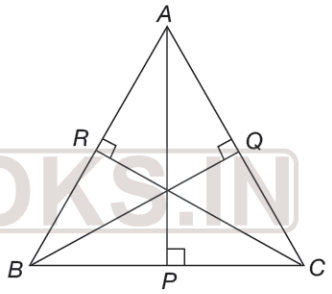
3. The sides of a triangle are in the ratio 4 : 6 : 11. Which of the following words best described the triangle?
(A) obtuse **(B)** isosceles
(C) acute **(D)** impossible

4. Let X be any point on the side BC of a triangle ABC. If XM, XN are drawn parallel to BA and CA meeting CA, BA in M, N respectively; MN meets CB produced in T. Then,



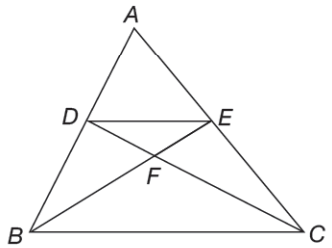
(A) $TB^2 = TX \times TC$ **(B)** $TC^2 = TB \times TX$
(C) $TX^2 = TB \times TC$ **(D)** $TX^2 = 2(TB \times TC)$

5. The altitudes of the given triangle are $AP = 3$, $BQ = 4$, and $CR = 5$. What is the ratio of $BC : AC : AB$?



(A) 20 : 15 : 12 **(B)** 20 : 12 : 15
(C) 15 : 12 : 20 **(D)** 3 : 4 : 5

6. In the given figure, $DE \parallel BC$ and $AD : DB = 5 : 4$. Find $\dfrac{\text{area}(\Delta DFE)}{\text{area}(\Delta CFB)}$.



(A) 5 : 9 **(B)** 25 : 16
(C) 25 : 81 **(D)** None of these

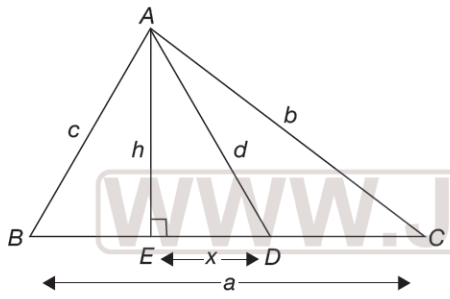
7. In a triangle ABC, if $\angle B = 90°$ and D is the point in BC such that $BD = 2\ DC$, then:

(A) $AC^2 = AD^2 + 3CD^2$
(B) $AC^2 = AD^2 + 5CD^2$
(C) $AC^2 = AD^2 + 7CD^2$
(D) $AC^2 = AB^2 + 5BD^2$

8. Let ΔXOY be a right-angled triangle with $\angle XOY = 90°$. Let M and N be the mid-points of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, the length XY is equal to
(A) 24 **(B)** 26
(C) 28 **(D)** 34

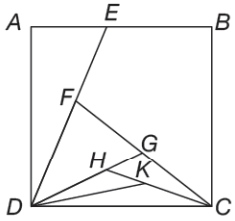
9. If the sides of a right triangle are 9, 12, and 15 cm long, then the sum of squares of medians is:
(A) 227.5 **(B)** 337.5
(C) 537.5 **(D)** None of these

10. In the following figure, $AE \perp BC$, D is the mid-point of BC, then x is equal to



(A) $\frac{1}{a}\left[b^2 - d^2 - \frac{a^2}{4}\right]$
(B) $\frac{h+d}{3}$
(C) $\frac{c+d-h}{2}$
(D) $\frac{a^2+b^2+d^2-c^2}{4}$

11. In the figure, the area of square $ABCD$ is 4 cm^2 and E any point on AB. F, G, H and K are the mid-point of DE, CF, DG, and CH, respectively. The area of ΔKDC is –



(A) $\frac{1}{4}$ cm^2 **(B)** $\frac{1}{8}$ cm^2
(C) $\frac{1}{16}$ cm^2 **(D)** $\frac{1}{32}$ cm^2

12. The line joining the mid-points of the diagonals of a trapezium has length 3. If the longer base is 97, then the shorter base is:
(A) 94 **(B)** 92
(C) 91 **(D)** 90

13. One side of a parallelogram has length 3, and another side has length 4. Let a and b denote the lengths of the diagonals of the parallelogram. Which of the following quantities can be determined from the given information?
I. $a + b$ II $a^2 + b^2$ III $a^3 + b^3$
(A) Only I **(B)** Only II
(C) Only III **(D)** Only I and II

14. A square $ABCD$ is constructed inside a triangle PQR having sides 10, 17, and 21 as shown in figure. Find the perimeter of the square $ABCD$.

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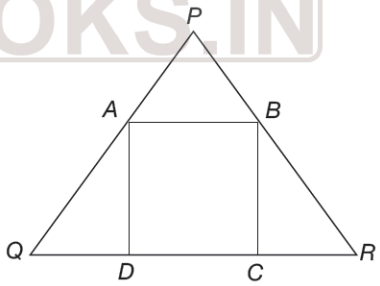


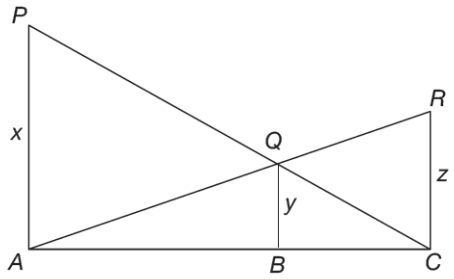
(A) 28 **(B)** 23.2
(C) 25.4 **(D)** 28.8

15. $ABCD$ is a trapezium in which $AB \parallel CD$. If $\angle ADC = 2\angle ABC$, $AD = a$ cm and $CD = b$ cm, then the length (in cm) of AB is:
(A) $\frac{a}{2} + 2b$ **(B)** $a + b$
(C) $\frac{2}{3}a + b$ **(D)** $a + \frac{2}{3}b$

16. E is the mid-point of diagonal BD of a parallelogram $ABCD$. If the point E is joined to a point F on DA such that $DF = \frac{1}{3}DA$, then the ratio of the area of ΔDEF to the area of quadrilateral $ABEF$ is:

(A) 1 : 3 **(B)** 1 : 4
(C) 1 : 5 **(D)** 2 : 5

17. In figure, PA, QB and RC are each perpendicular to AC. Then: $\frac{1}{x} + \frac{1}{z} =$



(A) 1 **(B)** y
(C) $\frac{1}{y}$ **(D)** None

18. In a triangle ABC, the internal bisector of the angle A meets BC at D. If $AB = 4$, $AC = 3$, and $\angle A = 60°$, then the length of AD is:

(A) $2\sqrt{3}$ **(B)** $\frac{12\sqrt{3}}{7}$
(C) $\frac{15\sqrt{3}}{8}$ **(D)** None of these

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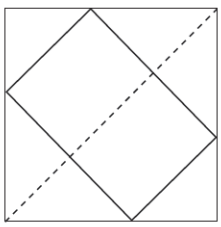
19. In a triangle ABC, a straight line parallel to BC intersects AB and AC at point D and E, respectively. If the area of ADE is one–fifth of the area of ABC and BC = 10 cm, then DE equals:

(A) 2 cm **(B)** $2\sqrt{5}$ cm
(C) 4 cm **(D)** $4\sqrt{5}$ cm

20. A rectangle is inscribed in a square creating four isosceles right triangles. If the total area of these four triangles is 200. The length of the diagonal of the rectangle is:



(A) 10 **(B)** 15
(C) 20 **(D)** 25

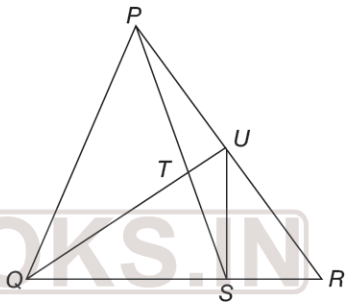
21. The area of a rhombus is 28 cm^2 and one of its diagonals in 4cm. Its perimeter is:

(A) $4\sqrt{53}$ **(B)** 36 cm
(C) $2\sqrt{53}$ **(D)** $\sqrt{53}$ cm

22. E is the mid-point of diagonal BD of a parallelogram $ABCD$. If the point E is joined to a point F on DA such that $DF = \frac{1}{3}DA$, then the ratio of the area of ΔDEF to the area of quadrilateral $ABEF$ is:

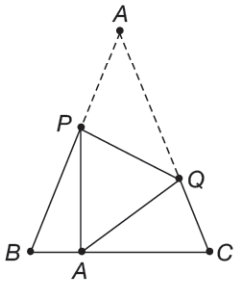
(A) 1 : 3 **(B)** 1 : 4
(C) 1 : 5 **(D)** 2 : 5

23. In the figure below, S is a point on OR and U is a point on PR. The line segments PS and QU intersect at the point T. It is given that $PT = TS$ and $QS = 2RS$. If the area of ΔPQR is 150 cm^2 and the area of ΔPSU is x cm^2. Find the value of x.



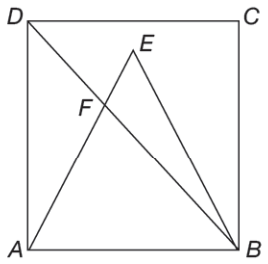
(A) 15 **(B)** 18
(C) 20 **(D)** 22

24. Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on BC as shown. If $BA' = 1$ and $A'C = 2$. If the length of crease PQ is $\frac{k\sqrt{21}}{20}$, find k.



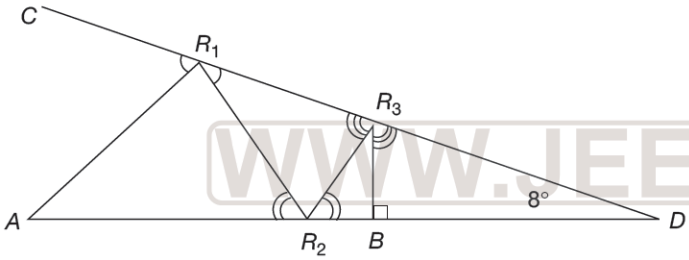
(A) 10 **(B)** 8
(C) 9 **(D)** 7

25. Vertex E of equilateral triangle ABE is in the interior of square $ABCD$, and F is the point of intersection of diagonal BD and line segment AE. If length AB is $\sqrt{1+\sqrt{3}}$ and the area of ΔABF is given by A. Find $(4A)^2$.



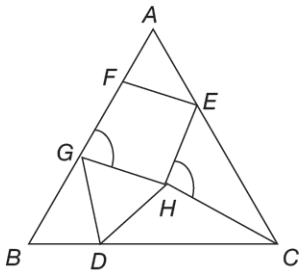
(A) 10 **(B)** 12
(C) 15 **(D)** 20

26. A ray of light originates from point A and travels in a plane, being reflected n times between lines AD and CD, before striking a point B (which may be on AD or CD) perpendicularly and retracing its path to A. (At each point of reflection, the light makes two equal angles as indicated in the adjoining figure. The figure shows the light path for $n = 3$.) If $\angle CDA = 8°$, what is the largest value n can have?



(A) 10 **(B)** 11
(C) 12 **(D)** 13

27. In ΔABC shows below, $AB = AC$, F is a point on AB and E a point on AC such that $AF = EF$, H is a point in the interior of ΔABC, D is a point on BC and G is a point on AB such that $EH = CH = DH = GH = DG = BG$. Also, $\angle CHE = \angle HGF$. The measure of $\angle BAC$ in degree is _____.



(A) 15° **(B)** 20°
(C) 30° **(D)** 35°

28. $ABCD$ is a rectangle in which $AB = 8$, $AD = 9$. E is on AD such that $DE = 4$. H is on BC such that $BH = 6$. EC and AH cut at G. GF is drawn perpendicular to AD produced. Then $GF =$

(A) 20 **(B)** 22
(C) 18 **(D)** 15

29. A triangle has sides with lengths 13 cm, 14 cm, and 15 cm. A circle whose centre lies on the longest side touches the other two sides. The radius of the circle is (in cm) _____.

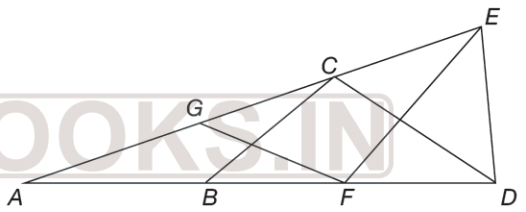
(A) 56/11 **(B)** 56/9
(C) 56/7 **(D)** 56/13

30. The internal bisector of $\angle A$ of ΔABC meet BC at P. $b = 2c$ in usual notation if $9AP^2 + 2a^2 = k \cdot c^2$. Find k.

(A) 12 **(B)** 15
(C) 18 **(D)** 20

PART – II KVPY Level Problems

1. In the figure, $AB = BC = CD = DE = EF = FG = GA$, then find $\angle DAE$ (approximately)

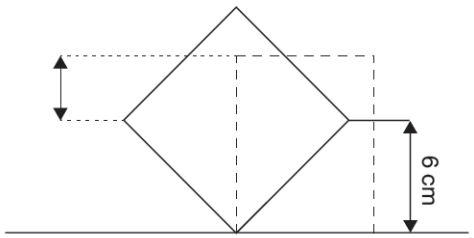


(A) 24° **(B)** 25°
(C) 26° **(D)** None of these

2. If the sum of all angles except one of a convex polygon is 2180°, then the number of sides of the polygon is:

(A) 19 **(B)** 17
(C) 15 **(D)** 13

3. A square board side 10 centimeters, standing vertically, is tilted to the left so that the bottom–right corner is: raised 6 centimeters from the ground.



By what distance is the top–left corner lowered from its original position?

(A) 1 cm **(B)** 2 cm
(C) 3 cm **(D)** 0.5 cm

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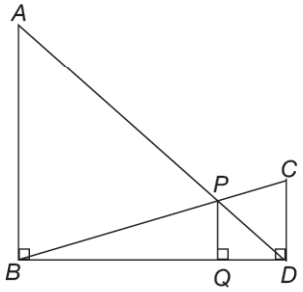
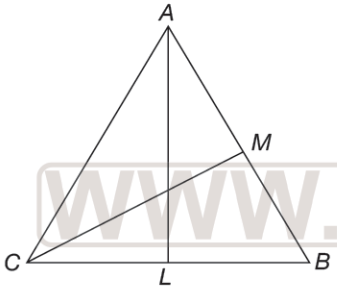
4. In the diagram given below, $\angle ABD = \angle CDB = \angle PQD = 90°$. If $AB : CD = 3 : 1$, the ratio of $CD : PQ$ is:



(A) 1 : 0.69 **(B)** 1 : 0.75
(C) 1 : 0.72 **(D)** None of the above.

5. In the figure below, AL is perpendicular to BC and CM is perpendicular to AB. If $CL = AL = 2BL$, find MC/AM.

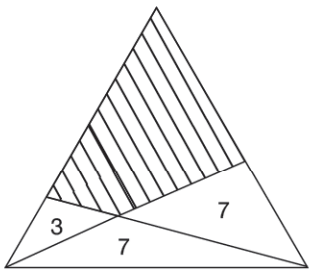


(A) 2 **(B)** 3
(C) 4 **(D)** Cannot be determined

6. $ABCD$ (in order) is a rectangle with $AB = CD = \frac{12}{5}$ and $BC = DA = 5$. Point P is taken on AD such that $\angle BPC = 90°$. The value of $(BP + PC)$ is equal to:

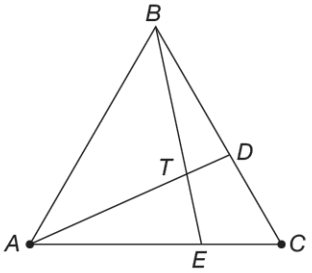
(A) 5 **(B)** 6
(C) 7 **(D)** 8

7. A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown. What is the area of the shaded quadrilateral?



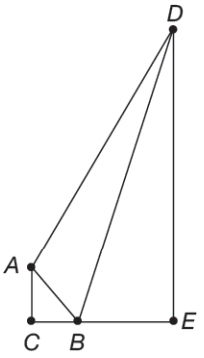
(A) 12 **(B)** 15
(C) 18 **(D)** 20

8. In ΔABC point D and E lie on $\overline{BC}$ and $\overline{AC}$, respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at T then $AT/DT = 3$ cm and $BT/ET = 4$, what is $\left(\frac{11CD}{BD}\right)$?



(A) 1 **(B)** 2
(C) 3 **(D)** 4

9. Triangle ABC has a right angle at C, $AC = 3$, and $BC = 4$. Triangle ABD has right angle at A and $AD = 12$. Points C and D are on opposite sides of $\overline{AB}$. The line through D parallel to $\overline{AC}$ meets $\overline{CB}$ extended at E. If $\frac{DE}{DB} = \frac{m}{n}$ where m and n are relatively prime positive integers, then $\frac{m+n}{2}$ is:



(A) 52 **(B)** 62
(C) 64 **(D)** 80

10. A thin piece of wood of uniform density in the shape of an equilateral triangle. with side length 3 inches weighs 12 gm. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length of 5 inches. If W is the weight in gm, of the second piece. Then find $[W]$, where $[x]$ denotes the greatest integer less than or equal to x.

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(A) 33 **(B)** 35
(C) 36 **(D)** 37

11. The isosceles right triangle ABC has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. If the area of ΔCDE can be written as $\frac{a(2-\sqrt{b})}{2}$, find the product of a and b.

(A) 75 **(B)** 72
(C) 70 **(D)** 80

12. Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I denote the intersection of the internal angle bisectors of ΔABC. What is BI?

(A) 10 **(B)** 15
(C) 18 **(D)** 20

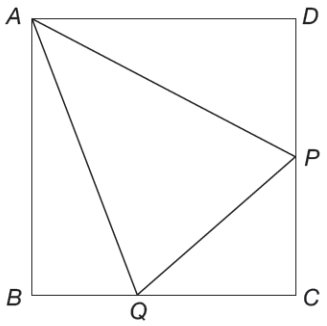
13. A has a pair of triangles with corresponding sides proportional, and B has a pair of pentagons with corresponding sides proportional.
S_1 = A's triangles must be similar
S_2 = D's pentagons must be similar
Which of the following statement is correct?

(A) S_1 is true, but S_2 is not true.
(B) S_2 is true, but S_1 is not true.
(C) Both S_1 and S_2 are true.
(D) Neither S_1 nor S_2 is true.

14. The side of a triangle are of length 20, 21, and 29 units. The sum of the lengths of altitude will be

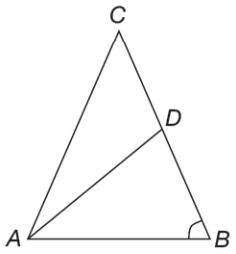
(A) $\frac{1609}{29}$ units **(B)** 49 units
(C) $\frac{1609}{21}$ units **(D)** 40 units

15. In the figure, $ABCD$ is a square of side 1 dm and $\angle PAQ = 45°$. The perimeter (in dm) of the triangle PQC is



(A) 2 **(B)** $1+\sqrt{2}$
(C) $2\sqrt{2}-1$ **(D)** $1+\sqrt{3}$

16. In the figure, ABC is a triangle in which AD bisects $\angle A$, $AC = BC$, $\angle B = 72°$, and $CD = 1$ cm. Length of BD (in cm) is

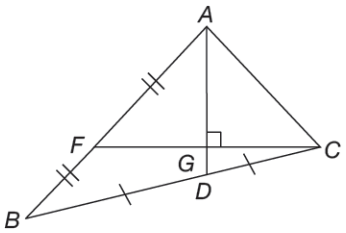


(A) 1 **(B)** $\frac{1}{2}$
(C) $\frac{\sqrt{5}-1}{2}$ **(D)** $\frac{\sqrt{3}+1}{2}$

17. If the hypotenuse of a right-angled triangle is 41 cm and the area of the triangle is 180 sq cm, then the difference between the lengths of the legs of the triangle must be:

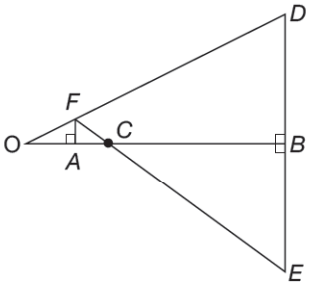
(A) 22 cm **(B)** 25 cm
(C) 27 cm **(D)** 31 cm

18. Consider ΔABC, the medians AD and CF intersect at right angles at G. If $BC = 3$ cm and $AB = 4$ cm, then the length of AC is (in cm):



(A) 0 **(B)** $\sqrt{5}$
(C) $\sqrt{7}$ **(D)** 3

19. In figure, OB is perpendicular bisector of line segments DE, $FA \perp OB$ and FE intersect OB at the point C, then find the ratio $\frac{OC}{OA} + \frac{OC}{OB}$



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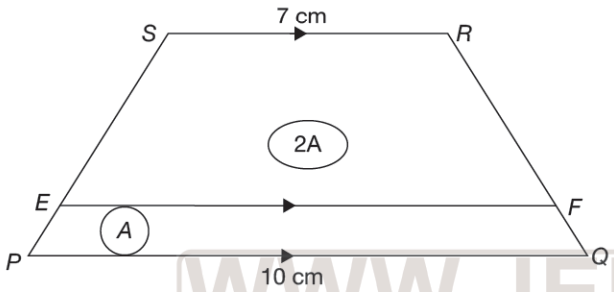
(A) 1 **(B)** 3
(C) 2 **(D)** 4

20. Let ABC be an acute angled triangle in which D, E, F are points on BC, CA, AB, respectively, such that $AD \perp BC$, $AE = EC$, and CF bisect $\angle C$ internally. Suppose CF meets AD and DE in M and N, respectively. If $FM = 2$, $MN = 1$, $NC = 3$, find the perimeter of the triangle ABC.

(A) $9\sqrt{3}$ **(B)** $6\sqrt{3}$
(C) $12\sqrt{3}$ **(D)** $8\sqrt{3}$

21. In the given figure, the area of trapezium $PQFE$ is half of the area of trapezium $EFRS$. If $SR \,||\, EF \,||\, PQ$, SR and PQ is 7 cm and 10 cm, respectively, then $EF = ?$

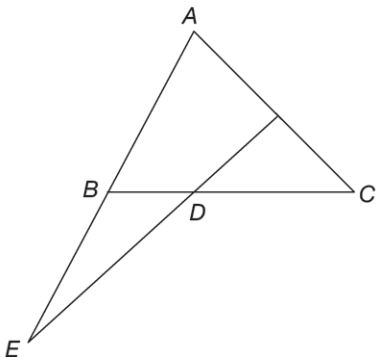


(A) $\sqrt{83}$ cm **(B)** $\sqrt{79}$ cm
(C) $\sqrt{87}$ cm **(D)** $\sqrt{73}$ cm

22. The length of sides of triangle are integers and its perimeter is 14. How many such distinct triangles are possible?

(A) 6 **(B)** 5
(C) 4 **(D)** 3

23. In the given figure $\frac{BD}{CD} = \frac{3}{4}$ and $AE = 6BE$, then $AF =$ _____



(A) 2/9 **(B)** 4/6
(C) 3/8 **(D)** 5/9

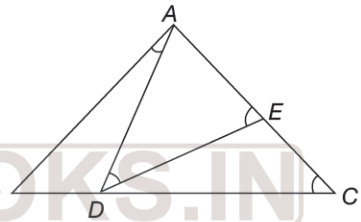
24. Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. If the ratio of the area of the other small right triangle to the area of the square is $\frac{1}{\text{km}}$, then find k

(A) 1 **(B)** 2
(C) 3 **(D)** 4

25. The altitudes of triangle are 12, 15 and 20 units. The largest angle in the triangle is:

(A) 75° **(B)** 90°
(C) 120° **(D)** 135°

26. Triangle ABC is isosceles with $AB = AC$. The measure of angle BAD is 30° and $AD = AE$. The measure of angle EDC is:

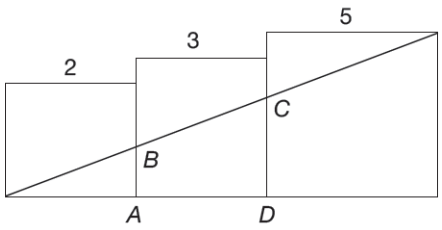


(A) 5° **(B)** 10°
(C) 15° **(D)** 20°

27. A square $DEAF$ is constructed inside a 30° – 60° – 90° triangle ABC with the hypotenuse $BC = 4$, D on side BC, E on side AC, and F on side AB. The length of the side of the square is:

(A) $3 - \sqrt{3}$ **(B)** $3 - \sqrt{2}$
(C) 2 **(D)** 1.5

28. Three squares have the dimensions indicated in the diagram. The area of the quadrilateral $ABCD$ is:



(A) $\frac{21}{4}$ **(B)** $\frac{15}{4}$
(C) $\frac{42}{4}$ **(D)** Data not sufficient

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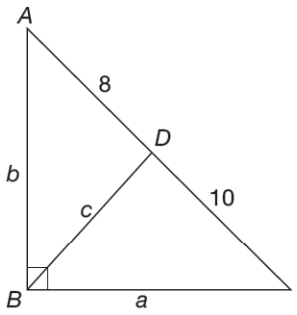
29. A rhombus is inscribed in triangle ABC in such a way that one of its vertices is A and two of its sides lie along AB and AC, where $AC = 6$, $AB = 12$, and $BC = 8$, the side of the rhombus is:

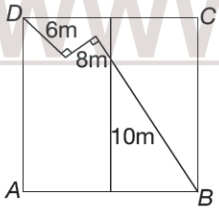
(A) 2 **(B)** 3
(C) 4 **(D)** 5

30. In right-angled $\Delta ABC = \angle B = 90°$, $BD \perp AC$, $AB = b$, $BD = c$, $AD = 8$, $DC = 10$, then find 'b'.



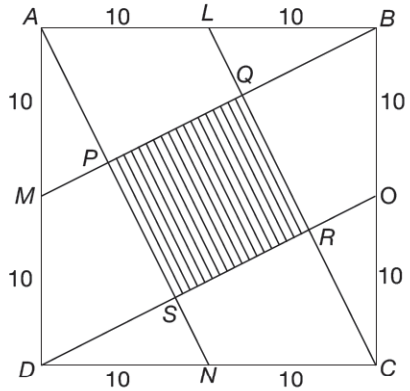
(A) $4\sqrt{5}$ **(B)** 12
(C) $6\sqrt{5}$ **(D)** $\sqrt{18}$

31. Find the area of the square $ABCD$.



(A) 160 m^2 **(B)** 140 m^2
(C) 125 m^2 **(D)** 120 m^2

32. Let $ABCD$ be a square of side 20 cm. The area of the square $PQRS$ (in cm)^2 interior to $ABCD$, shown in figure in



(A) 60 **(B)** 80
(C) 100 **(D)** 400

33. Let D be point on the side BC of a triangle ABC such that $\angle ADC = \angle BAC$. If $AC = 21$ cm, then the side of an equilateral triangle whose area is equal to the area of the rectangle sides BC and DC is

(A) $14 \times 3^{1/2}$ **(B)** $42 \times 3^{-1/2}$
(C) $14 \times 3^{3/4}$ **(D)** $42 \times 3^{1/2}$

34. Point, P inside ΔABC. Let segment APD, BPE, CPF are drawn with D on BC, E on CA, F on AB given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, $CF = 20$. Find area of ΔABC.

(A) 100 **(B)** 102
(C) 105 **(D)** 108

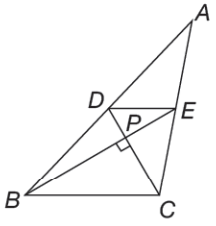
35. $ABCD$ is a square and there is a point P inside it. $AP = 12$, $BP = 26$ and side of square is 30. And centroid of ΔABP, ΔBCP, ΔCDP, ΔDAP are vertices of convex quadrilateral. Find area of quadrilateral.

(A) 200 **(B)** 150
(C) 120 **(D)** 180

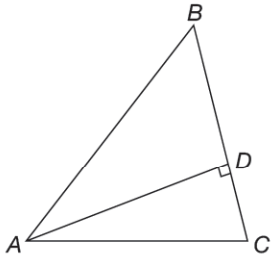
PART – III PRMO Level Problems

1. When extending AB, BC, CA of ΔABC to B', C', A', respectively, such that $AB' = 2AB$, $CC' = 2BC$, $AA' = 3CA$. If area of ΔABC is l, find the area of $\Delta A'B'C$.

2. In ΔABC, D and E are the mid-points of the sides AB and AC respectively, CD and BE intersect at P with $\angle BPC = 90°$. Suppose BD = $8\sqrt{5}$ and CE = $6\sqrt{5}$. Find BC.



3. In ΔABC, $\angle BAC = 45°$, D is a point on BC such that AD is perpendicular to BC. If $BD = 3$ cm and $DC = 2$ cm, and the area of the ΔABC is x cm^2. Find the value of x.



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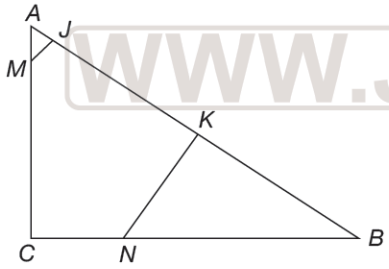
4. Let ABC be any triangle. Let D and E be the points respectively in the segments of AB and BC such that $AD = 7DB$ and $BE = 10EC$. Assume that AE and CD meet at point F. Determine $[k]$, where k is the real number such that $AF = k \times FE$.

5. In an acute-angled triangle ABC, points D, E, and F are the feet of the perpendiculars from A, B, and C onto BC, AC, and AB, respectively. Suppose $\sin A = \frac{3}{5}$ and $BC = 39$. Find the length of AH, where H is the intersection AD with BE.

6. Let O be an interior point of ΔABC. Extend AO to meet the side BC at D. Similarly, extend BO and CO to meet CA and AB, respectively at E and F. If $AO = 30$, $FO = 20$, $BO = 60$, $DO = 10$, and $CO = 20$, find EO.

7. In ΔABC, $AB = 13$, $AC = 5$, and $BC = 12$. Points M and N lie on $\overline{AC}$ and $\overline{BC}$, respectively, with CM = CN = 4. Points J and K are on $\overline{AB}$ so that $\overline{MJ}$ and $\overline{NK}$ are perpendicular to $\overline{AB}$. What is the whole part of area of pentagon $CMJKN$?



8. In triangle ABC, side AC and the perpendicular bisector of BC meet in point D, and BD bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, the area of triangle $ABD = a\sqrt{5}$ then the value of a?

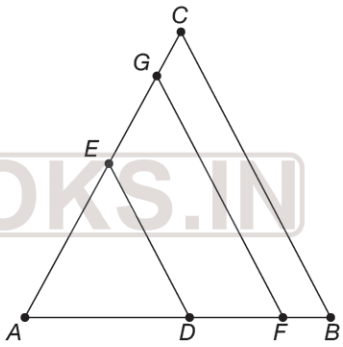
9. In ΔABC, we have $AB = 1$ and $AC = 2$. Side BC and the median from A to BC have the same length. If the length of BC is $\sqrt{a}$ then the value of a?

10. Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the mid-point of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and AB, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b and c are positive integers and b is not divisible by the square of any prime. What is the value of $a + b + c$?

11. In ΔPAT, $\angle P = 36°$, $\angle A = 56°$, and $PA = 10$. Points U and G lie on sides TP and TA, respectively, so $PU = AG = 1$. Let M and N be the mid-points of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?

12. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3\,BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. The ratio of the area of $\Delta A'B'C'$ to the area of ΔABC is $Q : 1$. What is Q?

13. Triangle ABC is equilateral with $AB = 1$. Points E and G are on $\overline{AC}$ and points D and F are on $\overline{AB}$ such that both $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$. Furthermore, triangle ADE and trapezoids $DFGE$ and $FBCG$ all have the same perimeter. $DE + FG$ can be written as $\frac{r}{s}$ where r and s are co-prime. What is $r + s$?



14. The angles in a particular triangle are in arithmetic progression, and the side lengths are 4, 5, x. The sum of the possible values of x equals $a + \sqrt{b} + \sqrt{c}$ where a, b, and c are positive integers. What is $a + b + c$?

15. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$ and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

16. Points B and C lie on $\overline{AD}$. The length of $\overline{AB}$ is 4 times the length of $\overline{BD}$, and the length of $\overline{AC}$ is 9 times the length of $\overline{CD}$. The length of $\overline{BC}$ × S = length of $\overline{AD}$. What is S?

17. Point P is inside equilateral ΔABC. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB^2?

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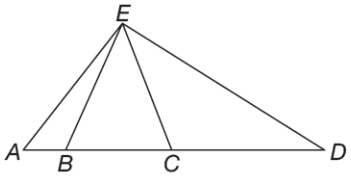
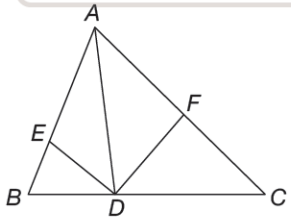
18. In triangle ABC, $AB = (b^2 - 1)$ cm, $BC = a^2$ cm and $AC = 2a$ cm, where a and b are positive integers greater than 1. Find the value of $a - b$.

19. In the figure below, ADE is a triangle with $\angle AED = 120°$, and B and C are points on side AD such that BCE is an equilateral triangle. If $AB = 4$ cm, $CD = 16$ cm and $BC = x$ cm, find the value of x.

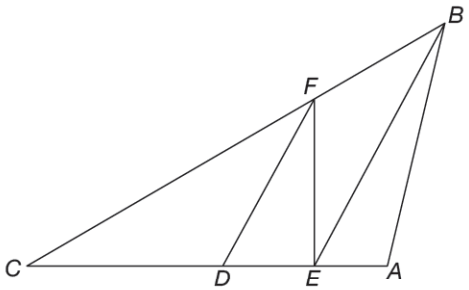


20. In the triangle ABC, $AB = 8$, $BC = 7$, and $CA = 6$. Let E be the point on BC such that $\angle BAE = 3\angle EAC$. Find $\dfrac{4AE^2}{5}$.

21. In the figure below, ABC is a triangle and D is a point on side BC, Point E is on side AB such that DE is the angle bisector of $\angle ADB$ and point F is on side AC such that DF is the angle bisector of $\angle ADC$. Find the value of $\dfrac{AE}{EB}\cdot\dfrac{BD}{DC}\cdot\dfrac{CF}{FA}$.

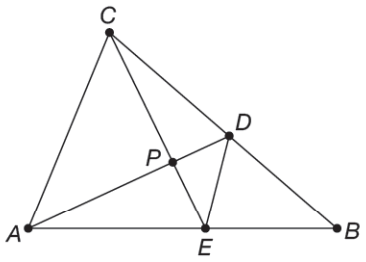


22. In the figure below, ABC is a triangle with $AB = 10$ cm and $BC = 40$ cm. Points D and E lie on side AC and point F on side BC such that EF is parallel to AB and DF is parallel to EB. Given that BE is an angle bisector of $\angle ABC$ and that $AD = 13.5$ cm, find the length of CD in cm.



23. In triangle ABC, $AB = 13$, $BC = 14$ and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$ and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$ and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\dfrac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

24. In ΔABC, medians $\overline{AD}$ and $\overline{CE}$ intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $2[AEDC]$?



25. A flagpole is originally 5 meters tall. A hurricane snaps the flagpole at a point x meters above the ground so that the upper part, still attached to the stump, touches the ground 1 meter away from the base. What is $[x]$ where [:] is the greatest integer function?

26. Distinct points A, B, C, and D lie on a line, with $AB = BC = CD = 1$. Points E and F lie on a second line, parallel to the first, with $EF = 1$. A triangle with positive area has three of the six points as its vertices. How many possible values are there for the area of the triangle?

27. Two equilateral triangles are contained in a square whose side length is $2\sqrt{3}$. The bases of these triangles are the opposite sides of the square and their intersection is a rhombus. If the area of the rhombus is $8\sqrt{k} - 12$, find k.

28. A triangle ΔABC is inscribed in a circle of radius 1, with $\angle BAC = 60°$. Altitudes AD and BE of ΔABC intersect at H. Find the smallest possible value of the length of the segment AH.

29. Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

30. In ΔABC, $AB = AC = 28$ and $BC = 20$. Points D, E, and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram ADEF?

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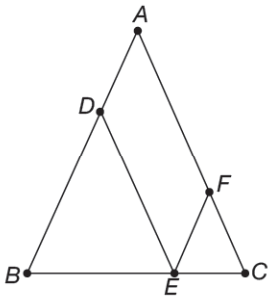
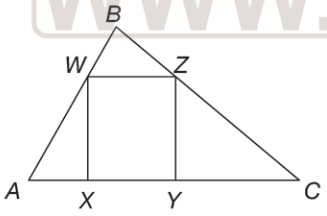


31. In ΔABC, $\angle C = 90^\circ$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. The perimeter of the triangle can be written as $a(1+\sqrt{2})$. Find the product of a and b.

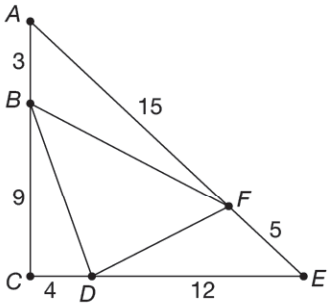
32. Point P is inside equilateral ΔABC. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB^2?

33. Right ΔABC has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in ΔABC with X and Y on $\overline{AC}$, W on $\overline{AB}$, and Z on $\overline{BC}$. What is the side length of the square to the nearest integer?

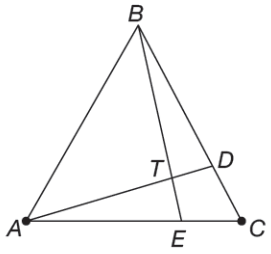


34. In a triangle with integer side lengths, one side is three times as long as the second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

35. In right triangle ΔACE, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on $\overline{AC}$, $\overline{CE}$, and $\overline{EA}$, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the difference of the area of ΔACE to that of ΔDBF?



36. In ΔABC, points D and E lie on $\overline{BC}$ and $\overline{AC}$, respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at T then $AT/DT = 3$ and $BT/ET = 4$. What is $\left(\dfrac{11CD}{BD}\right)$?



37. A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. Total N number of toothpicks would be needed to construct a large equilateral triangle if the base row of the triangle consists of 2003 small equilateral triangles. Then find the sum of all digits of N.

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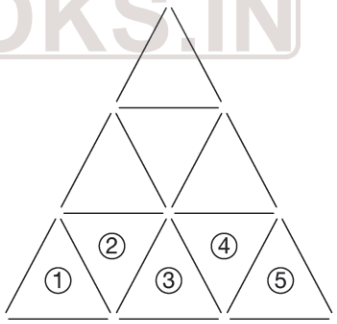


38. Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find $\left(\dfrac{3HC}{JE}\right)$.

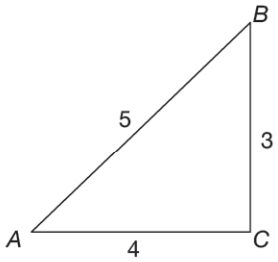
39. In triangle ABC, $\angle ABC = 120^\circ$, $\overline{AB} = 3$ and $\overline{BC} = 4$. If perpendiculars constructed to $\overline{AB}$ at A and to $\overline{BC}$ at C meet at D, then $\sqrt{3}$ CD equals?

40. Two congruent $30^\circ - 60^\circ - 90^\circ$ triangles are placed so that they overlap partly and their hypotenuses coincide. If the hypotenuse of each triangle is 12, and the area common to both triangles is given by A. Find the value of $\sqrt{3}$ A·

41. In ΔABC with right angle at C, altitude CH and median CM trisects the right angle. If the area of ΔCHM is 10, then the area of ΔABC is

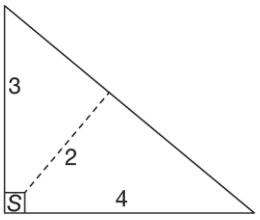
42. In a triangle with side of lengths a, b, and c, $(a + b + c)(a + b - c) = 3ab$. The measure of the angle opposite to the side of length c is?

43. A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. Then the length in inches of the crease is L. Find the value of $8L$.



44. Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

45. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse E is 2 units. Let the fraction of the field planted can be written as $\frac{x}{y}$. Find the value of $10(y - x)$ where x, y are co-prime numbers.



46. Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the mid-point of $\overline{AB}$, and let E be the mid-point of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

47. A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angled vertex of the triangle. A square with side length y is inscribed in another right-angled triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. Find the value of $(7x + 37y)$.

48. Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. If the area of ΔADE is given by P. Find $25P$.

49. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. If the ratio of the area of $\Delta A'B'C'$ to the area of ΔABC is $x : 1$. Find the value of x.

50. The isosceles right triangle ABC has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. If the area of ΔCDE can be written as $\frac{a(2-\sqrt{b})}{2}$. Find the product of a and b.

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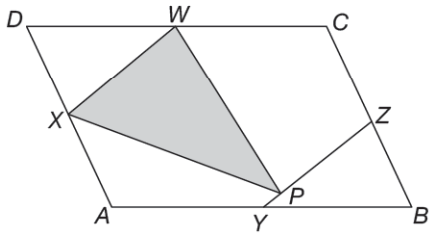
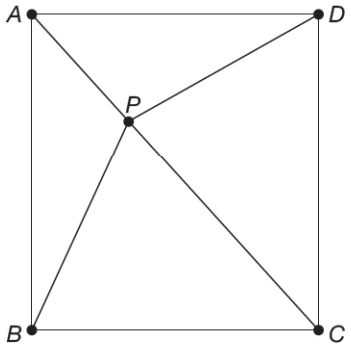
51. In the diagram below, $ABCD$ is a square. The points A, B, and G are collinear. The line segments AC and DG intersect at E and the line segments DG and BC intersect at F. Suppose that $DE = 15$ cm, $EF = 9$ cm, and $FG = x$ cm. Find the value of x.

52. W, X, Y, and Z are the mid-points of the four sides of parallelogram $ABCD$. P is a point on the line segment YZ. What percent of the area of parallelogram $ABCD$ is triangle PXW?

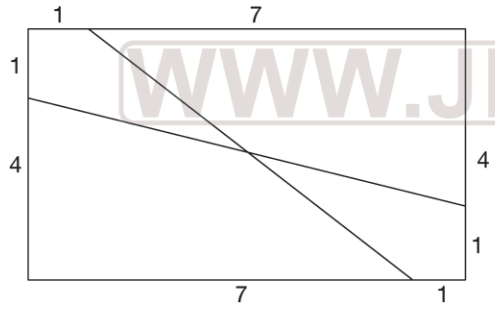


53. Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of ΔABP, ΔBCP, ΔCDP, and ΔDAP are the vertices of a convex quadrilateral. The area of that quadrilateral is A. What is $\frac{A}{4}$?



54. The ratio of the short side of a certain rectangle to the long side is equal to the ratio of the long side to the diagonal. The square of the ratio of the short side to the long side of this rectangle is in the form of $\frac{\sqrt{a}-b}{c}$. What is $(a + b - c)^2$?

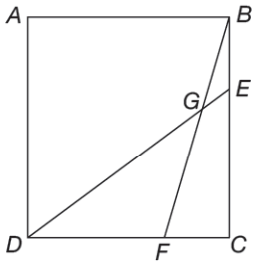
55. The area of the shaded region of the given 8×5 rectangle is an improper fraction with co-prime digits. What is the sum of numerator and denominator?



56. For some positive integers p, there is a quadrilateral $ABCD$ with positive integer side lengths, perimeter p, right angles at B and C, $AB = 2$, and $CD = AD$. How many different values of $p < 2015$ are possible?

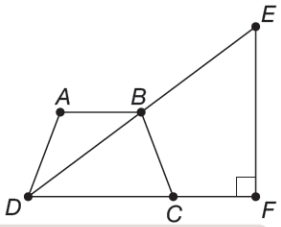
57. Trapezoid $ABCD$ has $AD \parallel BC$, $BD = 1$, $\angle DBA = 23°$, and $\angle BDC = 46°$. The ratio $BC : AD$ is 9 : 5. $CD = \frac{p}{q}$ where $gcd\ (p, q) = 1$. What is $p^2 + q^2$?

58. Given that $ABCD$ is a square. Points E and F lie on the side BC and CD, respectively, such that $BE = CF = \frac{1}{3}AB$. G is the intersection of BF and DE. If $\frac{Area\ of\ ABGD}{Area\ of\ ABCD} = \frac{m}{n}$ is in the lowest term, find the value of $m + n$.

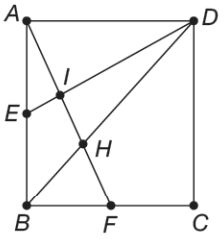


59. A regular octagon $ABCDEFGH$ has sides of length two. Then the area of ΔADG is given by $(a + b\sqrt{c})$. Find the product of a, b, & c.

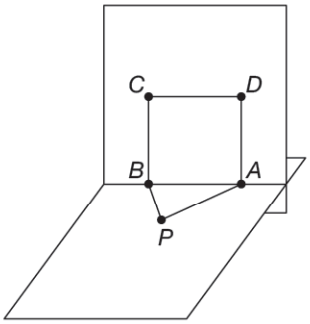
60. In the figure, $ABCD$ is an isosceles trapezoid with side lengths $AD = \overline{BC} = 5$, $AB = 4$, and $DC = 10$. The point $\overline{C}$ is on $\overline{DF}$ and B is the mid-point of hypotenuse $\overline{DE}$ in the right triangle DEF. Then $CF =$



61. If $ABCD$ is a 2×2 square, E is the mid-point of $\overline{AB}$, F is the mid-point of $\overline{BC}$, $\overline{AF}$, and $\overline{DE}$ intersect at I, and BD and AF intersect at H, and the area of quadrilateral $BEIH$ is $\frac{k}{2k+1}$, find k.



62. Triangle PAB and square $ABCD$ are in perpendicular planes. Given that $PA = 3$, $PB = 4$, and $AB = 5$, what is PD^2?



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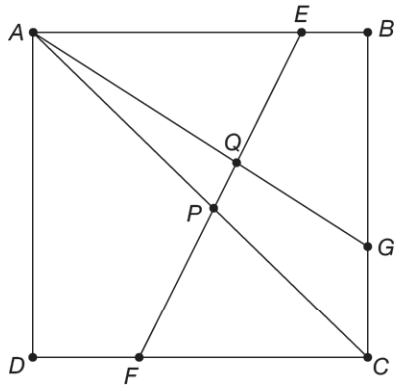
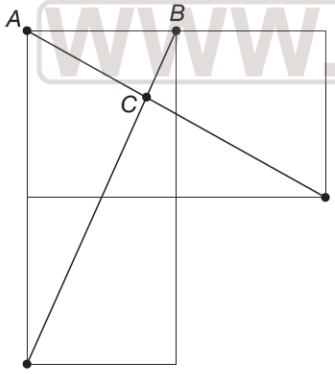
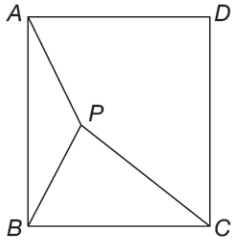
63. Rectangle ABCD has AB = 5 and BC = 4. Point E lies on $\overline{AB}$ so that EB = 1, point G lies on $\overline{BC}$ so that CG = 1 and point F lies on $\overline{CD}$ so that DF = 2. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $91\left(1-\frac{PQ}{EF}\right)$?



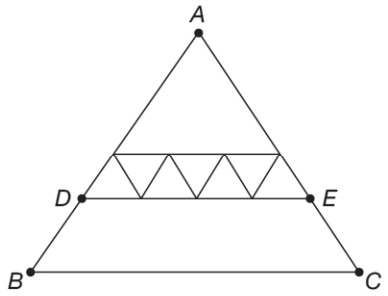
64. Three unit squares and two line segments connecting two pairs of vertices are shown. If the area of ΔABC is N, find $5N$.



65. In the following diagram, ABCD is a square with $PA = a$, $PB = 2a$ and PC = $3a$. Find the value of x where $x° = 180° - \angle APB$.



66. All of the triangles in the diagram below are similar to isosceles triangle ABC, in which AB = AC. Each of the 7 smallest triangles has area 1, and ΔABC has area 40. What is the area of trapezoid DBCE?



EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. L, M, and N are mid-points of sides AB, BC, and CA of triangle ABC. If area of triangle ABC is 48 sq. units, the area of triangle LMN will be: **[NSEJS-2008]**

(A) 6 units **(B)** 8 units
(C) 12 units **(D)** 24 units

2. The number of isosceles triangles with integer sides such that no side is greater than 4 units is: **[NSEJS-2008]**

(A) 8 **(B)** 9
(C) 16 **(D)** 12

3. Three line segments in a plane have lengths a, b, and c. No two of these are parallel. If $\sqrt{a}+\sqrt{b}=\sqrt{c}$, then: **[NSEJS-2009]**

(A) can form an acute-angled triangle
(B) can form a right-angled triangle
(C) can form an isosceles triangle
(D) cannot form a triangle

4. If the altitudes of a triangle are in the ratio 2 : 3 : 4, then the lengths of the corresponding sides are in the ratio: **[NSEJS-2010]**

(A) 2 : 3 : 4 **(B)** 6 : 4 : 3
(C) 3 : 2 : 4 **(D)** 3 : 2 : 1

5. All the three sides of a Δ ABC have lengths in integral units, with AB = 2001 units and BC = 1002 = units. The possible number of triangles with this condition is: **[NSEJS-2010]**

(A) 2001 **(B)** 2002
(C) 2003 **(D)** 2004

6. AD, BE, and CF are the medians of ΔABC. The sum of lengths of segments BE and CF is: **[NSEJS-2010]**

(A) $< \frac{3}{2}BC$ **(B)** $> \frac{5}{3}BC$

(C) $> \frac{3}{2}BC$ **(D)** $< \frac{2}{3}BC$

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7. The sides of a triangle are in the ratio 6 : 8 : 9. Therefore: **[NSEJS-2010]**

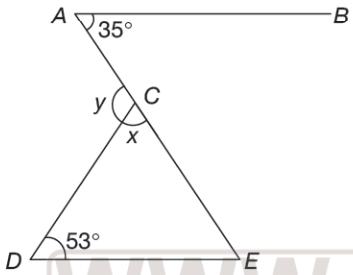
(A) Angles of the triangle are in the ratio 6 : 8 : 9
(B) It is an acute-angled triangle
(C) It is a right-angled triangle
(D) It is an obtuse-angled triangle

8. The perimeter of an isosceles right-angled triangle is $2p$. Its area is: **[NSEJS-2010]**

(A) $\left(2+\sqrt{2}\right)p^2$ **(B)** $\left(2-\sqrt{2}\right)p^2$

(C) $\left(3-2\sqrt{2}\right)p^2$ **(D)** $\left(3+2\sqrt{2}\right)p^2$

9. In the figure shown, AB is parallel to DE. The difference between angles x and y is: **[NSEJS-2010]**



(A) 0° **(B)** 4°
(C) 10° **(D)** 12°

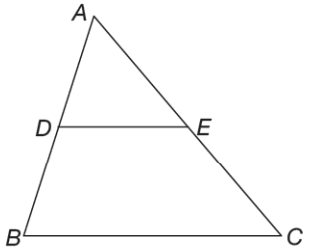
10. In ΔABC, D is a point on side AC such than $\angle ABD = 1/2\ \angle ABC$. If $AB = 36$, $BC = 48$, $CD = 28$, then the length DA will be: **[NSEJS-2011]**

(A) 20 **(B)** 21
(C) 22 **(D)** 24

11. An equilateral triangle has area A cm^2. A regular hexagon of maximum area is cut off from the triangle. If the area of the hexagon is 320 cm^2, area of A is **[NSEJS-2011]**

(A) 640 cm^2 **(B)** 480 cm^2
(C) 600 cm^2 **(D)** 400 cm^2

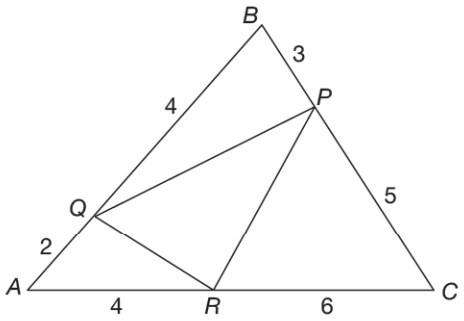
12. In the figure shown below, $DE \| BC$ and $AD = 3x - 2$, $AE = 5x - 4$, $BD = 7x - 5$ and $CE = 5x - 3$. Therefore, the value of x is: **[NSEJS-2011]**



(A) only I **(B)** only $\frac{7}{10}$

(C) 1 or $\frac{7}{10}$ **(D)** $\frac{10}{7}$

13. In the adjoining figure $AQ = 2$, $QB = 4$, $BP = 3$, $PC = 5$, $CR = 6$ and $RA = 4$. Find the area of triangle PQR. **[NSEJS-2012]**



(A) 4.8 **(B)** 5.2
(C) 5.8 **(D)** 6.2

14. Consider triangles having integer sides such that no side is greater than 4 units. How many such triangles are possible? **[NSEJS-2012]**

(A) 13 **(B)** 17
(C) 24 **(D)** 64

15. Let m be the number of distinct (non-congruent) integer – sided triangles each with perimeter 15 and n be the number of distinct (non-congruent) integer-sided triangles each with perimeter 16. Then $m - n$ equals: **[NSEJS-2013]**

(A) –2 **(B)** 0
(C) 2 **(D)** –4

16. Let ABC be a triangle in which $AB = AC$. Let D be a point on AC such that BD bisects angle B. Value of the ration $\frac{AB}{BC}$ is between: **[NSEJS-2013]**

(A) 1.0 and 1.5 **(B)** 1.5 and 2.0
(C) 2.0 and 2.5 **(D)** 2.5 and 3.0

17. A piece of wire 60 cm long is cut into two parts, one of them being 24 cm long. Each part is then bent to form a square. The ratio of the area of the larger to the smaller square is: **[NSEJS-2014]**

(A) 9/4 **(B)** 7/4
(C) 3/2 **(D)** 11/3

18. The adjacent figure is a modification of the Switzerland flag to suit the problem! Five identical small

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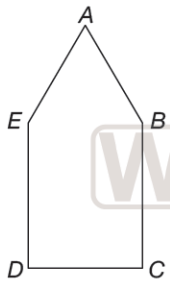
squares from the central cross. The length of each side of the big square is 10 m. If the area of the white cross is 20% of the area of the flag, then the length of the side of the small square is: **[NSEJS-2014]**

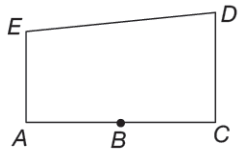
(A) 2m **(B)** 2.25
(C) 1.6m **(D)** 1.75m

19. If the distance between A and B is 230 km, B and C is 120 km, C and A is 350 km. Also, if the distance between C and D is 200 km, distance between D and B is 330 km and distance from A to E is 100 km and distance between D and E is 570 km. The diagram (not drawn to scale) that represents this graphically is: **[NSEJS-2014]**

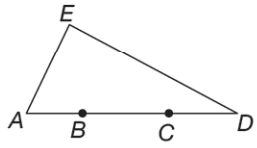
(A)



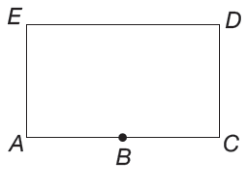
(B)



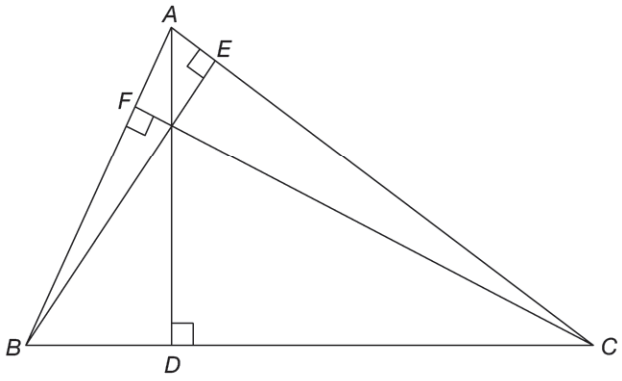
(C)



(D)



20. In the adjoining figure segment AD, BE, and CF are the altitudes of triangle ABC. Find $AD \times BC$ if $AB \times AC = 409.6$, $BE \times CF = 202.5$. **[NSEJS-2015]**



(A) 225 **(B)** 256
(C) 288 **(D)** 312

21. The adjacent sides of a parallelogram are 30 cm and 20 cm. The length of one of the diagonal is 40 cm. What is the length of the other diagonal? **[NSEJS-2015]**

(A) 60 cm **(B)** $10\sqrt{10}$ cm
(C) $20\sqrt{5}$ cm **(D)** $8\sqrt{30}$ cm

22. If $ABCD$ is a rhombus and $\angle ABC = 60°$ then **[NSEJS-2016]**

(A) The points A, B, C, D are concyclic
(B) The quadrilateral has exactly half the area of the square with same sides as $ABCD$
(C) The quadrilateral has area $\frac{\sqrt{3}}{2} AB^2$
(D) The diagonals of the quadrilateral $ABCD$ are equal and bisect each other at right angle

23. A triangle has perimeter 316 and its sides are of integer length. The maximum possible area of such a triangle is achieved for **[NSEJS-2016]**

(A) Single triangle
(B) Two triangles
(C) Three triangles
(D) More than three triangle

24. In triangle ABC, segment AD, segment BE and segment CF are altitudes. If $AB \times AC = 172.8$ cm^2 and $BE \times CF = 108.3$ cm^2, then $AD \times BC =$ _____. **[NSEJS-2017]**

(A) 136.8 cm^2 **(B)** 132.4 cm^2
(C) 129.2 cm^2 **(D)** 128.6 cm^2

25. Diagonals of a quadrilateral bisect each other. Therefore, the quadrilateral must be a _____. **[NSEJS-2017]**

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(A) parallelogram (B) rhombus
(C) rectangle (D) square

26. In a square $ABCD$, a point P is inside the square such that ABP is an equilateral triangle. The segment AP cuts the diagonal BD in E. Suppose $AE = 2$. The area of $ABCD$ is **[NSEJS-2018]**

(A) $4 + 2\sqrt{3}$ (B) $5 + 2\sqrt{3}$
(C) $4 + 4\sqrt{3}$ (D) $5 + 4\sqrt{3}$

27. Let $ABCD$ be a rectangle. Let X and Y be points, respectively, on AB and CD such that $AX : XB = 1 : 2 = CY : YD$. Join AY and CX; let BY intersect CX in K; let DX intersect AY in L. If m/n denotes the ratio of the area of $XKYL$ to that of $ABCD$, then $m + n$ equals **[NSEJS-2018]**

(A) 9 (B) 11
(C) 13 (D) 15

28. In a triangle ABC, let AD be the median from A; let E be a point on AD such that $AE : ED = 1 : 2$; and let BE extended meets AC in F. The ratio of AF/FC is **[NSEJS-2018]**

(A) 1/6 (B) 1/5
(C) 1/4 (D) 1/3

29. In a triangle ABC, a point D on AB is such that $AD : AB = 1 : 4$ and DE is parallel to BC with E on AC. Let M and N be the mid points of DE and BC respectively. What is the ratio of the area of the quadrilateral $BNMD$ to that of triangle ABC? **[NSEJS-2018]**

(A) 1/4 (B) 9/32
(C) 7/32 (D) 15/32

30. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on the line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of AEB? **[NSEJS - 2019]**

(A) 10 sq. units (B) 15/2 sq. units
(C) 25/2 sq. units (D) 20 sq. units

PART – II Previous Years' KVPY Problems

1. Let t_n denote the number of integral-sided triangles with distinct sides chosen from $\{1, 2, 3, \ldots, n\}$. Then, $t_{20} - t_{19}$ equals: **[KVPY-2009]**

(A) 81 (B) 153
(C) 163 (D) 173

2. The sides of a quadrilateral are all positive integers and three of them are 5, 10, 20. How many possible value are there for the fourth side? **[KVPY-2009]**

(A) 29 (B) 31
(C) 32 (D) 34

3. In a ΔABC, the altitudes from B and C on to the opposite sides are not shorter than their respective opposite sides. Then, one of the angles of ABC is **[KVPY-2009]**

(A) 30° (B) 45°
(C) 60° (D) 72°

4. In a ΔABC, $AB = AC = 37$. Let D be a point on BC such that $BD = 7$, $AD = 33$. The length of CD is: **[KVPY-2009]**

(A) 7 (B) 11
(C) 40 (D) not determinable

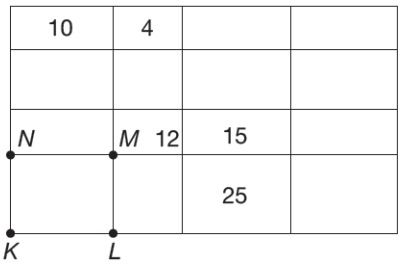
5. The sides of a ΔABC are positive integers. The smallest side has length 1. Which of the following statements is true? **[KVPY-2010]**

(A) The area of ΔABC is always a rational number
(B) The area of ΔABC is always an irrational number
(C) The perimeter of ΔABC is an even integer
(D) The information provided is not sufficient to conclude any of the statements A, B, or C above

6. Consider a square $ABCD$ of side 12 and let M, N be the mid-points of AB, CD, respectively. Take a point P on MN and let $AP = r$, $PC = s$. Then, the area of the triangle whose sides are r, s, 12 is **[KVPY-2010]**

(A) 72 (B) 36
(C) $\frac{rs}{2}$ (D) $\frac{rs}{4}$

7. A rectangle is divided into 16 sub-rectangles as in the figure, the number in each sub-rectangle represents the area of that sub-rectangle. What is the area of the rectangle $KLMN$? **[KVPY-2010]**



(A) 20 (B) 30
(C) 40 (D) 50

8. In a ΔABC, D and E are points on AB, AC, respectively, such that DE is parallel to BC. Suppose BE, CD intersect of O. If the areas of the triangles ADE and ODE

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are 3 and 1, respectively. Find the area of the area of the ΔABC, with justification. **[KVPY-2010]**

(A) 9 **(B)** 10
(C) 11 **(D)** 12

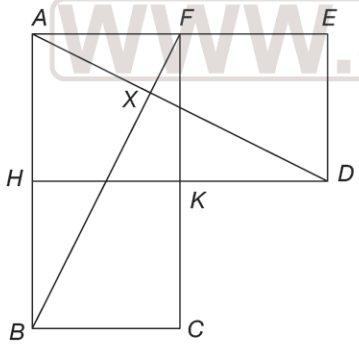
9. Let ABC be a triangle with $\angle B = 90°$. Let AD be the bisector of $\angle A$ with D on BC. Suppose $AC = 6$ cm and the area of the ΔADC is 10 cm^2. Then, the length of BD in cm is equal to **[KVPY-2011]**

(A) $\frac{3}{5}$ **(B)** $\frac{3}{10}$
(C) $\frac{5}{3}$ **(D)** $\frac{10}{3}$

10. The sides of a triangle are distinct positive integers in an arithmetic progression. If the smallest side is 10, the number of such triangles is **[KVPY-2012]**

(A) 8 **(B)** 9
(C) 10 **(D)** infinitely many

11. In the figure, $AHKF$, $FKDE$, and $HBCK$ are unit squares, AD and BF intersect in X. Then, the ratio of the areas of triangles AXF and ABF is: **[KVPY-2012]**



(A) $\frac{1}{4}$ **(B)** $\frac{1}{5}$
(C) $\frac{1}{6}$ **(D)** $\frac{1}{8}$

12. In a ΔABC, it is known that $AB = AC$. Suppose D is the mid-point of AC and $BD = BC = 2$. Then, the area of the ΔABC is: **[KVPY-2012]**

(A) 2 **(B)** $2\sqrt{2}$
(C) $\sqrt{7}$ **(D)** $2\sqrt{7}$

13. The number of values of b for which there is an isosceles triangle with sides of lengths $b + 5$, $3b - 2$, and $6 - b$ is **[KVPY-2013]**

(A) 0 **(B)** 1
(C) 2 **(D)** 3

14. In a ΔABC with $\angle A < \angle B < \angle C$, points D, E, F are on the interior of segments BC, CA, AB, respectively. Which of the following triangles cannot be similar to ΔABC? **[KVPY-2013]**

(A) ΔABD **(B)** ΔBCE
(C) ΔCAF **(D)** ΔDEF

15. In a rectangle $ABCD$, points X and Y are the mid-points of AD and DC, respectively. Lines BX and CD when extended intersect at E, lines BY and AD when extended intersect at F. If the area of $ABCD$ is 60, then the area of BEF is: **[KVPY-2013]**

(A) 60 **(B)** 80
(C) 90 **(D)** 120

16. Let $ABCD$ be a square and let P be a point on segment CD such that $DP : PC = 1 : 2$. Let Q be a point on segment AP such that $\Delta BQP = 90°$. Then, the ratio of the area of quadrilateral $PQBC$ to the area of the square $ABCD$ is: **[KVPY-2013]**

(A) $\frac{31}{60}$ **(B)** $\frac{37}{60}$
(C) $\frac{39}{60}$ **(D)** $\frac{41}{60}$

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17. In a quadrilateral $ABCD$, which is not a trapezium, it is known that $\angle DAB = \angle ABC = 60°$. Moreover, $\angle CAB = \angle CBD$. Then, **[KVPY-2014]**

(A) $AB = BC + CD$
(B) $AB = AD + CD$
(C) $AB = BC + AD$
(D) $AB = AC + AD$

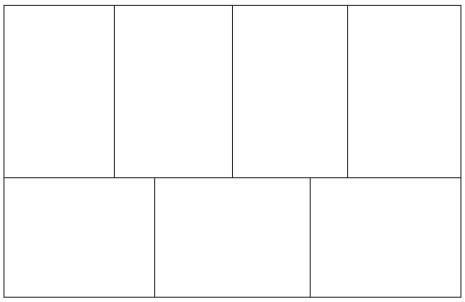
18. The angle bisectors BD and CE of a ΔABC are divided by the incentre I in the ratios 3 : 2 and 2 : 1, respectively. Then, the ratio in which I divides the angle bisector through A is: **[KVPY-2014]**

(A) 3 : 1 **(B)** 11 : 4
(C) 6 : 5 **(D)** 7 : 4

19. In a ΔABC with $\angle A = 90°$, P is a point on BC such that $PA : PB = 3 : 4$. If $AB = \sqrt{7}$ and $AC = \sqrt{5}$, then $BP : PC$ is: **[KVPY-2014]**

(A) 2 : 1 **(B)** 4 : 3
(C) 4 : 5 **(D)** 8 : 7

20. In the figure given below, a rectangle of perimeter 76 units is divided into 7 congruent rectangles.



What is the perimeter of each of the smaller rectangles? **[KVPY-2015]**

(A) 38 **(B)** 32
(C) 28 **(D)** 19

21. In a ΔABC, points X and Y are on AB and AC, respectively, such that XY is parallel to BC. Which of the two following equalities always hold? (Here $[PQR]$ denotes the area of ΔPQR). **[KVPY-2015]**

I. $[BCX] = [BCY]$

II. $[ACX] \cdot [ABY] = [AXY] \cdot [ABC]$

(A) Neither I nor II **(B)** Only I
(C) Only II **(D)** Both I and II

22. Let P be an interior point of a ΔABC. Let Q and R be the reflections of P in AB and AC, respectively. If Q, A, R are collinear, then $\angle A$ equals **[KVPY-2015]**

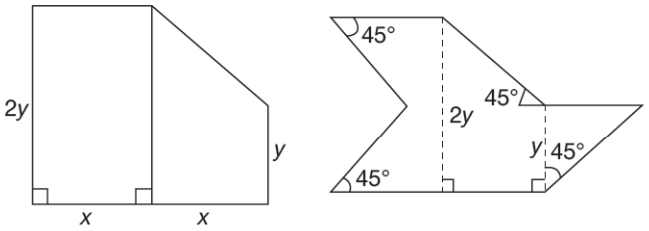
(A) 30° **(B)** 60°
(C) 90° **(D)** 120°

23. Let $ABCD$ be a square of side length 1. Let P, Q, R, S be points in the interiors of the sides AD, BC, AB, CD, respectively, such that PQ and RS intersect at right angles. If $PQ = \frac{3\sqrt{3}}{4}$, then RS equals: **[KVPY-2015]**

(A) $\frac{2}{\sqrt{3}}$ **(B)** $\frac{3\sqrt{3}}{4}$

(C) $\frac{\sqrt{2}+1}{2}$ **(D)** $4-2\sqrt{2}$

24. In the figure given below, if the areas of the two regions are equal, then which of the following is true? **[KVPY-2015]**



(A) $x = y$ **(B)** $x = 2y$
(C) $2x = y$ **(D)** $x = 3y$

25. A triangular corner is cut from a rectangular piece of paper and the resulting pentagon has sides 5, 6, 8, 9, 12 in some order. The ratio of the area of the pentagon to the area of the rectangle is **[KVPY-2015]**

(A) $\frac{11}{18}$ **(B)** $\frac{13}{18}$

(C) $\frac{15}{18}$ **(D)** $\frac{17}{18}$

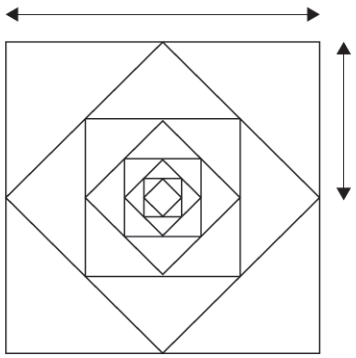
26. Let $ABCD$ be a trapezium with AD parallel to BC. Assume there is a point M in the interior of the segment BC such that $AB = AM$ and $DC = DM$. Then, the ratio of the area of the trapezium to the area of ΔAMD is: **[KVPY-2015]**

(A) 2
(B) 3
(C) 4
(D) not determinable from the data

27. Let $ABCD$ be a trapezium, in which AB is parallel to CD, $AB = 11$, $BC = 4$, $CD = 6$, and $DA = 3$. The distance between AB and CD is: **[KVPY-2016]**

(A) 2
(B) 2.4
(C) 2.8
(D) Not determinable with the data

28. Let S_1 be the sum of areas of the squares whose sides are parallel to coordinate axes. Let S_2 be the sum of areas of the slanted squares as shown in the figure. Then, $\frac{S_1}{S_2}$ is equal to **[KVPY-2016]**



(A) 2 **(B)** $\sqrt{2}$

(C) 1 **(D)** $\frac{1}{\sqrt{2}}$

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29. A quadrilateral has distinct integer side lengths. If the second-largest side has length 10, then the maximum possible length of the largest side is: **[KVPY-2017]**

(A) 25 **(B)** 26
(C) 27 **(D)** 28

30. Let ABC be a triangle with $\angle C = 90°$. Draw CD perpendicular to AB. Choose points M and N on sides AC and BC, respectively, such that DM is parallel to BC and DN is parallel to AC. If $DM = 5$, $DN = 4$, then AC and BC are, respectively, equal to: **[KVPY-2017]**

(A) $\frac{41}{4}, \frac{41}{5}$ **(B)** $\frac{39}{4}, \frac{39}{5}$
(C) $\frac{38}{4}, \frac{38}{5}$ **(D)** $\frac{37}{4}, \frac{37}{5}$

31. Let $ABCD$ be a square and E be a point outside $ABCD$ such that E, A, C are collinear in that order. Suppose $EB = ED = \sqrt{130}$ and the areas to ΔEAB and square $ABCD$ are equal. Then, the area of square $ABCD$ is **[KVPY-2017]**

(A) 8 **(B)** 10
(C) $\sqrt{120}$ **(D)** $\sqrt{125}$

32. Suppose BC is a given line segment in the plane and T is a scalene triangle. The number of points A in the plane such that the triangle with vertices A, B, C (in same order) is similar to triangle T is **[KVPY-2017]**

(A) 4 **(B)** 6
(C) 12 **(D)** 24

33. Let ABC be a triangle and M be a point on side AC closer to vertex C than A. Let N be a point on side AB such that MN is parallel to BC and let P be a point on side BC such that MP is parallel to AB. If the area of the quadrilateral $BNMP$ is equal to $\frac{5}{18}$ of the area of ΔABC, then the ration AM/MC equals **[KVPY-2017]**

(A) 5 **(B)** 6
(C) $\frac{18}{5}$ **(D)** $\frac{15}{2}$

34. Consider four triangles having sides (5, 12, 9), (5, 12, 11), (5, 12, 13), and (5, 12, 15). Among these, the triangle having maximum area has sides. **[KVPY-2017]**

(A) (5, 12, 9) **(B)** (5, 12, 11)
(C) (5, 12, 13) **(D)** (5, 12, 15)

35. Let $ABCD$ be a trapezium with parallel sides AB and CD such that the circle S with AB as its diameter touches CD. Further, the circle S passes through the mid-points of the diagonals AC and BD of the trapezium. The smallest angle of the trapezium is: **[KVPY-2017]**

(A) $\frac{\pi}{3}$ **(B)** $\frac{\pi}{4}$
(C) $\frac{\pi}{5}$ **(D)** $\frac{\pi}{6}$

36. Consider a ΔPQR in which the relation $QR^2 + PR^2 = 5\ PQ^2$ holds. Let G be the point of intersection of medians PM and QN. Then, $\angle QGM$ is always **[KVPY-2018]**

(A) less than 45°
(B) obtuse
(C) a right angle
(D) acute and larger than 45°

37. Let a, b, c be the side-lengths of a triangle and l, m, n be the lengths of its medians. Put $K = \frac{l+m+n}{a+b+c}$. Then, as a, b, c vary, K can assume every value in the interval: **[KVPY-2018]**

(A) $\left(\frac{1}{4}, \frac{2}{3}\right)$ **(B)** $\left(\frac{1}{2}, \frac{4}{5}\right)$
(C) $\left(\frac{3}{4}, 1\right)$ **(D)** $\left(\frac{4}{5}, \frac{5}{4}\right)$

38. Let PQR be a triangle is which $PQ = 3$. From the vertex R, draw the altitude RS to meet PQ at S. Assume that $RS = \sqrt{3}$ and $PS = QR$ Then, PR equals: **[KVPY-2018]**

(A) $\sqrt{5}$ **(B)** $\sqrt{6}$
(C) $\sqrt{7}$ **(D)** $\sqrt{8}$

39. Let PQR be an acute-angled triangle in which $PQ < QR$ from the vertex Q draw the altitude QQ_1, the angle bisector QQ_2 and the median QQ_3, with Q_1, Q_2, Q_3 lying on PR. Then, **[KVPY-2018]**

(A) $PQ_1 < PQ_2 < PQ_3$ **(B)** $PQ_2 < PQ_1 < PQ_3$
(C) $PQ_1 < PQ_3 < PQ_2$ **(D)** $PQ_3 < PQ_1 < PQ_2$

40. All the vertices of a rectangle are of the form (a, b) with a, b integers satisfying the equation $(a-8)^2 - (b-7)^2 = 5$. Then, the perimeter of the rectangle is **[KVPY-2018]**

(A) 20 **(B)** 22
(C) 24 **(D)** 26

41. Suppose $ABCD$ is a trapezium whose sides and height are integers and AB is parallel to CD. If the area

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of $ABCD$ is 12 and the sides are distinct, then $|AB - CD|$ **[KVPY-2018]**

(A) is 2
(B) is 4
(C) is 8
(D) Cannot be determined from the data

42. In a triangle ABC, $\angle BAC = 90°$; AD is the altitude from A on to BC. Draw DE perpendicular to AC and DF perpendicular to AB. Suppose $AB = 15$ and $BC = 25$. Then length of EF is: **[KVPY – 2019]**

(A) 12 **(B)** 10
(C) $5\sqrt{3}$ 770 **(D)** $5\sqrt{5}$

43. Let P be an interior point of a convex quadrilateral $ABCD$ and K, L, M, N be the mid-points of AB, BC, CD, DA, respectively. If Area $(PKAN) = 25$, Area $(PLBK) = 36$, and Area $(PMDN) = 41$, then Area $(PLCM)$ is: **[KVPY – 2019]**

(A) 20 **(B)** 29
(C) 52 **(D)** 54

PART – III Previous Years' PRMO Problems

1. A triangle with perimeter 7 has integer-side length. What is the maximum possible area of such a triangle? **[PRE-RMO-2012]**

2. In ΔABC, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of correct letters? **[PRE-RMO-2012]**

3. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are online segment CD so that $DF = 1$ and $GC = 2$. Line AF and BG intersect at E. What is the area of ΔAEB? **[PRE-RMO-2012]**

4. $ABCD$ is a square and $AB = 1$. Equilateral triangle AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY? **[PRE-RMO-2012]**

5. Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P and Q be the mid-points of the diagonals AC and BD. If $AD = 16$ and $BC = 20$, what is the length of PQ? **[PRE-RMO-2013]**

6. Let ABC be an equilateral triangle. Let P and S be point on AB and AC, respectively, and let Q and R be points on BC such that $PQRS$ is a rectangle. If $PQ = \sqrt{3}\ PS$ and the area of $PQRS$ is $28\sqrt{3}$, what is the length of PC? **[PRE-RMO-2013]**

7. Let A_1, B_1, C_1, D_1 be the mid-points of the sides of a convex quadrilateral $ABCD$ and let A_2, B_2, C_2, D_2 be the mid-points of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$? **[PRE-RMO-2013]**

8. In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is 13 : 18 and $BC = 12$ then what is the length of AC? **[PRE-RMO-2013]**

9. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle? **[PRE-RMO-2014]**

10. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$. If the area of triangle AXY is 10 then what is the area of triangle ABC? **[PRE-RMO-2014]**

11. Let XOY be a triangle with $\angle XOY = 90°$. Let M and N be the mid-points of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY? **[PRE-RMO-2014]**

12. A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any interior point, and the side of the square are parallel to thi sides of the two given rectangle. What is the smallest possible area of the square? **[PRE-RMO-2015]**

13. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one sides has length 12? **[PRE-RMO-2015]**

14. In a rectangle $ABCD$, E is the mid-point of AB, F is a point on AC such that BF is perpendicular to AC and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find DB. **[PRE-RMO-2015]**

15. Suppose the altitudes of a triangle are 10, 12 and 15. What is the semi-perimeter? **[PRE-RMO-2015]**

16. What is the number of triples (a, b, c) of positive such that (i) $a < b < c < 10$ and (ii) a, b, c, 10 from the sides of a quadrilateral? **[PRE-RMO-2017]**

17. Let P be an interior point of a triangle ABC whose side lengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through

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P parallel to CA meets BC in M and BA in N The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length. **[PRE-RMO-2017]**

18. Let $ABCD$ be a rectangle and let E and F be points on CD and BC, respectively, such that area $(ADE) = 16$, area$(CEF) = 9$ and area$(ABF) = 25$. What is the area of triangle AEF? **[PRE-RMO-2017]**

19. Consider the areas of the four triangle obtained by drawing the diagonals AC and BD of trapezium $ABCD$. The product of these areas, takes two at a time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square roots of the maximum possible area of the trapezium of the nearest integer. **[PRE-RMO-2017]**

20. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$. **[PRE-RMO-2018]**

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2, and G_3 be the centroids of the triangles HBC, HCA, and HAB, respectively. If the area of triangle $G_1G_2G_3$ is units, what is the area of triangle ABC? **[PRE-RMO-2018]**

22. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If $AD = 7$ and $BE = 9$, find the integer nearest to the area of triangle ABC. **[PRE-RMO-2019]**

23. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer? **[PRE-RMO-2019]**

24. Let ABC be an acute-angled triangle with $AB = 15$ and $BC = 8$. Let D be a point on AB such that $BD = BC$. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible value of AE, find [a] the integer part of α. **[PRE-RMO-2019]**

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ANSWER KEYS

Know Your Progress - 1

1. Euclid II[nd] axiom, a = 40 **3.** 126° **4.** 54° **5.** (i) infinite (ii) one
7. 79°12′ **8.** $\angle COD = 80°$ **9.** 54° **10.** 260°

Know Your Progress - 2

1. 20 **2.** $x = 45°$, $y = 26°$ **6.** Infinite

Know Your Progress - 3

1. 30 **3.** $PN = 15$ cm, $RM = 10.67$ cm **7.** C **8.** A **9.** C **10.** B **11.** A **12.** C
13. C **14.** B **15.** C

Know Your Progress - 4

3. 5 cm **8.** D **9.** D **10.** B

Know Your Progress - 5

8. 1 : 2 **9.** B **10.** B

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. B	**2.** B	**3.** D	**4.** C	**5.** A	**6.** C	**7.** B	**8.** B	**9.** B	**10.** A
11. B	**12.** C	**13.** B	**14.** B	**15.** B	**16.** C	**17.** C	**18.** B	**19.** B	**20.** C
21. A	**22.** C	**23.** C	**24.** D	**25.** B	**26.** A	**27.** B	**28.** A	**29.** B	**30.** C

PART – II KVPY Level Problems

1. C	**2.** C	**3.** B	**4.** B	**5.** B	**6.** C	**7.** C	**8.** D	**9.** C	**10.** A
11. A	**12.** B	**13.** A	**14.** A	**15.** A	**16.** C	**17.** D	**18.** B	**19.** C	**20.** C
21. A	**22.** C	**23.** A	**24.** D	**25.** B	**26.** C	**27.** A	**28.** A	**29.** C	**30.** B
31. A	**32.** B	**33.** C	**34.** D	**35.** A					

PART – III PRMO Level Problems

1. 24	**2.** 20	**3.** 15	**4.** 77	**5.** 52	**6.** 20	**7.** 18	**8.** 14	**9.** 2	**10.** 12
11. 80	**12.** 37	**13.** 34	**14.** 36	**15.** 21	**16.** 10	**17.** 48	**18.** 0	**19.** 8	**20.** 27
21. 1	**22.** 24	**23.** 21	**24.** 27	**25.** 2	**26.** 3	**27.** 3	**28.** 1	**29.** 12	**30.** 56
31. 24	**32.** 48	**33.** 2	**34.** 43	**35.** 54	**36.** 4	**37.** 27	**38.** 5	**39.** 10	**40.** 36
41. 40	**42.** 60	**43.** 15	**44.** 13	**45.** 20	**46.** 75	**47.** 72	**48.** 54	**49.** 37	**50.** 75
51. 16	**52.** 25	**53.** 50	**54.** 16	**55.** 15	**56.** 31	**57.** 41	**58.** 23	**59.** 24	**60.** 4
61. 7	**62.** 34	**63.** 81	**64.** 1	**65.** 45	**66.** 24				

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. C	**2.** D	**3.** D	**4.** B	**5.** C	**6.** C	**7.** B	**8.** C	**9.** B	**10.** B
11. B	**12.** A	**13.** C	**14.** A	**15.** C	**16.** BONUS (> 0.5)		**17.** A	**18.** A	
19. BONUS ($DB > BC + CD$)			**20.** C	**21.** B	**22.** C	**23.** A	**24.** A	**25.** A	**26.** A
27. B	**28.** C	**29.** D	**30.** C						

PART – II Previous Years' KVPY Problems

1. A	**2.** A	**3.** B	**4.** C	**5.** B	**6.** B	**7.** D	**8.** D	**9.** D	**10.** B
11. B	**12.** C	**13.** C	**14.** A	**15.** C	**16.** D	**17.** C	**18.** B	**19.** A	**20.** C
21. D	**22.** C	**23.** B	**24.** B	**25.** D	**26.** B	**27.** B	**28.** A	**29.** B	**30.** A
31. B	**32.** C	**33.** A	**34.** C	**35.** D	**36.** C	**37.** C	**38.** C	**39.** A	**40.** A

PART – III Previous Years' PRMO Problems

1. 15	**2.** 3	**3.** 12.5	**5.** 2	**6.** 14	**7.** 208	**8.** 36	**9.** 49	**10.** 45	**11.** 26
12. 25	**13.** 84	**14.** 24	**15.** Discounted		**16.** 73	**17.** Discounted		**18.** 30	**19.** 13
20. 24	**21.** 63	**22.** 47	**23.** 90	**24.** Bonus					

Discounted: maximum marks were reduced

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Chapter 7 Plane Geometry-II

Learning Outcomes

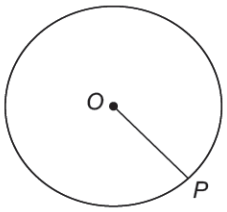
After reading this chapter, you will be able to understand

- Circle
- Secant and Tangent
- Area Related to Circle
- Surface Area and Volume

Circle

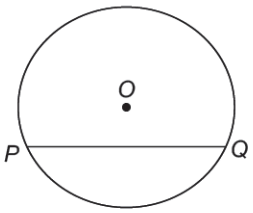
Definitions

(a) Circle: The collection of all the points in a plane, which are at a fixed distance from a fixed point in the plane, is called a circle.
The fixed point is called the centre of the circle and the fixed distance is called the radius of the circle.

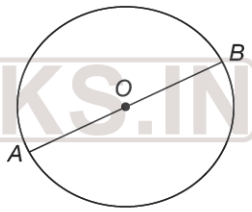


In figure, O is the centre and the length OP is the radius of the circle. So the line segment joining the centre and any point on the circle is called a radius of the circle.

(b) Chord: If we take two points P and Q on a circle, then the line segment PQ is called a chord of the circle.

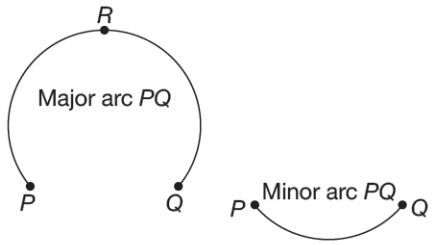


(c) Diameter: The chord that passes through the centre of the circle, is called the diameter of the circle.



A diameter is the longest chord and all diameters of same circle have the same length, which is equal to two times the radius. In figure, AOB is a diameter of circle.

(d) Arc: A piece of a circle between two points is called an arc. The longer one is called the major arc PQ and the shorter one is called the minor arc PQ. The minor arc PQ is also denoted by $\overset{\frown}{PQ}$ and the major arc $\overset{\frown}{QP}$ by QP. When P and Q are ends of a diameter, then both arcs are equal and each is called a semi-circle.



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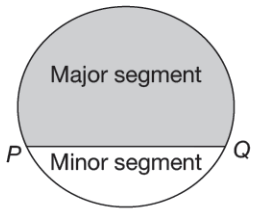
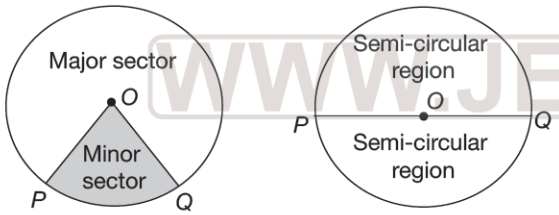
(e) **Circumference:** The length of the complete circle is called its circumference.

(f) **Segment:** The region between a chord and either of its arcs is called a segment of the circular region or simply a segment of the circle. There are two types of segments that are the major segment and the minor segment (as in figure).



(g) **Sector:** The region between an arc and the two radii joining the centre to the end points of an arc is called a sector. Minor arc corresponds to the minor sector and the major arc corresponds to the major sector. When two arcs are equal, then both segments and both sectors become the same and each is known as a semi-circular region.



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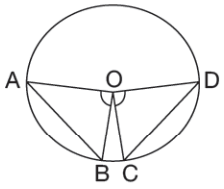
Theorem Related to Chords

Theorem: Equal chords of a circle subtend equal angles at the centre.



Given: AB and CD are the two equal chords of a circle with centre O.

To Prove: $\angle AOB = \angle COD$

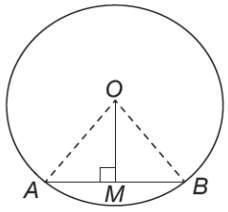
Proof: In ΔAOB and COD,

$OA = OC$	[Radii of a circle]
$OB = OD$	[Radii of a circle]
$AB = CD$	[Given]
$\therefore \Delta AOB \cong \Delta COD$	[By *SSS* congruency]
$\therefore \angle AOB = \angle COD$	[By CPCT]

Hence, Proved.

Converse: If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.

Theorem: The perpendicular from the centre of a circle to a chord bisects the chord.



Given: A circle with centre O. AB is a chord of this circle. $OM \perp AB$.

To Prove: $MA = MB$.

Construction: Join OA and OB.

Proof: In right triangles OMA and OMB,

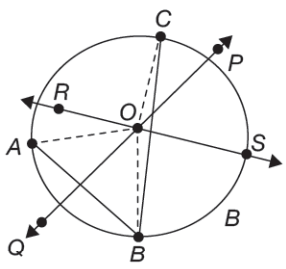
$OA = OB$	[Radii of a circle]
$OM = OM$	[Common]
$\angle OMA = \angle OMB$	[90° each]
$\therefore \Delta OMA \cong \Delta OMB$	[By RHS]
$\therefore MA = MB$	[By CPCT]

Hence, Proved.

Converse: The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

Theorem: There is one and only one circle passing through three given non-collinear points.

Proof: Take three points A, B, and C, which are not in the same line, or in other words, they are not collinear [as in figure]. Draw perpendicular bisectors of AB and BC say, PQ and RS, respectively. Let these perpendicular bisectors intersect at one-point O (Note that PQ and RS will intersect because they are not parallel) [as in figure].



$\because$ O lies on the perpendicular bisector PQ of AB.

$\therefore$ $OA = OB$ [$\because$ Every point on the perpendicular bisector of a line segment is equidistant from its end points]

Similarly,

$\because$ O lies on the perpendicular bisector RS of BC.

$$\therefore OB = OC$$

[$\because$ Every point on the perpendicular bisector of a line segment is equidistant from its end points]

So, $OA = OB = OC$

i.e., the points A, B, and C are at equal distances from the point O.

So, if we draw a circle with centre O and radius OA, it will also pass through B and C. This shows that there is a circle passing through the three points A, B, and C. We know that two lines (perpendicular bisectors) can intersect at only one point, so we can draw only one circle with radius OA. In other words, there is a unique circle passing through A, B, and C.

Hence, Proved.

Theorem: Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).

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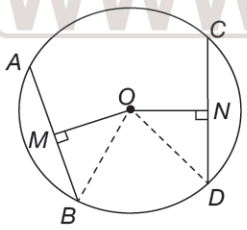


Given: A circle have two equal chords AB and CD. i.e., $AB = CD$ and $OM \perp AB$, $ON \perp CD$.

To Prove: $OM = ON$

Construction: Join OB and OD

Proof: $AB = CD$ (Given)

[$\because$ The perpendicular drawn from the centre of a circle to a chord bisects the chord]

$$\therefore \frac{1}{2}AB = \frac{1}{2}CD \Rightarrow BM = DN$$

In ΔOMB and ΔOND

$\angle OMB = \angle OND = 90°$ [Given]

$OB = OD$ [Radii of same circle]

$BM = DN$ [Proved above]

$\therefore \Delta OMB \cong \Delta OND$ [By R.H.S. congruency]

$\therefore OM = ON$ [By CPCT]

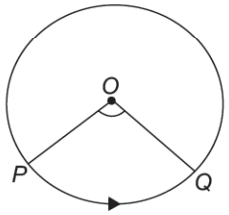
Hence, Proved.

Remark

Chords equidistant from the centre of a circle are equal in length.

Angle Subtended by an Arc of a Circle

In figure, the angle subtended by the minor arc PQ at O is $\angle POQ$ and the angle subtended by the major arc PQ at O is reflex angle $\angle POQ$.

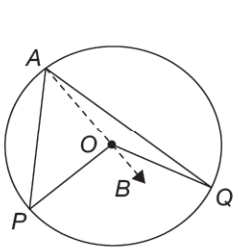


Theorem: The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

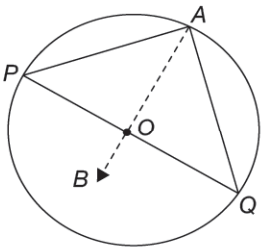
Given: An arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle.

To Prove: $\angle POQ = 2\angle PAQ$.

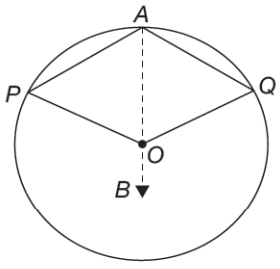
Construction: Join AO and extend it to a point B.



(A)



(B)



(C)

Proof: There arise three cases:-

(A) arc PQ is minor
(B) arc PQ is a semi-circle
(C) arc PQ is major.

In all the cases,

$$\angle BOQ = \angle OAQ + \angle AQO \quad \ldots(i)$$

[$\because$ An exterior angle of a triangle is equal to the sum of the two interior opposite angles]

In ΔOAQ,

$$OA = OQ \quad \text{[Radii of a circle]}$$

$$\therefore \angle OQA = \angle OAQ \quad \ldots(ii)$$

[Angles opposite equal sides of a triangle are equal]

From (i) and (ii)

$$\angle BOQ = 2\angle OAQ \quad \ldots(iii)$$

Similarly,

$$\angle BOP = 2\angle OAP \quad \ldots(iv)$$

Adding (iii) and (iv), we get

$$\angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$$

$$\Rightarrow \angle POQ = 2\angle PAQ \quad \ldots(v)$$

Note

For the case (C), where PQ is the major arc, (v) is replaced by reflex angles.

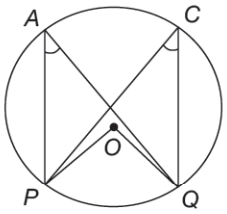
Thus, reflex $\angle POQ = 2\angle PAQ$.

Theorem: Angles in the same segment of a circle are equal.

Proof: Let P and Q be any two points on a circle to form a chord PQ, A, and C any other points on the remaining part of the circle and O be the centre of the circle. Then,

$$\angle POQ = 2\angle PAQ \quad \ldots(i)$$

$$\text{And } \angle POQ = 2\angle PCQ \quad \ldots(ii)$$



[Angle subtended at the centre is double than the angle subtended by it on the remaining part of the circle]

From equation (i) and (ii)

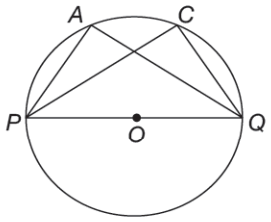
$$2\angle PAQ = 2\angle PCQ$$

$$\Rightarrow \angle PAQ = \angle PCQ.$$

Hence, Proved.

Theorem: Angle in the semi-circle is a right angle.

Proof: $\because$ $\angle PAQ$ is an angle in the segment, which is a semi-circle.



$$\therefore \angle PAQ = \frac{1}{2}\angle POQ = \frac{1}{2} \times 180° = 90°$$

[$\because$ POQ is straight line angle or $\angle POQ = 180°$]

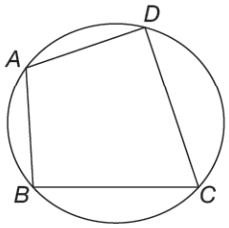
If we take any other point C on the semi-circle, then again we get

$$\angle PCQ = \frac{1}{2}\angle POQ = \frac{1}{2} \times 180° = 90°.$$

Hence, Proved.

Cyclic Quadrilateral

A quadrilateral $ABCD$ is called cyclic if all the four vertices of it lie on a circle.



Theorem: The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.

Given: A cyclic quadrilateral $ABCD$.

To Prove: $\angle A + \angle C = \angle B + \angle D = 180°$.

Construction: Join AC and BD.

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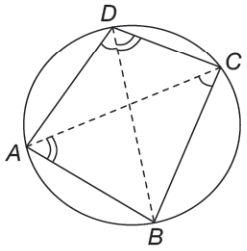
Proof:

$$\angle ACB = \angle ADB \quad \ldots(i)$$



And $\angle BAC = \angle BDC \quad \ldots(ii)$

[Angles of same segment of a circle are equal]
Adding equation (i) and (ii)

$$\therefore \angle ACB + \angle BAC = \angle ADB + \angle BDC$$

$$\angle ACB + \angle BAC = \angle ADC.$$

Adding $\angle ABC$ to both sides, we get

$$\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC.$$

$$\therefore \angle ADC + \angle ABC = 180°$$

$$\text{i.e., } \angle D + \angle B = 180°$$

$$\therefore \angle A + \angle C = 360° - (\angle B + \angle D) = 180°$$

[$\because \angle A + \angle B + \angle C + \angle D = 360°$]

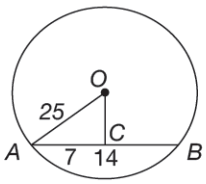
Hence, Proved.

Example 1

The radius of a circle is 25 cm and the length of one of its chords is 14 cm. Find the distance of the chord from the centre.

Solution

Let O be the center of the circle of radius 25 cm and AB is the chord of length 14 cm



$$OC \perp AB$$

$$AC = \frac{AB}{2} = \frac{14}{2} = 7$$

[Line perpendicular from centre to chord bisect the chord]

In ΔAOC

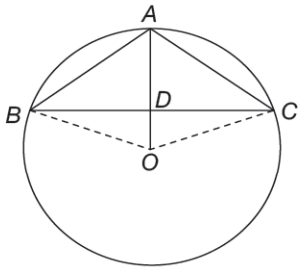
$$(OC)^2 + (AC)^2 = (AO)^2$$

$$(OC)^2 = (25)^2 - (7)^2$$

$$OC = 24 \text{ cm.}$$

Example 2

In figure, $\overset{\frown}{AB} \cong \overset{\frown}{AC}$ and O is the centre of the circle. Prove that OA is the perpendicular bisector of BC.



Solution

Given: In figure, $\overset{\frown}{AB} \cong \overset{\frown}{AC}$ and O is the centre of the circle.

To Prove: OA is the perpendicular bisector of BC.

Construction: Join OB and OC.

Proof:

$$\because \overset{\frown}{AB} \cong \overset{\frown}{AC} \text{ [Given]}$$

$$\therefore \text{chord } AB = \text{chord } AC.$$

[$\because$ If two arcs of a circle are congruent, then their corresponding chords are equal]

$$\therefore \angle AOB = \angle AOC \quad \ldots(i)$$

[$\because$ Equal chords of a circle subtend equal angles at the centre]

In ΔOBD and ΔOCD,

$$\angle DOB = \angle DOC \quad \text{[From (i)]}$$

$$OB = OC \quad \text{[Radii of the same circle]}$$

$$OD = OD \quad \text{[Common]}$$

$$\therefore \Delta OBD \cong \Delta OCD \quad \text{[By } SAS \text{ congruency]}$$

$$\therefore \angle ODB = \angle ODC \quad \text{[By CPCT]} \quad \ldots(ii)$$

And, $BD = CD \quad \text{[By CPCT]} \quad \ldots(iii)$

But $\angle BDC = 180°$

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$\therefore \angle ODB + \angle ODC = 180°$

$\Rightarrow \angle ODB + \angle ODB = 180°$ [From equation (ii)]

$\Rightarrow 2\angle ODB = 180°$

$\Rightarrow \angle ODB = 90°$

$\therefore \angle ODB = \angle ODC = 90°$ [From (ii)] (iv)

So, by (iii) and (iv), OA is the perpendicular bisector of BC. Hence, Proved.

Example 3

AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E. Prove that $EB = ED$.

Solution

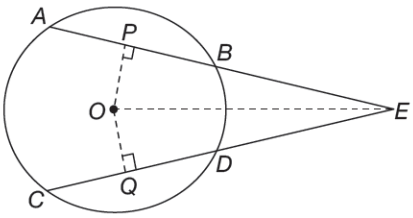
Given: AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E.

To Prove: $EB = ED$.

Construction: From O draw $OP \perp AB$ and $OQ \perp CD$. Join OE.

Proof: $\because AB = CD$ [Given]

$\therefore OP = OQ$ [$\because$ Equal chords of a circle are equidistant from the centre]



Now in ΔOPE and ΔOQE,

$\angle OPE = \angle OQE$ [Each 90°]

$OE = OE$ [Common]

$OP = OQ$ [Proved above]

$\therefore \Delta OPE \cong \Delta OQE$ [By *RHS* congruency]

$\therefore PE = QE$ [By CPCT]

$\Rightarrow PE - \frac{1}{2}AB = QE - \frac{1}{2}CD$

[$\because AB = CD$ (Given)]

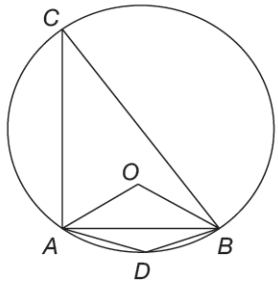
$\Rightarrow PE - PB = QE - QD \Rightarrow EB = ED.$

Hence, Proved.

Example 4

A chord of a circle is equal to the radius of the circle, find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

Solution



Let AB be a chord of the circle with centre at O such that $OA = OB = AB$. (All are equal to radius of a circle)

ΔOAB is equilateral.

$\therefore \angle AOB = 60° \quad \Rightarrow \angle ACB = \frac{1}{2}\angle AOB = 30°$

Consider arc ACB.

Clearly, it makes $360° - 60° = 300°$ at the centre O.

$\angle ADB = \frac{1}{2}$ Reflex $\angle AOB = \frac{1}{2}(300°) = 150°$

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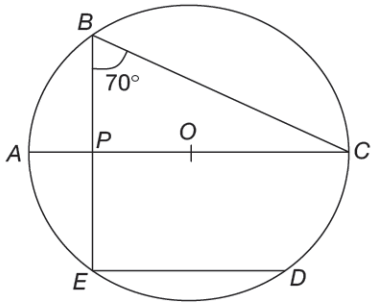
Example 5

In the given figure, the chord ED is parallel to the diameter AC. Find $\angle CED$.

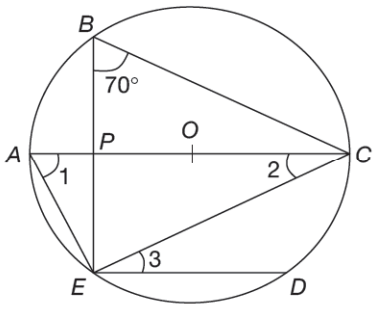


Solution

$\angle CBE = \angle 1$ [Angles in the same segment]

$\angle 1 = 70°$ [$\because \angle CBE = 70°$] ...(i)

$\angle AEC = 90°$ [Angle in a semi-circle is a right angle]...(ii)



Now, in ΔAEC,

$$\angle 1 + \angle AEC + \angle 2 = 180°$$

$$\therefore 70° + 90° + \angle 2 = 180°$$

$$\Rightarrow \angle 2 = 180°$$

$$\Rightarrow 160° = 20°$$

Thus $\angle 2 = 20°$ …(iii)

Also, $ED \,||\, AC$ [Given]

$\therefore \angle 2 = \angle 3$ [Alternate angles]

$\therefore \angle 3 = 20°$

Hence, $\angle CED = 20°$.

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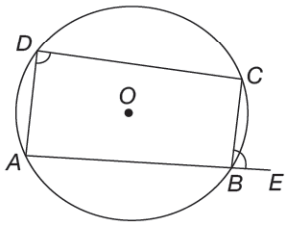
Example 6

If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Solution

Let $ABCD$ be a cyclic quadrilateral inscribed in a circle with centre O. The side AB of quadrilateral $ABCD$ is produced to E. Then, we have to prove that

$$\angle CBE = \angle ADC.$$



Since the sum of opposite pairs of angles of a cyclic quadrilateral is 180°

$$\therefore \angle ABC + \angle ADC = 180°$$

But $\angle ABC + \angle CBE = 180°$

[Q$\angle ABC$ and $\angle CBE$ form a linear pair]

$$\therefore \angle ABC + \angle ADC = \angle ABC + \angle CBE$$

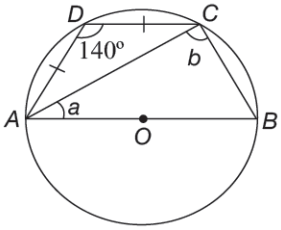
$$\Rightarrow \angle ADC = \angle CBE$$

or $\angle CBE = \angle ADC.$

Hence, Proved

Example 7

In the given figure, $AD = DC$ and O is the centre of circle. Find the value of a and b.



Solution

In ΔADC

$$AD = DC$$

$$\angle DAC = \angle DCA = x \text{ (say)}$$

$$140° + x + x = 180°$$

$$2x = 40° \quad \Rightarrow x = 20°$$

$$\angle ADC + \angle ABC = 180°$$

[Opposite angles in a cyclic quadrilateral are supplementary]

$$140° + \angle ABC = 180°$$

$$\angle ABC = 40°$$

$$\angle b = 90° \quad \text{[Angle in a semi-circle]}$$

$$\angle a = 180° - \angle ABC - \angle b = 180° - 40° - 90°$$

$$\angle a = 180° - 130° = 50°.$$

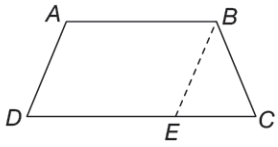
So, $a = 50°$ and $b = 90°$.

Example 8

If the nonparallel side of a trapezium are equal, prove that it is cyclic.

Solution

Given: $ABCD$ is a trapezium whose two non-parallel sides AD and BC are equal.



To Prove: Trapezium $ABCD$ is a cyclic.

Construction: Draw $BE \,||\, AD$.

Proof: $\because AB \,||\, DE$ [Given]

and $AD \,||\, BE$ [By construction]

Quadrilateral $ABED$ is a parallelogram.

$\therefore \angle BAD = \angle BED$ [Opp. angles of a ||gm] ...(i)

And, $AD = BE$ [Opp. sides of a ||gm] ...(ii)

But $AD = BC$ [Given] ...(iii)

From (ii) and (iii),

$$BE = BC$$

$\therefore \angle BCE = \angle BEC$ [Angles opposite to equal sides] ...(iv)

$$\angle BEC + \angle BED = 180^\circ \quad \text{[Linear Pair Axiom]}$$

$$\Rightarrow \angle BCE + \angle BAD = 180^\circ \quad \text{[From (iv) and (i)]}$$

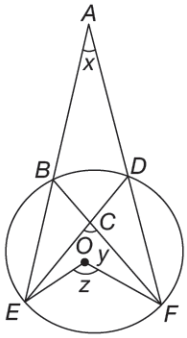
$\Rightarrow$ Trapezium $ABCD$ is cyclic.

[$\because$ If a pair of opposite angles of a quadrilateral is 180°, then the quadrilateral is cyclic]

Hence, Proved.

Example 9

In figure, O is the centre of the circle. Prove that $\angle x + \angle y = \angle z$.



Solution

$$\angle EBF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$$

[$\because$ Angle subtended by an arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]

$$\therefore \angle ABF = 180^\circ - \frac{1}{2}\angle z \text{ [Linear Pair Axiom]} \quad ...(i)$$

$$\angle EDF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$$

[$\because$ Angle subtended by any arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]

$$\therefore \angle ADE = 180^\circ - \frac{1}{2}\angle z \text{ [Linear Pair Axiom]} \quad ...(ii)$$

$$\angle BCD = \angle ECF = \angle y \quad \text{[Vertically Opp. Angles]}$$

$$\angle BAD = \angle x$$

In quadrilateral $ABCD$

$$\angle ABC + \angle BCD + \angle CDA + \angle BAD = 2 \times 180^\circ$$

$$\Rightarrow 180^\circ - \frac{1}{2}\angle z + \angle y + 180^\circ - \frac{1}{2}\angle z + \angle x = 2 \times 180^\circ$$

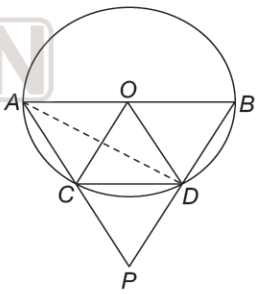
$$\Rightarrow \angle x + \angle y = \angle z.$$

Hence, Proved.

Example 10

AB is a diameter of the circle with centre O and chord CD is equal to radius OC. AC and BD produced meet at P. Prove that $\angle CPD = 60^\circ$.

Solution



Given: AB is a diameter of the circle with centre O and chord CD is equal to radius OC. AC and BD produced meet at P.

To Prove: $\angle CPD = 60^\circ$.

Construction: Join AD

Proof: In ΔOCD,

$$OC = OD \text{ [Radii of the same circle]} \quad ...(i)$$

$$OC = CD \quad \text{[Given]} \quad ...(ii)$$

From (i) and (ii),

$$OC = OD = CD$$

$\therefore \Delta OCD$ is an equilateral triangle.

$$\therefore \angle COD = 60^\circ$$

$$\therefore \angle CAD = \frac{1}{2}\angle COD = \frac{1}{2}\angle(60^\circ) = 30^\circ$$

[$\because$ Angle subtended by any arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]

$$\Rightarrow \angle PAD = 30^\circ \quad ...(iii)$$

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And, $\angle ADB = 90°$ [Angle in a semi-circle]…(iv)

$\Rightarrow \angle ADB + \angle ADP = 180°$ [Linear Pair Axiom]

$\Rightarrow 90° + \angle ADP = 180°$ [From (iv)]

$\Rightarrow \angle ADP = 90°$ …(v)

In ΔADP,

$\angle APD + \angle PAD + \angle ADP = 180°$

$\Rightarrow \angle APD + 30° + 90° = 180°$ [From (iii) and (v)]

$\Rightarrow \angle APD + 120° = 180°$

$\Rightarrow \angle APD = 180° - 120° = 60°$

$\Rightarrow \angle CPD = 60°$.

Hence, Proved.

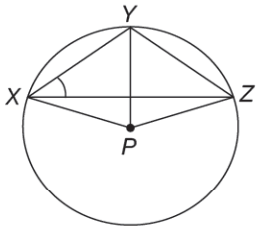
Know Your Progress - 1

1. A circular park of radius 20 m is situated in a colony. Three boys Ankur, Syed, and David are sitting at equal distances on its boundary each having a toy telephone in his hands to talk to each other. Find the length of the string of each phone.

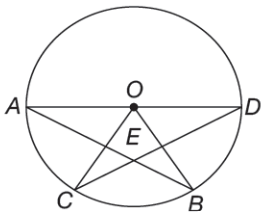
2. AB and CD are two parallel chords of a circle such that $AB = 10$ cm and $CD = 24$ cm. If the chords are on the opposite side of the centre and the distance between them is 17 cm, find the radius of the circle.

3. In a circle of radius 5 cm, AB and AC are two chords such that $AB = AC = 6$ cm. Find the length of the chord BC.

4. In the given figure, P is the centre of the circle. Prove that: $\angle XPZ = 2\,(\angle XZY + \angle YXZ)$.

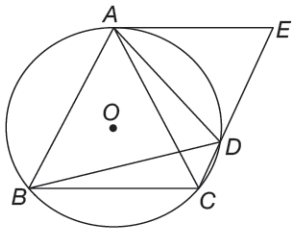


5. In a circle with centre O, chords AB and CD intersects inside the circumference at E. Prove that $\angle AOC + \angle BOD = 2\angle AEC$.

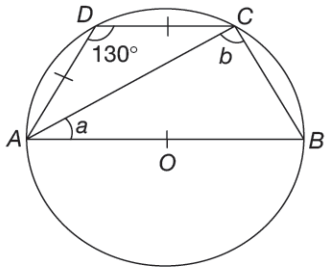


6. Bisectors of angles A, B, and C of a triangle ABC intersect its circumcircle at D, E, and F, respectively. Prove that the angles of the triangle DEF are $90° - \frac{\angle A}{2}$, $90° - \frac{\angle B}{2}$ and $90° - \frac{\angle C}{2}$.

7. D is a point on the circumcircle of ΔABC in which $AB = AC$ such that B and D are on the opposite side of line AC. If CD is produced to a point E such that $CE = BD$, prove that $AD = AE$.



8. In figure, O is the centre of the circle. Find the value of a and b.



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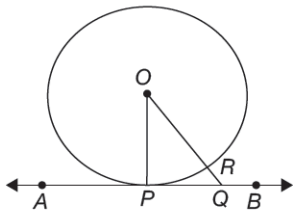
Secant and Tangent

Secant to a circle is a line that intersects the circle in two distinct points.

A tangent to a circle is a line that intersects the circle in exactly one point.

Theorem: A tangent to a circle is perpendicular to the radius through the point of contact.



Given: A circle $C(O, r)$ and a tangent AB at a point P.

To Prove: $OP \perp AB$

Construction: Take any point Q, other than P on the tangent AB. Join OQ. Suppose OQ meets the circle at R.

Proof: Among all line segments joining the point O to a point on AB, the shortest one is perpendicular to AB. So, to prove that $OP \perp AB$, it is sufficient to prove that OP is shorter than any other segment joining O to any point of AB.

Clearly $OP = OR$ (Radius)

$$\text{Now, } OQ = OR + RQ$$

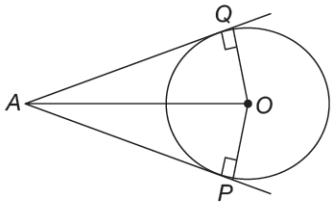
$$\Rightarrow OQ > OR$$

$$\Rightarrow OQ > OP \qquad (\because OP = OR)$$

Thus, OP is shorter than any other segment joining O to any point of AB.

Hence, $OP \perp AB$.

Theorem: Lengths of two tangents drawn from an external point to a circle are equal.



Given: AP and AQ are two tangents drawn from a point A to a circle C (O, r).

To Prove: $AP = AQ$

Construction: Join OP, OQ, and OA.

Proof: In ΔAOQ and ΔAPO

$$\angle OQA = \angle OPA$$

[Tangent at any point of a circle is perp. to radius through the point of contact]

$$AO = AO \qquad \text{[Common]}$$

$$OQ = OP \qquad \text{[Radius]}$$

So, by R.H.S. criterion of congruency $\Delta AOQ \cong \Delta AOP$

$$\therefore AQ = AP \qquad \text{[By CPCT]}$$

Hence, Proved.

Results

(i) If two tangents are drawn to a circle from an external point, then they subtend equal angles at the centre.

$$\angle QOA = \angle POA \qquad \text{[By CPCT]}$$

(ii) If two tangents are drawn to a circle from an external point, they are equally inclined to the segment, joining the centre to that point

$$\angle OAQ = \angle OAP \qquad \text{[By CPCT]}$$

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Segment of a Chord

Let AB be a chord of a circle and let P be a point on AB inside the circle. Then, P is said to divide AB internally into two segments PA and PB.

Theorem: If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by two segments of one chord is equal in area to the rectangle formed by the two segments of another chord.

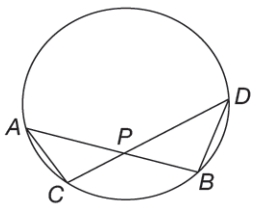


Fig. (i)

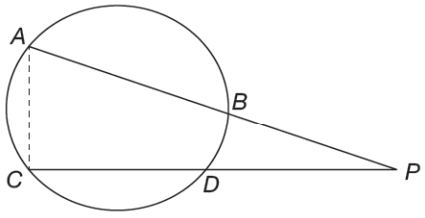


Fig. (ii)

Given: Two chords AB and CD of a circle $C(O, r)$ intersecting at P, inside in fig. (i) and outside in fig.(ii).

To Prove: $PA \cdot PB = PC \cdot PD$

Construction: Join AC and BD

Proof:

Case 1: When AB and CD intersect at P inside the circle.

In Δs APC and BPD,

$\angle APC = \angle BPD$ [Vertically opposite angles]

$\angle PAC = \angle PDB$ [Angles in the same segment]

$\therefore \Delta APC \sim \Delta BPD$

Hence, $\frac{PA}{PD} = \frac{PC}{PB} \Rightarrow PA \times PB = PC \times PD$

Case 2: When AB and CD when produced intersect at P outside the circle.

In ΔPBD and ΔPAC, $\angle PBD = \angle ACP$

[$ABCD$ is a cyclic quadrilateral and exterior angle is equal to the interior opposite angle]

$\angle PDB = \angle CAP$

[$ABCD$ is a cyclic quadrilateral and exterior angle is equal to the interior opposite angle]

$\therefore \Delta PBD \sim \Delta PCA$

Hence, $\frac{PB}{PC} = \frac{PD}{PA} \Rightarrow PA \times PB = PC \times PD$

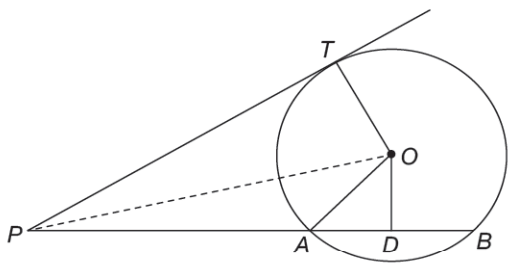
Theorem: If PAB is a secant to a circle intersecting the circle at A and B and PT is a tangent segment, then

$PA \times PB = PT^2$.

Given: A circle with centre O; PAB is a secant intersecting the circle at A and B and PT is a tangent segment to the circle.

To Prove: $PA \times PB = PT^2$.

Construction: Join OA, OP, and OT. Draw $OD \perp AB$.



Proof: Since $OD \perp$ chord AB

$\therefore AD = DB$.

Now $PA \times PB = (PD - AD)(PD + DB)$

$= (PD - AD)(PD + AD)$ [$\because AD = DB$]

$= PD^2 - AD^2$

$= (OP^2 - OD^2) - AD^2$

[From right-angled ΔODP]

$= OP^2 - (OD^2 + AD^2)$

$= OP^2 - OA^2$ [From right-angled ΔODA]

$= OP^2 - OT^2$ $\because OA = OT$ [Radii]

$= PT^2$ [From right-angled ΔOTP]

Hence, $PA \times PB = PT^2$.

Angle in the Alternate Segments

Let PAQ be a tangent to a circle at point A and AB be a chord. Then, the segment opposite to the angle formed by the chord of a circle with the tangent at a point is called the alternate segment for that angle.

Theorem: A line touches a circle and from the point of contact a chord is drawn. Prove that the angles which the chord makes with the given line are equal, respectively, to angles formed in the corresponding alternate segments.

Given: A circle with centre O and PAT is a tangent to the circle at A. A chord AB is drawn. Let C and D be the points on the circumference on opposite sides of AB.

To Prove: $\angle BAT = \angle ACB$ and $\angle BAP = \angle ADB$.

Construction: Draw the diameter AOE and join EB.

Proof: Since AE is the diameter,

$\therefore \angle ABE = 90°$ [Angle in semi-circle]

$\therefore \angle BEA + \angle EAB = 90°$ [Remaining $\angle s$ of ΔBEA] ...(i)

But since $EA \perp AT$ [$\because$ Radius is $\perp$ to the tangent]

$\therefore \angle EAB + \angle BAT = 90°$ ...(ii)

Hence from (i) and (ii),

$\angle BEA + \angle EAB = \angle EAB + \angle BAT$ ...(iii)

$\Rightarrow \angle BEA = \angle BAT$

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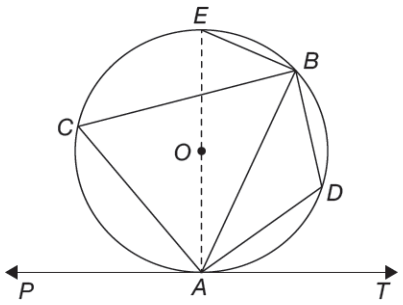
But $\angle BEA = \angle ACB$ [Angles in the same segment]

$$\angle ACB = \angle BAT$$

i.e. $\angle BAT = \angle ACB.$



This proves the first part.

Again, since $ADBC$ is a cyclic quadrilateral.

$$\therefore \angle ACB + \angle ADB = 180°$$

Also $\angle BAT + \angle BAP = 180°$ [Linear pair]

$$\therefore \angle ACB + \angle ADB = \angle BAT + \angle BAP$$

$\Rightarrow \angle BAT + \angle ADB = \angle BAT + \angle BAP$ [$\because \angle ACB = \angle BAT$]

$$\Rightarrow \angle ADB = \angle BAP$$

$$\therefore \angle BAP = \angle ADB$$

Hence, $\angle BAT = \angle ACB$ and $\angle BAP = \angle ADB$.

Common Tangents to Two Circles

Definition

A line which touches the two given circles is called common tangent to the two circles. Let $C(O_1, r_1)$, $C(O_2, r_2)$ be two given circles. Let the distance between centres O_1 and O_2 be d i.e., $O_1O_2 = d$.

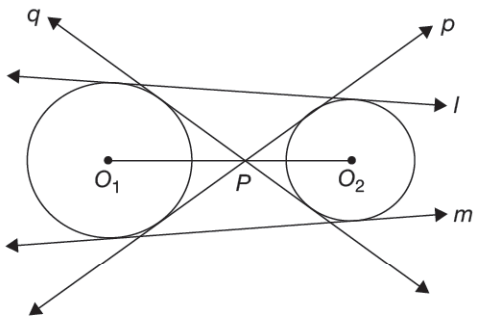


Fig. (i)

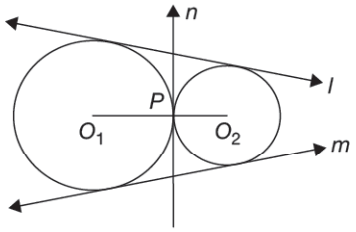


Fig. (ii)

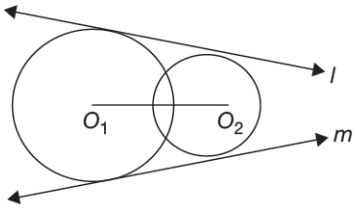


Fig. (iii)

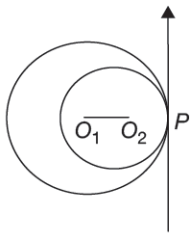


Fig. (iv)

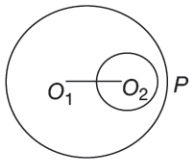


Fig. (v)

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(a) In fig. (i) $d > r_1 + r_2$ i.e., two circles do not intersect. In this case, four common tangents are possible. The tangent lines l and m are called direct common tangents and the tangent lines p and q are called indirect (transverse) common tangents.

(b) In fig. (ii), $d = r_1 + r_2$. In this case, two circles touch externally and there are three common tangents.

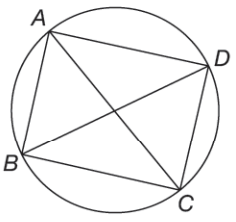
(c) In fig.(iii), $d < r_1 + r_2$. In this case two circles intersect in two distinct points and there are only two common tangents.

(d) In fig. (iv), $d = r_1 - r_2$ $(r_1 > r_2)$, in this case, two circles touch internally and there is only one common tangent.

(e) In fig. (v), the circle $C(O_2, r_2)$ lies wholly in the circle $C(O_1, r_1)$ and there is no common tangent.

Ptolemy's Theorem

In cyclic quadrilateral, the product of the diagonals is equal to the sum of the product of the opposite sides.



$$AC \times BD = AB \times DC + BC \times AD$$

Polygon

A closed plane figure bounded by line segments is called a polygon.

A polygon is named according to the number of sides it has:

No. of sides	3	4	5	6	7	8	10
Figure	Triangle	Quadrilateral	Pentagon	Hexagon	Heptagon	Octagon	Decagon

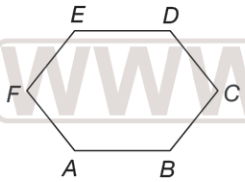
In general, a polygon having n sides is called 'n' sided polygon.

Diagonal of Polygon

Line segment joining any two non–consecutive vertices of a polygon is called its diagonal.

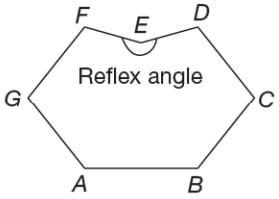
Convex Polygon

If all the interior angles of a polygon are less than 1800, it is called a convex polygon.



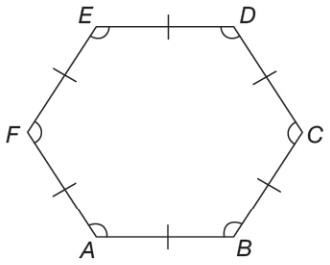
Concave Polygon

If one or more of the interior angles of a polygon is greater than 1800 i.e., reflex, it is called a concave polygon.



Regular Polygon

A polygon is called a regular polygon if all its sides have equal length and all the angles have equal measure.



Remarks

1. The sum of the interior angles of a convex polygon of n sides is $(2n - 4)$ right angles or $(2n - 4)$ 90°.
2. The sum of the exterior angles of a convex polygon is 4 right angles or 360°.
3. Each interior angle of a n-sided regular polygon is $\frac{(2n-4)\times 90^\circ}{n}$
4. Each exterior angle of a regular polygon of n sides $= \left(\frac{360^\circ}{n}\right)$.
5. If a polygon has n sides, then the number of diagonals of the polygon $= \frac{n(n-3)}{2}$.

Example 11

If all the sides of a parallelogram touches a circle, show that the parallelogram is a rhombus.

Solution

Given: Sides AB, BC, CD, and DA of a $||^{gm}$ $ABCD$ touch a circle at P, Q, R, and S, respectively.

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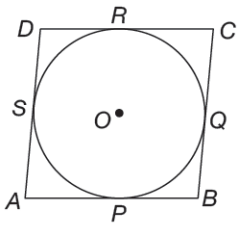
To Prove: $||^{gm}$ $ABCD$ is a rhombus.



Proof: $AP = AS$ …(i)

$BP = BQ$ …(ii)

$CR = CQ$ …(iii)

$DR = DS$ …(iv)

[Tangents drawn from an external point to a circle are equal]

Adding (1), (2), (3), and (4), we get

$\Rightarrow AP + BP + CR + DR = AS + BQ + CQ + DS$

$\Rightarrow (AP + BP) + (CR + DR) = (AS + DS) + (BQ + CQ)$

$\Rightarrow AB + CD = AD + BC$

$\Rightarrow AB + AB = AD + AD$

[In a $||^{gm}$ $ABCD$, opposite sides are equal]

$\Rightarrow 2AB = 2AD$ or $AB = AD$

But $AB = CD$ and $AD = BC$

[Opposite sides of a $||^{gm}$]

$\therefore AB = BC = CD = DA$

Hence, $||^{gm}$ $ABCD$ is a rhombus.

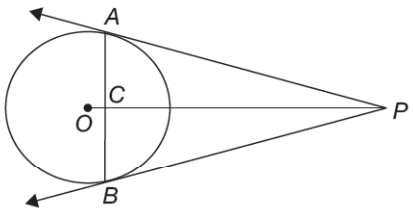
Example 12

Prove that the tangents at the extremities of any chord make equal angles with the chord.

Solution

Let AB be a chord of a circle with centre O, and let AP and BP be the tangents at A and B, respectively.

Suppose, the tangents meet at point P. Join OP. Suppose OP meets AB at C.



We have to prove that $\angle PAC = \angle PBC$

In triangles PCA and PCB,

$PA = PB$ [$\because$ Tangent from an external point are equal]

$\angle APC = \angle BPC$ [$\because$ PA and PB are equally inclined to OP]

And $PC = PC$ [Common]

So, by SAS criteria of congruence

$\Delta PAC \cong \Delta PBC$

$\Rightarrow \angle PAC = \angle PBC$ [By CPCT]

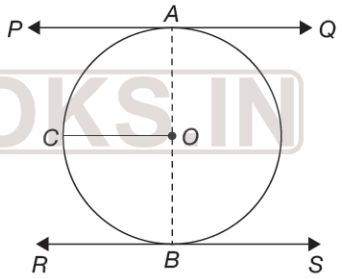
Example 13

Prove that the segment joining the points of contact of two parallel tangents passes through the centre.

Solution

Let PAQ and RBS be two parallel tangents to a circle with centre O. Join OA and OB. Draw $OC||PQ$.

Now, $PA \,||\, CO$



$\Rightarrow \angle PAO + \angle COA = 180°$

[Sum of co-interior angle is 180°]

$\Rightarrow 90° + \angle COA = 180°$ [$\because \angle PAO = 90°$]

$\Rightarrow \angle COA = 90°$

Similarly, $\angle COB = 90°$

$\therefore \angle COA + \angle COB = 90° + 90° = 180°$

Hence, AOB is a straight line passing through O.

Example 14

Two chords AB and CD of a circle intersect each other at O internally. If $AO = 3.5$ cm, $CO = 5$ cm, and $DO = 7$ cm, find OB.

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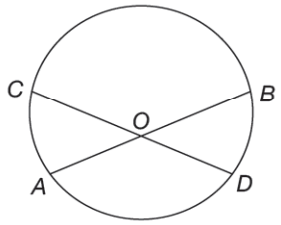
Solution:



We have,

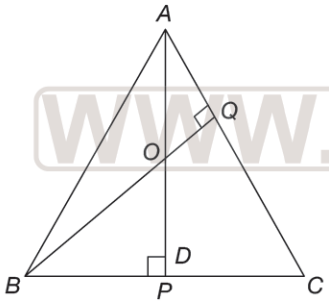
$$AO \times OB = CO \times OD \Rightarrow (3.5) \times OB = 5 \times 7 = 35$$

$$OB = \frac{35}{3.5} = 10 \text{ cm.}$$

Example 15

In ΔABC, $AP \perp BC$ and $BQ \perp AC$, which intersect each other at O. Prove that $AO \times OP = BO \times OQ$.

Solution



We have,

$$\angle APB = 90° \text{ and } \angle AQB = 90°$$
$$[\because AP \perp BC \text{ and } BQ \perp AC]$$

In ΔAOQ and ΔBOP

$$\angle AOQ = \angle BOP \text{ (Vertically opp. angles)}$$

$$\angle OQA = \angle OPB = 90°$$

By, AA Similarity

$$\Delta AOQ \sim \Delta BOP$$

$$\frac{AO}{OB} = \frac{OQ}{OP} \Rightarrow AO \times OP = OB \times OQ$$

Example 16

In ΔABC, the bisector of $\angle A$ intersects the base BC at the point D. Prove that $AB \times AC = BD \times DC + AD^2$.

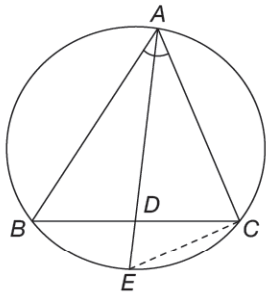
Solution

Draw a circumcircle of ΔABC. Produce AD to meet the circle at E. Join EC.

In Δs ABD and AEC, $\angle BAD = \angle EAC$
$[\because AD$ is the bisector of $\angle BAC]$

$\angle ABD = \angle AEC$ [$\angle s$ in the same segment]

$\therefore \Delta ABD \sim \Delta AEC.$



$$\therefore \frac{AD}{AC} = \frac{AB}{AE}$$

$$\Rightarrow AB \times AC = AD \times AE = AD \times (AD + DE)$$

$$AB \times AC = AD^2 + AD \times DE = AD^2 + BD \times BC$$
$$[\because AD \times DE = BD \times DC]$$

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Example 17

Two figures are given below:

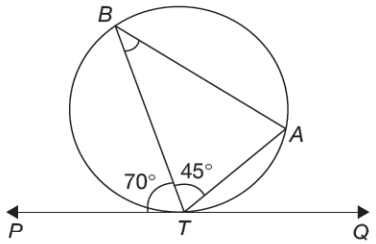


Fig. (i)

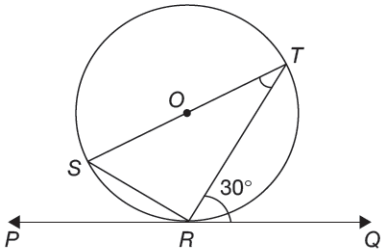


Fig. (ii)

(a) In fig (i) PQ is a tangent to the circle at T, $\angle BTA = 45°$ and $\angle PTB = 70°$. Find $\angle ABT$.

(b) In fig (ii) O is the centre and PQ is a tangent to circle at R. Calculate $\angle STR$.

Solution

(a) $\angle ATQ = 180° - (70° + 45°)$
[$\because$ Sum of the adjacent $\angle s = 180°$]
$= 180° - 115° = 65°$.
$\therefore \angle ABT = \angle ATQ = 65°$
[$\angle s$ in the alternate segment]

(b) $\angle SRT = 90°$ [Angle in a semi-circle]
Now $\angle RST = \angle TRQ = 30°$
[$\angle s$ in the alternate segments]
$\therefore \angle RTS = 90° - 30° = 60°$.

Example 18

$ABCD$ is a cyclic quadrilateral and PQ is a tangent to the circle at C. If BD is a diameter and $\angle DCQ = 40°$ and $\angle ABD = 60°$; find the measure of the following angles:

(i) $\angle DBC$ **(ii)** $\angle BCP$ **(iii)** $\angle ADB$.

Solution

Since BD is a diameter of the circle.

$\therefore \angle BAD = 90°$ and also $\angle BCD = 90°$

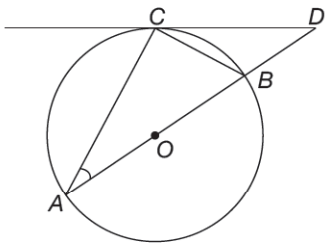
(i) $\angle DBC = \angle DCQ = 40°$
[$\angle s$ is the alternate segments]

(ii) $\angle BCP + \angle BCD + \angle DCQ = 180°$
$\Rightarrow \angle BCP + 90° + 40° = 180°$
$\therefore \angle BCP = 50°$

(iii) Similarly, from ΔBAD, $60° + \angle BAD + \angle ADB = 180°$
$\Rightarrow 60° + 90° + \angle ADB = 180°$
$\angle ADB = 30°$.

Example 19

AB is diameter and AC is a chord of a circle such that $\angle BAC = 30°$. The tangent at C intersects AB produced at D. Prove that $BC = BD$.



Solution

Since AB is the diameter of the circle

$\therefore \angle ACB = 90°$ [Angle in a semi-circle]

$\therefore \angle ABC = 90° - 30° = 60°$ ...(i)

Again, $\angle BCD = \angle CAB = 30°$
[$\angle s$ in the alternate segments]

Now in ΔCBD,

ext. $\angle CBA = \angle BCD + \angle BDC$

$\Rightarrow 60° = 30° + \angle BDC$

$\therefore \angle BDC = 30°$

Hence, $\angle BCD = \angle BDC$

$\Rightarrow BC = AD$

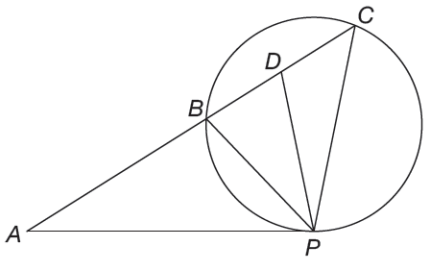
Example 20

In the adjoining figure, AP is a tangent to the circle at P. ABC is a secant such that PD is the bisector of $\angle BPC$. Prove that

(i) $\angle ADP = \angle APD$

(ii) $AP = AD$

(iii) $\angle BPD = \frac{1}{2}(\angle ABP - \angle APB)$



Solution

AP is a tangent to the circle and PB is a chord through the point of contact P.

$\therefore \angle 1 = \angle 2$ [$\angle s$ in the alternate segments]

$\Rightarrow \angle 1 + \angle 4 = \angle 2 + \angle 3$
[$\because \angle 3 = \angle 4$ since PD bisects $\angle CPB$]

$\Rightarrow \angle APD = \angle 2 + \angle 3$ ....(i)

But from ΔPCD, ext. $\angle PDA = \angle 2 + \angle 3$
Hence from (i), $\angle APD = \angle PDA$
Thus (i) is proved

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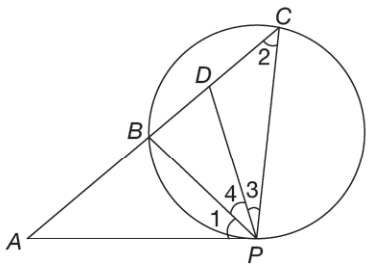


Again $\angle APD = \angle PDA$

$\Rightarrow AP = AD$

Further, from ΔPCB, ext. $\angle PBA$ = sum of the two opposite interior angles viz. $\angle 2 + (\angle 3 + \angle 4)$

$= \angle 1 + (\angle 4 + \angle 4)$ [$\because \angle 2 = \angle 1$ and $\angle 3 = \angle 4$]

$= \angle APB + 2\angle BPD \Rightarrow 2\angle BPD = \angle PBA - \angle APB$

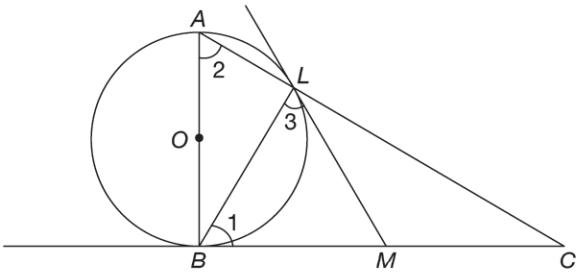
$\therefore \angle BPD = \frac{1}{2}(\angle PBA - \angle APB)$.

Example 21

In a right-angled ΔABC, a circle with side AB as diameter is drawn to intersect the hypotenuse AC in L. Prove that the tangent to the circle at L bisects the side BC.

Solution

Let the tangent at L meet BC in M. Join BL.



Firstly, $\angle ALB = 90°$ [$\angle$ in a semi-circle]

$\therefore \angle BLC = 90°$

Since the lengths of tangents from an external point are equal.

$\therefore MB = ML$

Now LM is a tangent to the circle at L and LB is a chord through the point of contact L

$\therefore \angle 3 = \angle 2$

Further, $\angle MLC = \angle BLC - \angle 3 = 90° - \angle 3$

and $\angle LCM = \angle ACB = 90° - \angle 2 = 90° - \angle 3$ [$\angle 2 = \angle 3$]

$\angle MLC = \angle LCM$

$\therefore ML = MC$

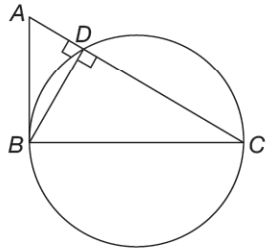
Thus $BM = LM = MC$.

Hence, M is the mid-point of BC.

Example 22

In a right triangle ABC, the perpendicular BD on the hypotenuse AC is drawn. Prove that

(i) $AC \times AD = AB^2$ **(ii)** $AC \times CD = BC^2$



Solution

We draw a circle with BC as diameter. Since $\angle BDC = 90°$.

$\therefore$ The circle on BC as diameter will pass through D again

$\therefore BC$ is a diameter and $AB \perp BC$.

$\therefore AB$ is a tangent to the circle at B.

Since AB is a tangent and ADC is a secant to the circle.

$\therefore AC \times AD = AB^2$

This proves (i)

Again

$AC \times CD = AC \times (AC - AD) = AC^2 - AC \times AD$

$= AC^2 - AB^2$ [Using (i)]

$= BC^2$ [ΔABC is a right triangle]

Hence, $AC \times CD = BC^2$.

This proves (ii).

Example 23

In the given figure, PAT is a tangent to the circumcircle of a ΔABC at the vertex A. A line parallel to PAT intersects the sides AB and AC at the points D and E, respectively.

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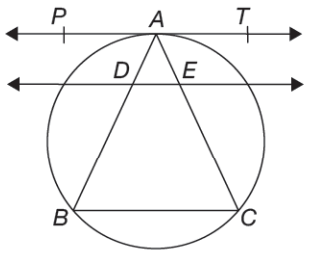


Prove that

(a) $\Delta ABC \sim \Delta AED$ **(b)** $AB \cdot AD = AC \cdot AE$

Solution

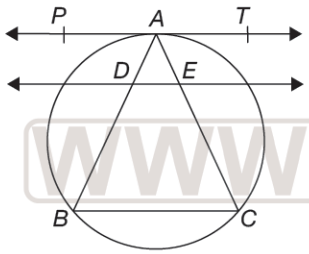
$\angle TAC = \angle TAE = \angle ABC$ …(i)

[∴ Angles in the alternate segment are equal

Similarly, $\angle PAB = \angle PAD = \angle ACB$ …(ii)

[∴ Angles in the alternate segment are equal]

Also $\angle TAE = \angle AED$ (alternate angles) …(iii)



$\angle PAD = \angle ADE$ [alternate angles] …(iv)

From (i) to (iv), we get

$\angle ABC = \angle AED$

$\angle ACB = \angle ADE$

$\therefore \Delta ABC \sim \angle AED$ [By AA similarity]

Since corresponding sides of similar triangles are proportional.

$$\frac{AB}{AE} = \frac{AC}{AD} \Rightarrow AB \times AD = AC \times AE$$

Hence, the required results are proved.

Example 24

Two circles touch externally at P and a common tangent touch them at A and B. Prove that

(i) the common tangent at P bisects AB.
(ii) AB subtends a right angle at P.

Solution

Let PT be the common tangent at any point P. Since the tangent to a circle from an external point are equal,

$\therefore TA = TP, TB = TP$

$\therefore TA = TB$

i.e., PT bisects AB at T

$TA = TP$ gives $\angle TAP = \angle TPA$ (from ΔPAT)

$TB = TP$ gives $\angle TBP = \angle TPB$ [from ΔPBT]

$\therefore \angle TAP + \angle TBP = \angle TPA + \angle TPB = \angle APB$

$\Rightarrow \angle TAP + \angle TBP + \angle APB = 2 \angle APB$

$\Rightarrow 2 \angle APB = 180°$ [sum of ∠s of a Δ = 180°]

$\Rightarrow \angle APB = 90°$

Hence, Proved.

Example 25

Two circles touch each other externally. Their radii are 9 cm and 4 cm and the distance between their centres is 13 cm. Find the length of their common tangent segment.

Solution

Given: Two circles $C(O, 9)$ and $C(O, 4)$ touch externally at S.

$\therefore OO' = 9 \text{ cm} + 4 \text{ cm} = 13 \text{ cm}$.

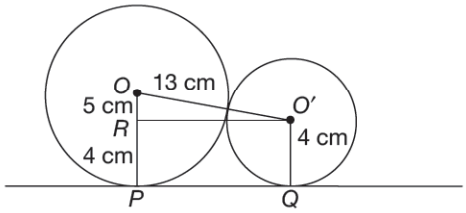
PQ is the common tangent segment.

Construction: Join OP and $O'Q$. Then $OP \perp PQ$ and $O'Q \perp PQ$. From O' draw $O'R \perp OP$.

Proof: Clearly $PQ = O'R$

Also, $RP = O'Q = 4$ cm.

$\therefore RP = OP - RP = 9 - 4 = 5$ cm.



Now from right-angled $\Delta ORO'$, $OO'^2 = OR^2 + O'R^2$

$\Rightarrow 13^2 = 5^2 + PQ^2$

$\Rightarrow PQ^2 = 13^2 - 5^2 = (13 - 5)(13 + 5)$

$\Rightarrow PQ^2 = 8 \times 18 = 9 \times 16$

$\Rightarrow PQ = 3 \times 4 = 12$ cm.

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Example 26

If the sum of interior angles of a polygon is 1620°, find its number of sides.

Solution

We know that sum of all interior angles of a n sided polygon = $(2n - 4)$ right angles

$$= [(2n - 4) \times 90°]$$

but given sum of interior angles = 1620°

$$\therefore (2n - 4) \times 900 = 1620°$$

$$\Rightarrow 2n - 4 = \frac{1620°}{90°}$$

$$\Rightarrow 2n - 4 = 18$$

$$\Rightarrow 2n = 22$$

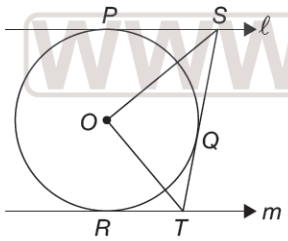
$$\Rightarrow n = 11$$

Know Your Progress - 2

1. Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that

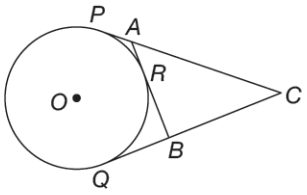
$$\angle PTQ = 2\angle OPQ$$

2. In figure, l and m are two parallel tangents at P and R. The tangent at Q makes an intercept ST between l and m. Prove that $\angle SOT = 90°$.



3. The radius of the incircle of a triangle is 4 cm and the segments into which one side is divided by the point of contact are 6 cm and 8 cm. Determine the other two sides of the triangle.

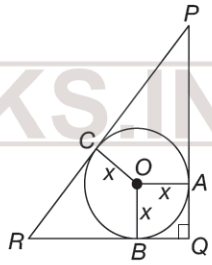
4. In figure, CP and CQ are tangents from an external point C to a circle with centre O. AB is another tangent which touches the circle at R. If $CP = 11$ cm and $BR = 4$ cm, find the length of BC.



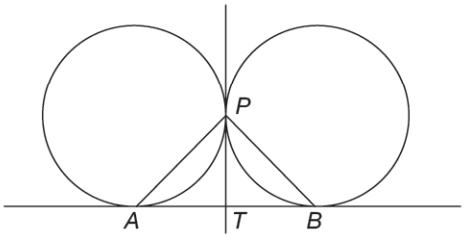
5. Prove that the lengths of the tangents drawn from an external point to a circle are equal. Using the above theorem prove that: If quadrilateral $ABCD$ is circumscribing a circle, then $AB + CD = AD + BC$.

6. Two concentric circles are of radius 5 cm and 3 cm. Find the length of the chord of the larger circle.

7. PQR is a right angled triangle with $PQ = 12$ cm and $QR = 5$ cm. A Circle with O and radius x is inscribed in ΔPQR. Find the value of x.



8. Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that $\angle PTQ = 2\angle OPQ$.



9. AX and BX are two adjacent sides of a regular polygon. If $\angle ABX = \frac{1}{3}\angle AXB$, then the number of sides of the polygon is:

(A) 6 **(B)** 7 **(C)** 9 **(D)** 5

10. The ratio of an interior angle of a regular pentagon to an exterior angle of a regular decagon is

(A) 4:1 **(B)** 3:1 **(C)** 2:1 **(D)** 7:3

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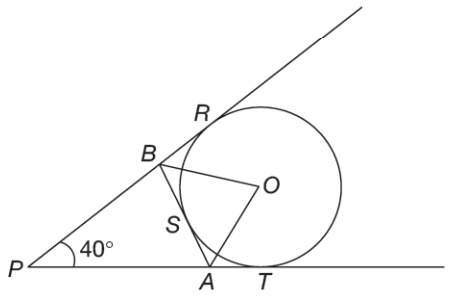
11. In the figure, ΔAPB is formed by three tangents to the circle with centre O. If $\angle APB = 40°$, then the measure of $\angle BOA$ is:



(A) 50° **(B)** 55° **(C)** 60° **(D)** 70°

12. Two circles with centres P and R touch each other externally at O. A line passing through O cuts the circles at T and S, respectively. Then,

(A) PT and RS are of equal length
(B) PT and RS are perpendicular to each other
(C) PT and RS are intersecting
(D) PT and RS are parallel

Area Related to Circles

Perimeter and Area of a Circle

In our daily life, we come across many objects that are circular in shape. For example, cycle wheels, bangles, circular paths etc. That is why the problem of finding perimeters and areas related to circular figures is of great importance.

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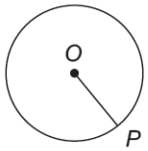
Circle

The collection of all points in a plane, which are at a fixed distance from a fixed point in the plane, is called a circle.



The fixed point is called the centre of the circle and the fixed distance is called the radius of the circle. The diameter of a circle is twice its radius.

In figure, O is the centre and the length OP is the radius of the circle.

The length of the boundary of a circle is called its circumference or its perimeter. We know that the ratio of the circumference of a circle to its diameter is always a constant. This constant ratio is denoted by the Greek letter π.

$$\frac{\text{Circumference}}{\text{Diameter}} = \pi$$

Circumference $= \pi \times 2r = 2\pi r$ [$\because$ Diameter $= 2r$]

The exact value of π is not known, because π is an irrational number. For all practical purposes, the value of π is approximately taken as $\frac{22}{7}$ or 3.14

If r is the radius of a circle, then

(i) Circumference $= 2\pi r$ or πd, where $d = 2r$ is the diameter of the circle.
(ii) Area $= \pi r^2$.
(iii) Area of semi-circle $= \frac{\pi r^2}{2}$.
(iv) Perimeter of the semi-circle $= \pi r + 2r$.
(v) Area enclosed by two concentric circles
$= \pi R^2 - \pi r^2 = \pi (R^2 - r^2) = \pi(R + r)(R - r)$
Where R and r and radii of two concentric circles.

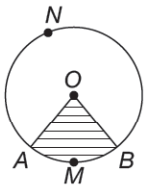
Some Important Points

(i) If two circles touch each other externally, then the distance between their centres is equal to sum of their radii.
(ii) If two circles touch each other internally, then the distance between their centres is equal to difference of their radii.
(iii) The distance moved by a rotating wheel in one revolution is equal to the circumference of the wheel.

Sector and Segment of a Cirlce

Sector of a Circle and its Area

The region bounded by an arc of a circle and its two bounding radii is called a sector of the circle.



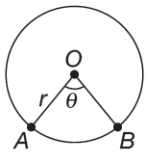
If the arc is a minor arc then the corresponding sector is called the minor sector

and the remaining part bounded by the major arc is called the major sector.

In figure, shaded region $OAMB$ is the minor sector and the remaining portion $OANB$ is the major sector.

Length of Arc and Area of Sector

Let r be the radius of the circle with centre O and AOB be a sector of the circle such that $\angle AOB = \theta$.



$0 < \theta < 180°$, then the arc AB is a minor arc of the circle.

Now, if θ increases, the length of arc AB is also increases and if θ becomes 180°, then arc AB becomes the circumference of a semi-circle.

When an arc subtends an angle 180° at the centre, then the length of the arc of the semi-circle $= \pi r$.

When an arc subtends angle θ at the centre, then length of the arc $\frac{\pi r}{180°} \times \theta = \frac{\pi r\theta}{180°}$.

If ℓ be the length of the arc AB, then $\ell = \frac{\pi r\theta}{180°}$

Again, when an arc subtends angle 180° at the centre, the corresponding sector is a semi-circular region of area $\frac{1}{2}\pi r^2$.

$\therefore$ When an arc subtends an angle 180° at the centre, then the area of the corresponding sector is $\frac{1}{2}\pi r^2$.

$\therefore$ When an arc subtends an θ at the centre, then area of the sector $= \frac{\pi r^2}{2} \times \frac{1}{180°} \times \theta = \frac{\pi r^2\theta}{360°}$

If A be the area of the sector, then $A = \frac{\pi r^2\theta}{360°}$ and length of an arc $\ell = \frac{\pi r\theta}{180°}$

Some Important Points

(i) Angle described by minute hand in 60 minutes $= 360°$.

$\therefore$ Angle described by minute hand in one minute $= \left(\frac{360°}{60}\right)° = 6°$.

Thus, minute hand rotates through an angle of 6° in one minute.

(ii) Angle described by hour-hand in -12 hours. $= 360°$.

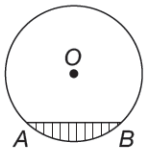
$\therefore$ Angle described by hour hand in one hour $= \left(\frac{360°}{12}\right)° = 30°$.

$\therefore$ Angle described by hour hand in one minute $= \left(\frac{30°}{60}\right)° = \left(\frac{1}{2}\right)°$.

Thus, hour hand rotates through an angle of $\left(\frac{1}{2}\right)°$ in one minute.

Segment of a Circle and its Area

The region enclosed by an arc and a chord is called the segment of the circle.



The segment containing the minor arc is called a minor segment and the remaining segment containing the major arc is called the major segment.

In the figure, the shaded region is the minor segment and the remaining part of the circle is major segment.

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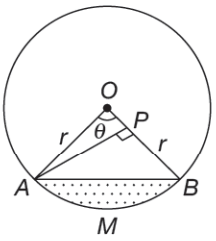
Area of a Segment of a Circle

Let r be the radius of a circle with centre O and let AB be an arc subtending an angle θ at the centre O. We shall find the area of the shaded segment AMB.



Let $AP \perp O$

Now, area of the segment AMB

$=$ Area of the sector $OAMB$ – area of ΔOAB

$$= \frac{\pi r^2\theta}{360°} - \frac{1}{2} \times OB \times AP$$

$$= \frac{\pi r^2\theta}{360°} - \frac{1}{2} r \times OA \times \sin\theta$$

$[\because$ From ΔAOP, $\sin\theta = \frac{AP}{OA} \Rightarrow AP = OA\sin\theta]$

$$= \frac{\pi r^2\theta}{360^\circ} - \frac{r^2}{2}\sin\theta \quad [\because OA = OB = r]$$

Hence, area of the segment $= \frac{\pi r^2\theta}{360^\circ} - \frac{1}{2}r^2\sin\theta$.

Application of Area Related to Circle

In our daily life we come across various plane figures, which are combinations of two or more plane figures.

For example, window designs, flower beds, circular paths etc. In this section, we shall discuss problems of combinations of plane figures.

Important Formula

(i) Heron's formula: Area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$
where s = semi-perimeter and a, b, c are the sides of the triangle.

(ii) Area of a right-angled triangle $= \frac{1}{2} \times$ base $\times$ altitude

(iii) Area of an equilateral triangle $= \frac{\sqrt{3}}{4}a^2$.

(iv) Area of a rectangle = Length $\times$ breadth

(v) Perimeter of a rectangle of sides 'a' and 'b' $= 2(a+b)$.

(vi) Area of a square of side 'a' $= a^2$.

(vii) Length of diagonal of a square of a side 'a' $= \sqrt{2}a$.

(viii) Perimeter of a square of side 'a' $= 4a$.

(ix) Area of a parallelogram = Base $\times$ Height

(x) Area of a rhombus $= \frac{1}{2}d_1d_2$.
Where d_1 and d_2 Vare the lengths of its diagonals.

(xi) Area of a trapezium $= \frac{1}{2}(a+b)h$.
Where a and b are lengths of two parallel sides and h is the distance between them.

Example 27

Find the area of a circle whose circumference is 22 cm.

Solution

Let r be the radius of the circle.

Then, circumference = 22 cm

$$\Rightarrow 2\pi r = 22 \Rightarrow 2\times\frac{22}{7}\times r = 22 \Rightarrow r = \frac{7}{2}\text{cm}$$

$$\therefore \text{Area of the circle} = \pi r^2 = \frac{22}{7}\times\frac{7}{2}\times\frac{7}{2}\text{cm}^2 = 38.5\text{ cm}^2.$$

Example 28

If the perimeter of a semi-circular protractor is 66 cm, find the diameter of the protractor

(Take $\pi = \frac{22}{7}$).

Solution

Let the radius of the protractor be r cm. Then, perimeter = 66 cm

$$\Rightarrow \pi r + 2r = 66$$

$$\Rightarrow r\left[\frac{22}{7} + 2\right] - 66$$

$$\Rightarrow r\left(\frac{36}{7}\right) - 66 \Rightarrow r - \frac{66\times 7}{36} \Rightarrow r - \frac{77}{6}\text{cm}$$

$$\therefore \text{Diameter} = 2r = 2r = \frac{77}{3}\text{cm}.$$

Example 29

Two circles touch externally. The sum of their areas is 130 π sq. cm. and the distance etween their centres is 14 cm. Find the radii of the circles.

Solution:

If two circles touch externally, then the distance between their centres is equal to the sum of their radii.

Let the radii of the two circles be r_1 cm and r_2 cm, respectively.

Let C_1 and C_2 be the centres of the given circles. Then,

$$C_1C_2 = r_1 + r_2 \quad [\because C_1C_2 = 14\text{ cm (given)}]$$

$$\Rightarrow 14 = r_1 + r_2$$

$$\Rightarrow r_1 + r_2 = 14 \qquad \ldots(i)$$

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It is given that the sum of the areas of two circles is equal to 130 π cm^2.

$$\therefore \pi r_1^2 + \pi r_2^2 = 130\,\pi$$

$$\Rightarrow r_1^2 + r_2^2 = 130 \quad \ldots(ii)$$

Now, $(r_1 + r_2)^2 = r_1^2 = r_1^2 + r + 2r_1r_2$

$$\Rightarrow 14^2 = 130 + 2r_1r_2 \quad \text{[Using (i) and (ii)]}$$

$$\Rightarrow 196 - 130 = 2r_1r_2$$

$$\Rightarrow r_1r_2 = 33 \quad \ldots(iii)$$

Now, $(r_1 - r_2)^2 = r_1^2 + r_1^2 + - 2r_1r_2$

$$\Rightarrow (r_1 - r_2)^2 = 130 - 2 \times 33 \quad \text{[Using (ii) and (iii)]}$$

$$\Rightarrow (r_1 - r_2)^2 = 64$$

$$\Rightarrow r_1 - r_2 = 8 \quad \ldots(iv)$$

Solving (i) and (iv), we get $r_1 = 11$ cm and $r_2 = 3$ cm.

Hence, the radii of the two circles are 11 cm and 3 cm.

Example 30

A race track is in the form of a ring whose inner circumference is 352 m and the outer circumference is 396 m. Find the width of the track.

Solution

Let the outer and inner radii of the ring be R metres and r metres respectively.

Then, $2\pi R = 396$ and $2\pi r = 352$

$$\Rightarrow 2 \times \frac{22}{7} \times R = 396 \text{ and } 2 \times \frac{22}{7} \times r = 352$$

$$\Rightarrow R = 396 \times \frac{7}{22} \times \frac{1}{2} \text{ and } r = 352 \times \frac{7}{22} \times \frac{1}{2}$$

$$\Rightarrow R = 63 \text{ m and } r = 56 \text{ m}$$

Hence, width of the track = $(R - r)$ m = $(63 - 56)$ m = 7 m.

Example 31

The diameter of a cycle wheel is 28 cm. How many revolutions will it make in moving 13.2 km?

Solution

Distance travelled by the wheel in one revolution

$$= 2\pi r = 2 \times \frac{22}{7} \times \frac{28}{2} = 88 \text{ cm}$$

and the total distance covered by the wheel

$$= 13.2 \times 1000 \times 100 \text{ cm}$$

$$= 1320{,}000 \text{ cm}$$

$\therefore$ Number of revolutions made by the wheel

$$= \frac{1320{,}000}{88} = 15{,}000.$$

Example 32

A wire is looped in the form of a circle of radius 28 cm. It is re-bent into a square form. Determine the length of the side of the square.

Solution

We have,

Length of the wire = Circumference of the circle

$$\text{Length of the wire} = \left\{2 \times \frac{22}{7} \times 28\right\} \text{cm}$$

$$\text{Length of the wire} = 176 \text{ cm} \quad \ldots(i)$$

Let the side of the square be x cm.

Then,

Perimeter of the square = Length of the wire

$$\Rightarrow 4x = 176$$

$$\Rightarrow x = 44 \text{ cm}$$

Hence, the length of the side of the square is 44 cm.

Example 33

A sector is cut from a circle of radius 21 cm. The angle of the sector is 150°. Find the length of its arc and area.

Solution

The length or arc l and area A of a sector of angle θ in a circle of radius r are given by

$$\ell = \frac{\theta}{360^\circ} \times 2\pi r \text{ and } A = \frac{\theta}{360^\circ} \times \pi r^2 \text{ respectively.}$$

Here, $r = 21$ cm and $\theta = 150$

$$\therefore l = \left\{\frac{150}{360} \times 2 \times \frac{22}{7} \times 21\right\} \text{cm} = 55 \text{ cm and}$$

$$A = \left\{\frac{150}{360} \times \frac{22}{7} \times (21)^2\right\} \text{cm}^2$$

Example 34

In figure, there are shown sector of two concentric circles of radii 7 cm and 3.5 cm. Find the area of the shaded region (use $\pi = \frac{22}{7}$).

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Solution

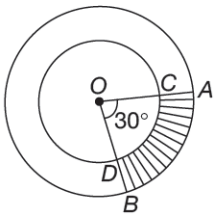
Let A_1 and A_2 be the areas of sectors OAB and OCD, respectively. Then,

A_1 = Area of a sector of angle 30° in a circle of radius 7 cm

$$\Rightarrow A_1 = \left\{\frac{30}{360}\times\frac{22}{7}\times 7^2\right\}$$

[Using: $A = \frac{\theta}{360}\times \pi r^2$]

$$\Rightarrow A_1 = \frac{77}{6}\text{ cm}^2$$



A_2 = Area of a sector of angle 30° in a circle of radius 3.5 cm.

$$\Rightarrow A_2 = \left\{\frac{30}{360}\times\frac{22}{7}\times(3.5)^2\right\}$$

$$\Rightarrow A_2 = \left\{\frac{1}{12}\times\frac{22}{7}\times\frac{7}{2}\times\frac{7}{2}\right\} = \frac{77}{21}\text{ cm}^2$$

$$\therefore \text{ Area of the shaded region} = A_1 - A_2 = \left(\frac{77}{6}-\frac{77}{24}\right)\text{cm}^2$$

$$= \frac{77}{24}\times(4-1)\text{ cm}^2 = \frac{77}{8}\text{ cm}^2 = 9.625\text{ cm}^2.$$

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Example 35

The minute hand of a clock is 10 cm long. Find the area of the face of the clock described by the minute hand between 9 *A*.M. and 9.35 *A*.M.

Solution

We have,

Angle described by the minute hand in one minute = 6°.

∴ Angle described by the minute hand in 35 minutes = (6 × 35)° = 210°

∴ Area swept by the minute hand in 35 minutes.

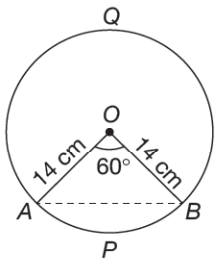
= Area of a sector of angle 210° in a circle of radius 10 cm

$$= \left\{\frac{210}{360}\times\frac{22}{7}\times(10)^2\right\}\text{cm}^2 = 183.3\text{ cm}^2.$$

Example 36

A chord of circle 14 cm makes an angle of 60° at the center of the circle. Find

(i) area of minor sector
(ii) area of the minor segment
(iii) area of the major sector
(iv) area of the major segment



Solution

Given, $r = 14$ cm, $\theta = 60°$

(i) Area of minor sector $OAPB = \frac{\theta}{360°}\pi r^2$

$$= \frac{60°}{360°}\,60°\times 3.14\times 14\times 14 = 102.57\text{ cm}^2$$

(ii) Area of minor segment $APB = \frac{\pi r^2\theta}{360°} - \frac{r^2}{2}\sin\theta$

$$= 102.57 - \frac{14\times 14}{2}\sin 60°$$

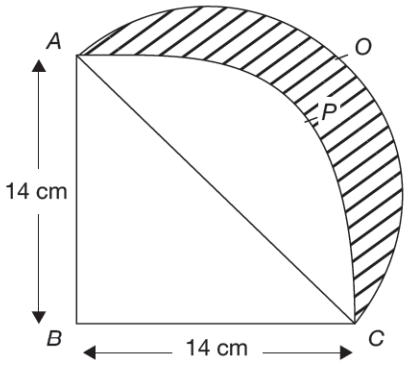
$$= 102.57 - 98\times\frac{\sqrt{3}}{2} = 17.80\text{ cm}^2.$$

(iii) Area of major sector = Area of circle. Area of minor sector $OAPB = \pi(14)^2 - 102.57$
$= 615.44 - 102.57 = 512.87\text{ cm}^2$.

(iv) Area of major segment AQB
= Area of circle − Area of minor segment APB
$= 615.44 - 17.80 = 597.64\text{ cm}^2$.

Example 37

In the given figure $ABCP$ is a quadrant of a circle of radius 14 cm. With AC as diameter, a semi-circle is drawn. Find the area of the shaded portion.



Solution

In right-angled triangle ABC,

we have

$$AC_2 = AB^2 + BC_2$$

$$AC_2 = 14^2 + 14^2$$

$$AC = \sqrt{2\times 14^2} = 14\sqrt{2}\text{ cm}$$

Now required Area

$$= \text{Area } APCQA$$

$$= \text{Area } ACQA - \text{Area ACPA}$$

= Area $ACQA$ – (Area $ABCPA$ – Area of ΔABC)

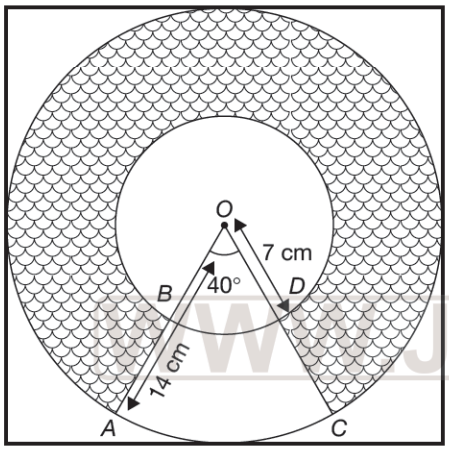
$$= \frac{1}{2} \times \pi \times \left(\frac{14\sqrt{2}}{2}\right)^2 - \left[\frac{1}{4} \times \pi (14)^2 + \frac{1}{2} \times 14 \times 14\right]$$

$$= \frac{1}{2} \times \frac{22}{7} \times 7\sqrt{2} \times 7\sqrt{2} - \frac{1}{4} \times \frac{22}{7} \times 14 \times 14 + 7 \times 14$$

$$= 154 - 154 + 98 = 98 \text{ cm}^2.$$

Example 38

Find the area of the shaded region in figure, where radii of the two concentric circles with centre O are 7 cm and 14 cm, respectively, and $\angle AOC = 40°$.



Solution

We have,

$$\text{Area of ring} = \pi (R^2 - r^2)$$

$$= \pi \times (14^2 - 7^2)$$

$$= 462 \text{ cm}^2$$

Area of the region $ABDC$

= Area of sector AOC – Area of sector BOD

$$= \left(\frac{40}{360} \times \frac{22}{7} \times 14 \times 14 - \frac{40}{360} \times \frac{22}{7} \times 7 \times 7\right) \text{cm}^2$$

$$= \left(\frac{1}{9} \times 22 \times 14 \times 2 - \frac{1}{9} \times 22 \times 7 \times 1\right) \text{cm}^2$$

Hence, required shaded area

$$= \left(462 - \frac{154}{3}\right) \text{cm}^2 = \frac{1232}{3} \text{ cm}^2 = 410.67 \text{ cm}^2$$

Example 39

In an equilateral triangle of side 24 cm, a circle is inscribed touching its sides. Find the area of the remaining portion of the triangle [Take $\sqrt{3} = 1.732$].

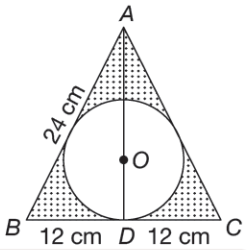
Solution

Let ABC be an equilateral triangle of side 24 cm, and let AD be perpendicular from A on BC. Since the triangle is equilateral, so AD bisects BC.

$$\therefore BD = CD = 12 \text{ cm}$$

The centre of the inscribed circle will coincide with the centroid of ΔABC.

$$\therefore OD = \frac{AD}{3}$$



In ΔABD, we have

$$AB^2 = AD^2 + BD^2 \text{ [Using Pythagoras Theorem]}$$

$$\Rightarrow 24^2 = AD^2 + 12^2$$

$$\Rightarrow AD = \sqrt{242 - 12^2} = \sqrt{(24-12)(24+12)} = \sqrt{36 \times 12} = 12\sqrt{3}.$$

$$\therefore OD = \frac{1}{3} AD = \left(\frac{1}{3} \times 12\sqrt{3}\right) \text{cm} = 4\sqrt{3} \text{ cm}$$

$$\text{Area of the incircle} = \pi (OD)^2 = \left\{\frac{22}{7} \times \left(4\sqrt{3}\right)^2\right\} \text{cm}^2 =$$

$$\left\{\frac{22}{7} \times 48\right\} \text{cm}^2 = 150.85 \text{ cm}^2$$

$$\text{Area of the triangle } ABC = \frac{\sqrt{3}}{4} (\text{Side})^2 (24)^4 =$$

$$\frac{\sqrt{3}}{4} (24)^4 = 249.4 \text{ cm}^2$$

$\therefore$ Area of the remaining portion of the triangle

$$= (249.4 - 150.85) \text{cm}^2 = 98.55 \text{ cm}^2.$$

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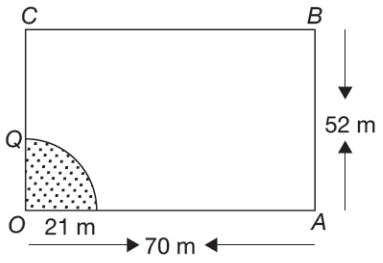
Example 40

A horse is placed for grazing inside a rectangular field 70 m by 52 m and is tethered to one corner by a rope 21 m long. On how much area can it graze?

Solution



Shaded portion indicates the area that the horse can graze. Clearly, shaded area is the area of a quadrant of a circle of radius $r = 21$ m.

$$\therefore \text{ Required area } = \frac{1}{4}\pi r^2$$

$$\Rightarrow \text{Required area } = \left\{\frac{1}{4}\times\frac{22}{7}\times(21)^2\right\}\text{cm}^2$$

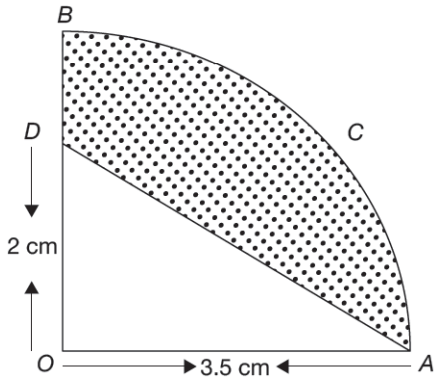
$$\Rightarrow \text{Required area } = \frac{693}{2}\text{ cm}^2\ 346.5\text{ cm}^2$$

Example 41

In figure, $AOBCA$ represents a quadrant of a circle of radius 3.5 cm with centre O. Calculate the area of the shaded portion (take $\pi = \frac{22}{7}$).

Solution

We have,



$$\text{Area of quadrant } AOBCA = \frac{1}{4}\pi r^2 = \frac{1}{4}\times\frac{22}{7}\times (3.5)^2 = 9.625\text{ cm}^2$$

$$\text{Area of } \Delta AOD = \frac{1}{2}\times \text{Base} \times \text{Height} = \frac{1}{2}(3.5)(2) = 3.5\text{ cm}^2$$

Hence, Area of the shaded portion = Area of quadrant – Area of $\Delta AOD = (9.625 - 3.5)\text{cm}^2 = 6.125\text{ cm}^2$

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Know Your Progress - 3

1. The difference between the circumference and the radius of a circle is 37 cm. Find the area of the circle.
2. A wheel makes 1000 revolutions in covering a distance of 88 km. Find the radius of the wheel.
3. A circular road runs around a circle. If the difference between the circumference of the outer circle and inner circle is 66 m, find the width of the road.
4. If the diameter of a circle is increased by 100%, find the percentage increase in its area.
5. The short and long hands of a clock are 4 cm and 6 cm long, respectively. Find the sum of the distances travelled by their tips in 2 days.
6. Find the area of a quadrant circle of circumference 22 cm.
7. The perimeter of a sector of a circle of radius 5.6 cm is 27.2 cm. Find the area of the sector.
8. A pendulum swings through an angle of 30° and describes an arc 8.8 cm in length. Find the length of the pendulum.
9. The perimeter of a sector of a circle of radius 7 cm is 44 cm. Find area of the sector.
10. An arc of a circle is of length 5π cm and the sector it bounds has an area of 25π cm^2. Find the radius of the circle.

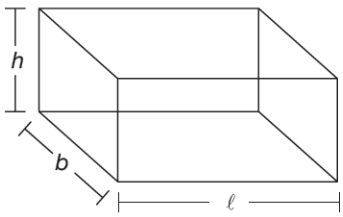
Surface Area and Volume

Mensuration is a branch of Mathematics that deals with the computation of geometric magnitudes, such as the length of a line segment, the area of a surface, and the volume of a solid. In this chapter, we shall deal with the computation of surface areas and volumes of solid figures.

Some Solid Figures

Cuboid

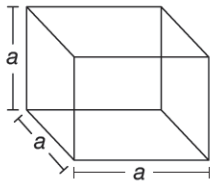
If l, b, and h denote respectively the length, breadth, and height of a cuboid, then:



(i) Total surface area of the cuboid (T.S.A.) = 2 (lb + bh + lh) sq. unit
(ii) Volume of the cuboid
= Area of the base × height = lbh cubic unit
(iii) Diagonal of the cuboid or longest rod
$= \sqrt{\ell^2 + b^2 + h^2}$ unit.
(iv) Area of four walls of a room
= 2 (l + b) h sq. unit

Cube

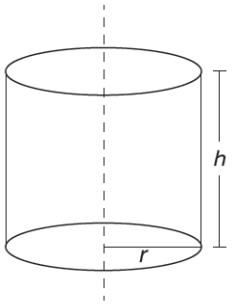
If the length of each edge of a cube is 'a' units, then:



(i) Total surface area of the cube (T.S.A.) = $6a^2$ sq. unit.
(ii) Lateral surface area (L.S.A.) = $4a^2$ sq. unit.
(iii) Volume of the cube = a^3 cubic unit.
(iv) Diagonal of the cube = $\sqrt{3}$ a unit.

Cylinder

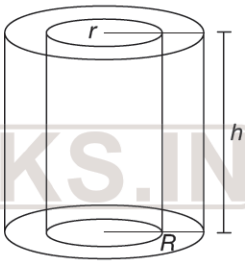
If r and h denote, respectively, the radius of the base and height of a right circular cylinder, then:



(i) Area of each circular end = πr^2 sq. unit
(ii) Curved surface area (C.S.A.)
= circumference × height = $2\pi rh$ sq. units
(iii) Total surface area (T.S.A.) = $2\pi r\ (h + r)$ sq. unit.
(iv) Volume = Area of the base × height = $\pi r^2 h$ cu. unit.

Hollow Cylinder

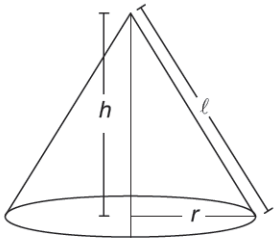
If R and r ($R > r$) denote, respectively, the external and internal radii of a hollow right circular cylinder, then:



(i) Area of each circular end = $\pi(R^2 - r^2)$ sq. unit.
(ii) Curved surface area (C.S.A.) = $2\pi\ (R + r)h$ sq. unit.
(iii) Total surface area (T.S.A.) = $2\pi\ (R + r)\ (R + h - r)$ sq. unit.
(iv) Volume of material = πh $(R^2 - r^2)$ cubic unit

Cone

If r, h, and l denote, respectively, the radius of base, height, and slant height of a right circular cone, then:



(i) $\ell^2 = r^2 + h^2$
(ii) Curved surface area (C.S.A.) = $\pi r\ell$ sq. unit.
(iii) Total surface area (T.S.A.) = $\pi r^2 + \pi r\ell$
= $\pi r(\ell + r)$ sq. unit.
(iv) Volume = $\frac{1}{3}\pi r^2 h$ cubic unit

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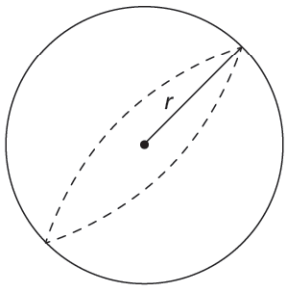
Sphere

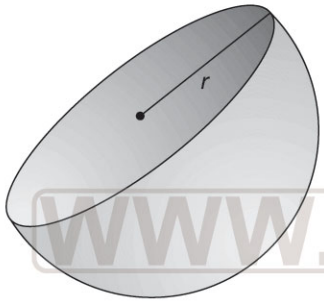
For a sphere of radius r, we have:



(i) Surface area = $4\pi r^2$ sq. unit.

(ii) Volume = $\frac{4}{3}\pi r^3$ cubic unit.

Hemisphere



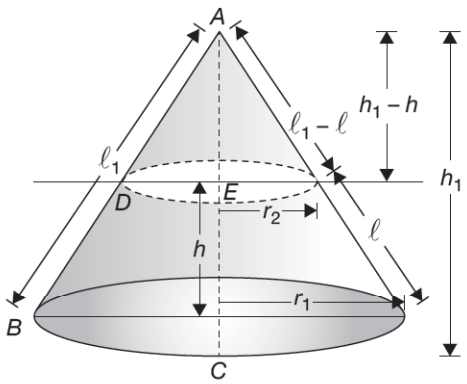
Curved surface area (C.S.A.) = $2\pi r^2$ sq. unit.

Total surface area (T.S.A.) = $3\pi r^2$ sq. unit.

Volume = $\frac{2}{3}\pi r^3$ cubic unit.

Frustum of a Cone

If a right circular cone is cut off along a plane parallel to its base, the portion of the cone between the plane and base of the cone is known as **'Frustum of Cone'.**



$$\Delta ABC \sim \Delta ADE$$

$$\therefore \frac{AC}{AE} = \frac{AB}{AD} = \frac{BC}{DE}$$

$$\frac{h_1}{h_1 - h} = \frac{\ell_1}{\ell_1 - \ell} = \frac{r_1}{r_2}$$

$$\text{Or } \frac{h_1}{h} = \frac{\ell_1}{\ell} = \frac{r_1}{r_1 - r_2}$$

(i) Volume of Frustum $= \frac{1}{3}\pi r_1^2 h_1 - \frac{1}{3}\pi r_2^2 (h_1 - h)$

$$= \frac{1}{3}\pi[r_1^2 h_1 - r_2^2(h_1 - h)]$$

$$= \frac{1}{3}\pi\left[r_1^2\left(\frac{r_1 h}{r_1 - r_2}\right) - r_2^2\left(\frac{r_1 h}{r_1 - r_2} - h\right)\right]$$

$$= \frac{1}{3}\pi h\left[\frac{r_1^3 - r_2^3}{r_1 - r_2}\right]$$

$$= \frac{1}{3}\pi h\left[r_1^2 + r_2^2 + r_1 r_2\right] \text{ cubic unit.}$$

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(ii) Curved Surface Area of Frustum $= \pi r_1 \ell_1 - \pi r_2(\ell_1 - \ell)$

$$= \pi\left[r_1\left(\frac{r_1 \ell}{r_1 - r_2}\right) - r_2\left(\frac{r_1 \ell}{r_1 - r_2} - \ell\right)\right]$$

$$= \pi\ell\left[\frac{r_1^2}{r_1 - r_2} - \frac{r_2^2}{r_1 - r_2}\right]$$

$= \pi\ell(r_1 + r_2)$ sq. unit.

(iii) Total surface area = $\pi\{(r_1 + r_2)\ \ell + r_1^2 + r_2^2\}$ sq. unit.

(iv) Slant height of the frustum = $\sqrt{h^2 + (r_1 - r_2)^2}$ unit.

(v) Height of the cone of which the frustum is a part $= \frac{hr_1}{r_1 - r_2}$ unit.

(vi) Slant height of the cone of which the frustum is a part = $\frac{\ell r_1}{r_1 - r_2}$ unit

(vii) Volume of the frustum

$= \frac{h}{3}\left\{A_1 + A_2 + \sqrt{A_1 A_2}\right\}$ cu. unit, where A_1 and A_2 denote the areas of circular bases of the frustum.

Example 42

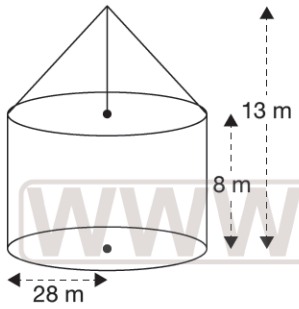
A circus tent is in the shape of a cylinder, upto a height of 8 m, surmounted by a cone of the same radius 28 m. If the total height of the tent is 13 m, find:

(i) total inner curved surface area of the tent.
(ii) cost of painting its inner surface at the rate of ₹3.50 per m^2.

Solution

According to the given statement, the rough sketch of the circus tent will be as shown:

(i) For the cylindrical portion:
$r = 28$m and $h = 8$ m
$\therefore$ Curved surface area $= 2\pi rh$
$= 2 \times \frac{22}{7} \times 28 \times 8\ m^2 = 1408\ m^2$



For conical portion:
$r = 28$ m and $h = 13$ m $- 8$ m $= 5$ m
$\therefore l^2 = h^2 + r^2 \Rightarrow l^2 = 5^2 + 28^2 = 809$
$\Rightarrow \quad l = \sqrt{809}$ m $= 28.4$ m
$\therefore$ Curved surface area $= \pi rl$
$= \frac{22}{7} \times 28 \times 28.4\ m^2 = 2499.2\ m^2$
$\therefore$ Total inner curved surface area of the tent.
= C.S.A. of cylindrical portion + C.S.A. of the conical portion
$= 1408\ m^2 + 2499.2\ m^2 = 3907.2\ m^2$

(ii) Cost of painting the inner surface
$= 3907.2 \times$ ₹3.50
= ₹13675.20

Example 43

A cylinder and a cone have same base area. But the volume of cylinder is twice the volume of cone. Find the ratio between their heights.

Solution

Since, the base areas of the cylinder and the cone are the same.

So their radius are equal (same).

Let the radius of their base be r and their heights be h_1 and h_2 respectively.

Clearly, volume of the cylinder $= \pi r^2 h_1$

and, volume of the cone $= \frac{1}{3}\pi r^2 h_2$

Given:

Volume of cylinder $= 2 \times$ volume of cone

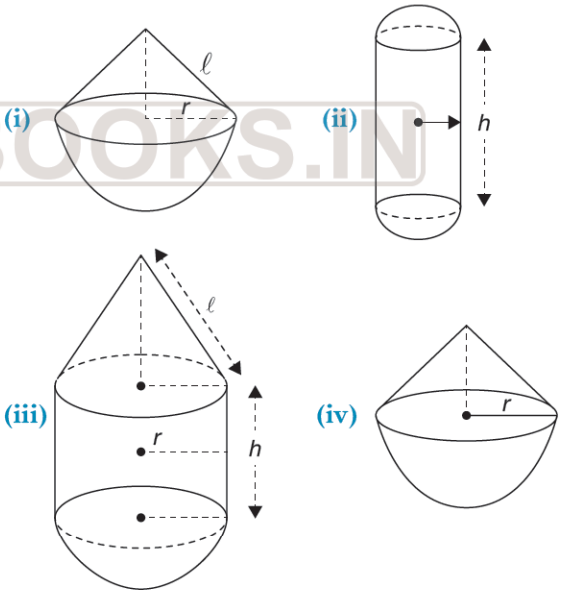
$$\Rightarrow \pi r^2 h_1 = 2 \times \frac{1}{3}\pi r^2 h_2$$

$$\Rightarrow h_1 = \frac{2}{3}h_2 \Rightarrow \frac{h_1}{h_2} = \frac{2}{3}$$

i.e., $h_1 : h_2 = 2 : 3$

Example 44

Find the formula for the total surface area of each figure given below:



Solution

(i) Required surface area
= C.S.A. of the hemisphere + C.S.A. of the cone
$= 2\pi r^2 + \pi rl = \pi r\,(2r + l)$ sq. unit

(ii) Required surface area
= 2 × C.S.A. of a hemisphere + C.S.A. of the cylinder
$= 2 \times 2\pi r^2 + 2\pi rh = 2\pi r\,(2r + h)$ sq. unit

(iii) Required surface area
= C.S.A. of the hemisphere + C.S.A. of the cylinder + C.S.A. of the cone
$= 2\pi r^2 + 2\pi rh + \pi rl = \pi r\,(2r + 2h + l)$ sq. unit

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(iv) If slant height of the given cone be l

$= l^2 = h^2 + r^2 \qquad \Rightarrow l = \sqrt{h^2 + r^2}$

And, required surface area

$= 2\pi r^2 + \pi rl = \pi r\,(2r + l)$

$= \pi r\left(2r + \sqrt{h^2 + r^2}\right)$ sq. unit

Example 45

The radius of a sphere increases by 25%. Find the percentage increase in its surface area.

Solution

Let the original radius be r.

$\Rightarrow$ Original surface area of the sphere $= 4\pi r^2$

Increase radius $= r + 25\%$ of r

$$= r + \frac{25}{100}r = \frac{5r}{4}$$

$\Rightarrow$ Increased surface area

$$= 4\pi\left(\frac{5r}{4}\right)^2 = \frac{25\pi r^2}{4}$$

Increase in surface area

$$= \frac{25pr^2}{4} - 4pr^2 = \frac{25\pi r^2 - 16\pi r^2}{4} = \frac{9\pi r^2}{4}$$

and, percentage increase in surface area

$$= \frac{\text{Increase in area}}{\text{Original area}} \times 100\%$$

$$= \frac{\frac{9\pi r^2}{4}}{4\pi r^2} \times 100\% = \frac{9}{16} \times 100\%$$

$$= 56.25\%$$

Example 46

Three solid spheres of radii 1 cm, 6 cm, and 8 cm are melted and recasted into a single sphere. Find the radius of the sphere obtained.

Solution

Let radius of the sphere obtained = R cm.

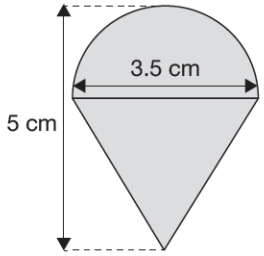
$$\therefore \ \frac{4}{3} \times \pi R^3 = \frac{4}{3}\pi(1)^3 + \frac{4}{3}\pi(6)^3 + \frac{4}{3}\pi(8)^3.$$

$$R^3 = 1 + 216 + 512 = 729 = 9^3$$

$$\therefore \ R = (9^3)^{1/3} = 9 \text{ cm}$$

Example 47

In given figure the top is shaped like a cone surmounted by a hemisphere. The entire top is 5 cm in height and the diameter of the top is 3.5 cm. Find the area it has to colour (take $\pi = \frac{22}{7}$)



Solution

T.S.A of the top = C.S.A of hemisphere + C.S.A of cone

Now, the curved surface of the hemisphere $= \frac{1}{2}(4\pi r^2) =$

$$2\pi r^2 = \left(2 \times \frac{22}{7} \times \frac{3.5}{2} \times \frac{3.5}{2}\right) \text{cm}^2$$

Also, the height of the cone

= height of the top – height (radius) of the hemispherical part

$$= \left(5 - \frac{3.5}{2}\right) \text{cm} = 3.25 \text{ cm}$$

So, the slant height of the cone (l)

$$= \sqrt{r^2 + h^2} = \sqrt{\left(\frac{3.5}{2}\right)^2 + (3.25)^2} \text{ cm}$$

$= 3.7$ cm (approx.)

Therefore, C.S.A of cone $= \pi rl$

$$= \left(\frac{22}{7} \times \frac{3.5}{2} \times 3.7\right) \text{cm}^2$$

This gives the surface area of the top as

$$\left(2 \times \frac{22}{7} \times \frac{3.5}{2} \times \frac{3.5}{2}\right) \text{cm}^2 + \left(\frac{22}{7} \times \frac{3.5}{2} \times 3.7\right) \text{cm}^2$$

$$= \frac{22}{7} \times \frac{3.5}{2}(3.5 + 3.7) \text{cm}^2$$

$$= \frac{11}{2} \times (3.5 + 3.7) \text{ cm}^2 = 39.6 \text{ cm}^2 \text{ (approx)}$$

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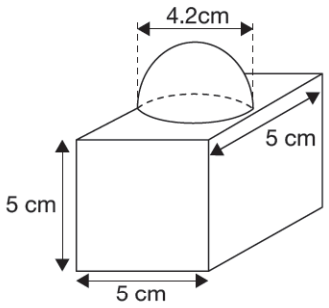
Example 48

The decorative block shown in figure is made of two solids - a cube and a hemisphere. The base of the block is a cube with edge 5 cm, and the hemisphere fixed on the top has a diameter of 4.2 cm. Find the total surface area of the block (take $\pi = \frac{22}{7}$)



Solution

The total surface area of the cube = 6 × (edge)^2 = 6 × 5 × 5 cm^2 = 150 cm^2. Note that the part of the cube where the hemisphere is attached is not included in the surface area.

So, the surface area of the block

= T.S.A of cube – Base area of hemisphere + C.S.A of hemisphere

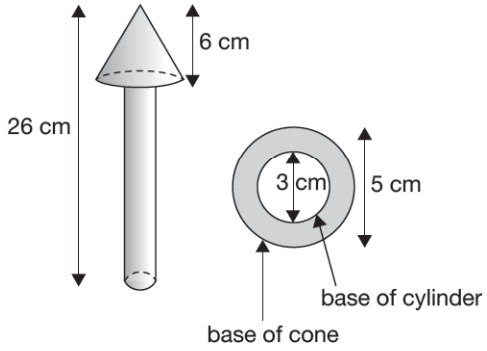
$$= 150 - \pi r^2 + 2\pi r^2 = (150 + \pi r^2) \text{ cm}^2$$

$$= 150 \text{ cm}^2 + \left(\frac{22}{7} \times \frac{4.2}{2} \times \frac{4.2}{2}\right) \text{cm}^2$$

$$= (150 + 13.86) \text{ cm}^2 = 163.86 \text{ cm}^2$$

Example 49

A wooden toy rocket is in the shape of a cone mounted on a cylinder, as shown in figure. The height of the entire rocket is 26 cm, while the height of the conical part is 6 cm. The base of the conical portion has a diameter of 5 cm, while the base diameter of the cylindrical portion is 3 cm. If the conical portion is to be painted orange and the cylindrical portion yellow, find the area of the rocket painted with each of these colours (take $\pi = 3.14$).



Solution

Denote radius of cone by r, slant height of cone by l, height of cone by h, radius of cylinder by r' and height of cylinder by h'. Then $r = 2.5$ cm, $h = 6$ cm, $r' = 1.5$ cm, $h' = 26 - 6 = 20$ cm and

$$l = \sqrt{r^2 + h^2} = \sqrt{2.5^2 + 6^2} \text{ cm} = 6.5 \text{ cm}$$

Here, the conical portion has its circular base resting on the base of the cylinder, but the base of the cone is larger than the base of the cylinder. So a part of the base of the cone (a ring) is to be painted.

So, the area to be painted orange

= C.S.A of the cone + base area of the cone – base area of the cylinder

$$= \pi rl + \pi r^2 - \pi(r')^2$$

$$= \pi[(2.5 \times 6.5) + (2.5)^2 - (1.5)^2] \text{ cm}^2$$

$$= \pi[20.25] \text{ cm}^2 = 3.14 \times 20.25 \text{ cm}^2$$

$$= 63.585 \text{ cm}^2$$

Now, the area to be painted yellow

= C.S.A of the cylinder + area of one base of the cylinder

$$= 2\pi r' h' + \pi(r')^2$$

$$= \pi r' (2h' + r')$$

$$= (3.14 \times 1.5)(2 \times 20 + 1.5) \text{ cm}^2 = 4.71 \times 41.5 \text{ cm}^2$$

$$= 195.465 \text{ cm}^2$$

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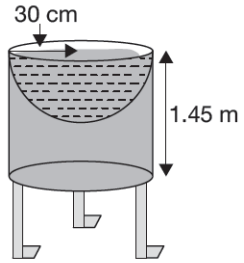
Example 9

A bird bath for garden is in the shape of a cylinder with a hemispherical depression at one end (see figure). The height of the cylinder is 1.45 m and its radius is 30 cm. Find the total surface area of the bird-bath (take $\pi = \frac{22}{7}$)



Solution

Let h be height of the cylinder and r the common radius of the cylinder and hemisphere. Then, the total surface area of the bird-bath

= C.S.A of cylinder + C.S.A of hemisphere + bottom of cylinder

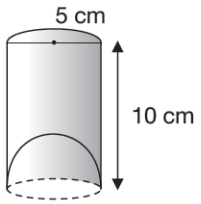
$$= 2\pi rh + 2\pi r^2 + \pi r^2 = \pi r(2h + 3r)$$

$$= \frac{22}{7} \times 30\,(2 \times 145 + 3 \times 30)\ \text{cm}^2$$

$$= 35828.6\ \text{cm}^2 = 3.58\ \text{m}^2$$

Example 51

A juice seller was serving his customers using glasses as shown in figure. The inner diameter of the cylindrical glass was 5 cm, but the bottom of the glass had a hemispherical raised portion which reduced the capacity of the glass. If the height of a glass was 10 cm. Find the apparent capacity of the glass and its actual capacity (use $\pi = 3.14$)



Solution

Since the inner diameter of the glass = 5 cm and height = 10 cm, the apparent capacity of the glass = $\pi r^2 h = 3.14 \times 2.5 \times 2.5 \times 10\ \text{cm}^3 = 196.25\ \text{cm}^3$

But the actual capacity of the glass is less by the volume of the hemisphere at the base of the glass.

i.e., it is less by $\frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times 3.14 \times 2.5 \times 2.5 \times 2.5\ \text{cm}^3 = 32.71\ \text{cm}^3$$

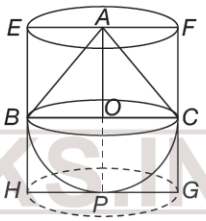
So, the actual capacity of the glass = apparent capacity of glass – volume of the hemisphere

$$= (196.25 - 32.71)\ \text{cm}^3$$

$$= 163.54\ \text{cm}^3$$

Example 52

A solid toy is in the form of a hemisphere surmounted by a right circular cone. The height of the cone is 2 cm and the diameter of the base is 4 cm. Determine the volume of the toy. If a right circular cylinder circumscribes the toy, find the difference of the volume of the cylinder and the toy (take $\pi = 3.14$)



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Solution

Let BPC be the hemisphere and ABC be the cone standing on the base of the hemisphere (see figure). The radius BO of the hemisphere (as well as of the cone) = $\frac{1}{2} \times 4\ \text{cm} = 2\ \text{cm}$

So, volume of the toy = $\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$

$$= \left[\frac{2}{3} \times 3.14 \times (2)^3 + \frac{1}{3} \times 3.14 \times (2)^2 \times 2\right] \text{cm}^3$$

$$= 25.12\ \text{cm}^3$$

Now, let the right circular cylinder $EFGH$ circumscribe the given solid. The radius of the base of the right circular cylinder

$= HP = BO = 2$ cm, and its height is

$EH = AO + OP = (2 + 2)$ cm $= 4$ cm

So, the volume required

= Volume of the right circular cylinder – volume of the toy

$$= (3.14 \times 2^2 \times 4 - 25.12)\ \text{cm}^3$$

$$= 25.12\ \text{cm}^3$$

Hence, the required difference of the two volumes = 25.12 cm^3.

Example 53

A cone of height 24 cm and radius of base 6 cm is made up of modeling clay. A child reshapes it in the form of a sphere. Find the radius of the sphere.

Solution

Volume of cone = $\frac{1}{3} \times \pi \times 6 \times 6 \times 24$ cm^3

If r is the radius of the sphere, then its volume is $\frac{4}{3}\pi r^3$.

Since the volume of clay in the form of the cone and the sphere remains the same, we have:

$$\frac{4}{3} \times \pi \times r^3 = \frac{1}{3} \times \pi \times 6 \times 6 \times 24$$

$$r^3 = 3 \times 3 \times 24 = 3^3 \times 2^3$$

$$r = 3 \times 2 = 6 \text{ cm}$$

Therefore, the radius of the sphere is 6 cm.

Example 54

Selvi's house has an overhead tank in the shape of a cylinder. This is filled by pumping water from a sump (an underground tank) which is in the shape of a cuboid. The sump has dimensions 1.57 m × 1.44 m × 95 cm. The overhead tank has its radius 60 cm and height 95 cm. Find the height of the water left in the sump after the overhead tank has been completely filled with water from the sump which had been full. Compare the capacity of the tank with that of the sump (use $\pi = 3.14$).

Solution

The volume of water in the overhead tank equals the volume of the water removed from the sump.

Now the volume of water in the overhead tank (cylinder) = $\pi r^2 h$

$$= 3.14 \times 0.6 \times 0.6 \times 0.95 \text{ m}^3$$

The volume of water in the sump when full

$$= l \times b \times h = 1.57 \times 1.44 \times 0.95 \text{ m}^3$$

The volume of water left in the sump after filling the tank

$$= [(1.57 \times 1.44 \times 0.95) - (3.14 \times 0.6 \times 0.6 \times 0.95)] \text{ m}^3$$

$$= (1.57 \times 0.6 \times 0.6 \times 0.95 \times 2) \text{ m}^3$$

So, the height of the water left in the sump

$$= \frac{\text{volume of water left in the sump}}{\ell \times b}$$

$$= \frac{1.57 \times 0.6 \times 0.6 \times 0.95 \times 2}{1.57 \times 1.44} \text{ m}$$

$$= 0.475 \text{ m} = 47.5 \text{ cm}$$

Also, $\frac{\text{Capacity of tank}}{\text{Capacity of sump}}$

$$= \frac{3.14 \times 0.6 \times 0.6 \times 0.95}{1.57 \times 1.44 \times 0.95} = \frac{1}{2}$$

Therefore, the capacity of the tank is half the capacity of the sump.

Example 55

A copper rod of diameter 1 cm and length 8 cm is drawn into a wire of length 18 m of uniform thickness. Find the thickness of the wire.

Solution

The volume of the rod = $\pi \times \left(\frac{1}{2}\right)^2 \times 8$ cm^3

$= 2\pi$ cm^3.

The length of the new wire of the same volume = 18 m = 1800 cm

If r is the radius (in cm) of cross section of the wire, its volume = $\pi \times r^2 \times 1800$ cm^3

Therefore, $\pi \times r^2 \times 1800 = 2p$

i.e., $r^2 = \frac{1}{900}$

i.e., $r = \frac{1}{30}$

So, the diameter of the cross section i.e., the thickness of the wire is $\frac{1}{15}$ cm, i.e., 0.67 mm (approx.)

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Know Your Progress - 4

1. Two cylinders have the radii in the ratio 1: 2 and heights are in the ratio 2: 1. Find the ratio of their volumes.
2. Find the length of longest stick that can be kept in a room of 3 m by 4 m by 5 m.
3. The radius of a metal sphere is 9 cm. It is melted and drawn into a wire of diameter 2 m. Find the length of the wire in metre.
4. A metallic sphere of radius 21 cm is melted and then recast into small cones each of radius 7 cm and height 6 cm. Find the number of cones made.
5. A biggest sphere is made from a cube of side 14 cm. Find the volume of solid cubic centimeter.
6. A solid 21 cm long, 20 cm broad, and 10 cm high has hemisphere fixed on the top. The radius of the hemisphere is 7 cm. Find the total surface area of the combined object.
7. A top is of the form of hemisphere mounted on a cone. The diameter of the conical portion is 12 cm and height 8 cm. Find the surface area and volume of the top.
8. 6 cylindircal milk powder tins are packed in a rectangular box, 3 on longer side, and 2 on breadth side.

   Find the volume of the empty space if the radius of the tin is 7 cm and height is 10 cm.
9. A frustum of a right circular cone has diameter of the base 10 cm and top 6 cm and height 5 cm find the volume.
10. The circumference of the end of the frustum of a right circular cone is 48 cm and the other end is 34 cm. the height of the frustum is 10 cm. Find its volume.

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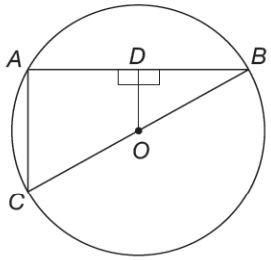
Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. OD is perpendicular to a chord AB of a circle whose centre is O. If BC is a diameter, if $CA = k \times OD$, then k equals to

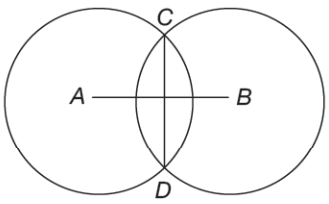


(A) 2 **(B)** 2.5
(C) 3 **(D)** 3.5

2. Two chords AB and CD of lengths 6 cm and 8 cm of a circle are parallel and are on the same side of its centre. If the distance between them is 1 cm. Find the radius of the circle.

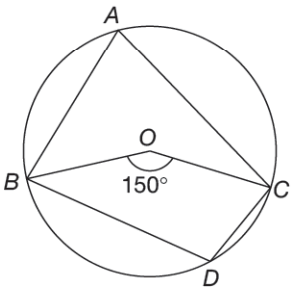
(A) 3 cm **(B)** 4 cm
(C) 5 cm **(D)** 2 cm

3. In the figure given above, A and B are the centers of the two congruent circles with radius 17 units. If $AB =$ 30 units, the length of the common chord DC is:



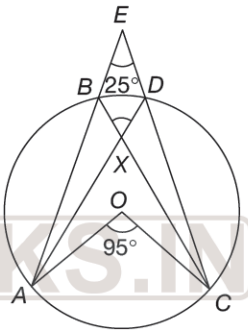
(A) 25 units **(B)** 18 units
(C) 10 units **(D)** 16 units

4. In figure, O is the centre of circle $\angle BOC = 150°$. Find $\angle BDC$



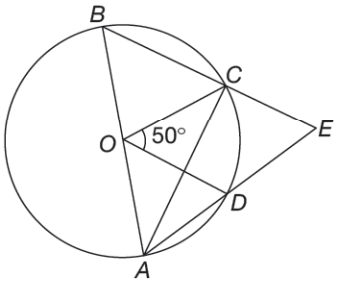
(A) 95° **(B)** 105°
(C) 100° **(D)** 110°

5. In figure, O is centre, then $\angle BXD =$



(A) 65° **(B)** 60°
(C) 70° **(D)** 55°

6. In figure, O is centre and $\angle AEC = 25° \angle AOC = 95°$, then $\angle AEB =$



(A) 60° **(B)** 65°
(C) 70° **(D)** 75°

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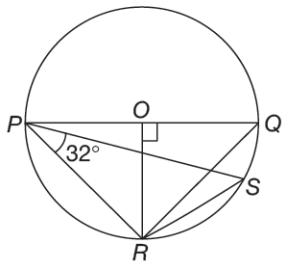
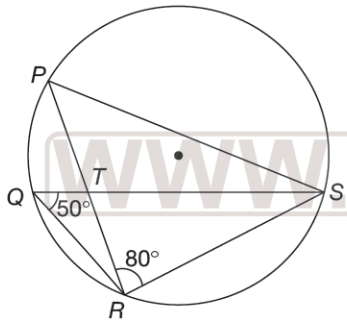
7. In figure, PQ is a diameter of a circle with centre at O and $OR \perp PQ$, where R is a point on the circle. If S is another point on the circle such that $\angle RPS = 32°$, then $\angle QRS$ is:



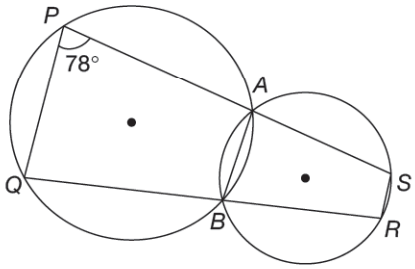
(A) 13° **(B)** 26°
(C) 45° **(D)** None of these

8. In figure, $PQRS$ is a cyclic quadrilateral. Its diagonals PR and QS intersect each other at T. If $\angle PRS = 80°$ and $\angle RQS = 50°$, calculate $\angle PSR$.



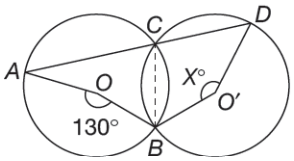
(A) 30° **(B)** 50°
(C) 70° **(D)** 90°

9. Find $\angle ASR$.



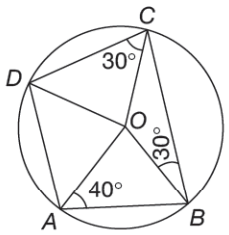
(A) 52° **(B)** 78°
(C) 102° **(D)** None of these

10. In the given figure, O, O' are centres of two circles intersecting at B and C. ACD is a straight line. Find x.



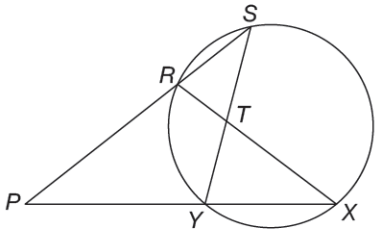
(A) 130° **(B)** 50°
(C) 40° **(D)** None of these

11. $ABCD$ is a cyclic quadrilateral inscribed in a circle with the centre O. Then $\angle OAD$ is equal to



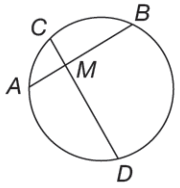
(A) 30° **(B)** 40°
(C) 50° **(D)** 60°

12. In the adjoining figure, the chords XY and SR are produced to meet outside the circle at P. The chords XR and YS meet inside the circle at T. If $X = x°$ and $\angle XTS = y°$, then $\angle P$ is equal to:



(A) $(x + y)°$ **(B)** $(y - x)°$
(C) $2(y - x)°$ **(D)** $(y - 2x)°$

13. In the circle shown $AB = 24$, and the perpendicular chord CD bisects AB. If DM is 4 times as long as CM then the length of BD, is



(A) $8\sqrt{5}$ **(B)** $12\sqrt{5}$
(C) $16\sqrt{5}$ **(D)** $20\sqrt{5}$

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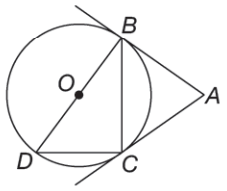
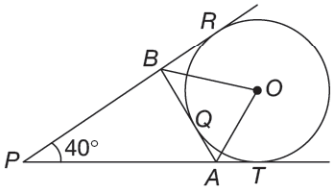
14. In the given figure, O is the centre of a circle and BD is a diameter. AB and AC are tangents touching the circle at B and C, respectively. If $\angle BAC = 70°$ then $\angle OBC$ is:



(A) 30° **(B)** 35°
(C) 40° **(D)** 45°

15. Triangle PAB is formed by three tangents to circle O and $\angle APB = 40°$, then angle AOB

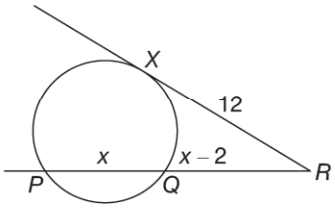


(A) 45° **(B)** 50°
(C) 60° **(D)** 70°

16. ABC is a right-angled triangle, right angled at B such that $BC = 6$ cm and $AB = 8$ cm. A circle with centre O is inscribed in ΔABC. The radius of the circle is:

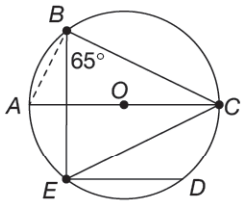
(A) 1 cm **(B)** 2 cm
(C) 3 cm **(D)** 4 cm

17. PQ is a chord of a circle. The tangent XR at X on the circle cuts PQ produced at R. If $XR = 12$ cm, $PQ = x$ cm, $QR = x - 2$ cm, then x in cm is:



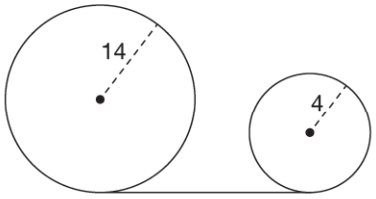
(A) 6 **(B)** 7
(C) 10 **(D)** 14

18. In the adjoining figure, chord ED is parallel to the diameter AC of the circle. If $\angle CBE = 65°$, then what is the value of $\angle DEC$?



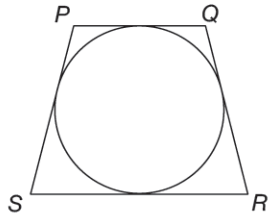
(A) 35° **(B)** 55°
(C) 45° **(D)** 25°

19. Two circles of radii 4 cm and 14 cm have a common external tangent of length 24 cm. The distance between the centres of these circles (in cm) is:



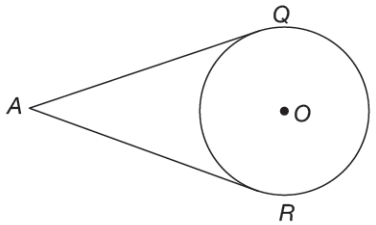
(A) 24 **(B)** 25
(C) 26 **(D)** 27

20. A circle is inscribing in trapezoid $PQRS$. If $PS = QR = 25$ cm, $PQ = 18$ cm what is the length of the diameter of the circle?



(A) 14 cm **(B)** 25 cm
(C) 24 cm **(D)** 674 cm

21. In the given figure AQ and AR are the tangents drawn from A to the circle with centre O. If the measure of $\angle QOR$ is 120°, then $\angle QAR$ is equal to:



(A) 45° **(B)** 50°
(C) 65° **(D)** 60°

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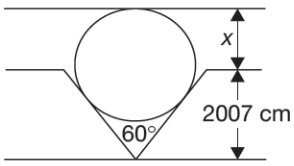
22. The sum of all angles except one of a convex polygon is 2190° (where the angles are less than 180°). Then the possible number of sides of the polygon is

(A) 13 **(B)** 15
(C) 17 **(D)** 19

23. In the adjoining figure the diameter of the circle is 2007 cm. Then x =



(A) $\frac{2007}{2}$ cm **(B)** $\frac{2007}{4}$ cm

(C) 2007 cm **(D)** $\frac{2}{3} \times 2007$ cm

24. Two circles touch externally at C. AB is a direct common tangent touching the circles at A, B. If AC = 14 cm, BC = 48 cm, then the length of the direct common tangent is _____.

(A) 50 **(B)** 45
(C) 48 **(D)** 40

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25. $ABCD$ is a square. With centres B and C and radius equal to the side of the square, circles are drawn to cut one another at E inside the square. $\angle BDE$ is equal to

(A) $22\frac{1}{2}°$ **(B)** 30°
(C) 15° **(D)** 37°

26. If two circles of radii 13 and 15 intersect and the length of the common chord is 24, then the distance between their centres is

(A) 10 **(B)** 12
(C) 14 **(D)** 16

27. C_1 and C_2 and two non-intersecting circles whose radii are in the ratio 1: 2. A third circle cuts the smaller circle at A and B and the bigger one at C and D. AB and CD intersect at P. The ratio of the lengths of the tangents form P to the circles C_1 and C_2 is:

(A) 1:4 **(B)** 1:8
(C) 1:1 **(D)** 1:2

28. The angles of a triangle are in the ratio 2:3:7. The length of the smallest side is 2012 cm. The radius of the circum circle of the triangle (in cm) is:

(A) 2013 **(B)** 2011
(C) 4024 **(D)** 2012

29. P_1 and P_2 are two regular polygons. The number of sides of P_1 and P_2, respectively, are in the ratio 3: 2 and the respective interior angles are in the ratio 10:9. Then the sum of the number of sides of P_1 and P_2 is _____.

(A) 15 **(B)** 20
(C) 25 **(D)** 30

30. Two circles touch both the arms of an angle whose measure is 60°. Both the circles also touch each other externally. The radius of the smaller circle is r. The radius of the bigger circle (in term of r) is

(A) $R = 3r$ **(B)** $R = 2r$
(C) $R = 4r$ **(D)** $R = 5r$

31. John's birthday cake is a delightful cylinder of radius 6 inches and height 3 inches, If these friends cut the cake into 8 equal sectors, the total surface area of a piece of birthday cake in sq. inches, is:

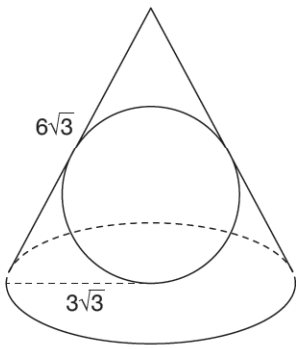
(A) $9(\pi + 4)$ **(B)** $\frac{27\pi}{2}$

(C) $4(\pi + 4)$ **(D)** $\frac{9}{2}(3\pi + 8)$

32. If a cone is cut into two parts by a horizontal plane passing through the mid-point of its axis, the ratio of the volumes of the upper part and the cone is:

(A) 1 : 2 **(B)** 1 : 4
(C) 1 : 6 **(D)** 1 : 8

33. A sphere is inscribed in a cone of radius $3\sqrt{3}$ and slant height $6\sqrt{3}$. The radius of the sphere, is:



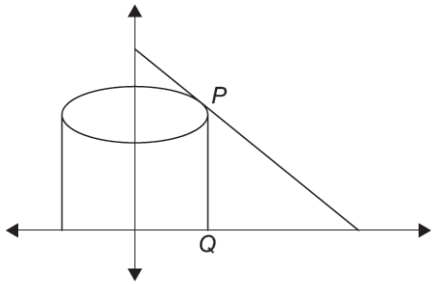
(A) 3 **(B)** $3\sqrt{3}$

(C) $6\sqrt{3}$ **(D)** $\frac{3\sqrt{3}}{2}$

34. A cone, a right cylinder, and a hemisphere stand on equal bases and have the same height. Their volumes are in the ratio:

(A) 1 : 2 : 3 **(B)** 1 : 3 : 2
(C) 2 : 3 : 1 **(D)** 2 : 1 : 3

35. Let P (4, k) be any point on the line $y = 6 - x$. If the vertical segment PQ is rotated about y-axis, the volume of the resulting cylinder is:



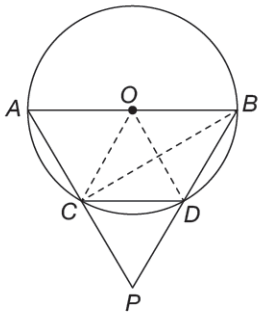
(A) 32π **(B)** 16π
(C) $\frac{32}{3}\pi$ **(D)** 8π

PART – II KVPY Level Problems

1. AB and AC are two chords of the circle of radius r. If $AB = 2AC$ and the perpendicular drawn from the centre on those chords are of lengths a and b. If $b^2 = k(a^2 + 3r^2)$ then $k =$
(A) 1 **(B)** 2
(C) 3 **(D)** 1/4

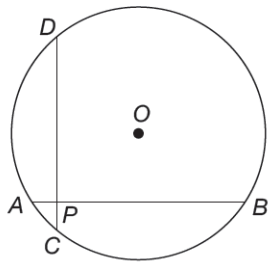
2. Let P be a point on the circumference of a circle. Perpendiculars PA and PB are drawn to points A and B on two mutually perpendicular diameters. If $AB = 36$ cm, the diameter of the circle is:
(A) 16 cm **(B)** 24 cm
(C) 36 cm **(D)** 72 cm

3. In the given figure, AB is the diameter of a circle C (O, r). Chord CD is equal to radius OC. If AC and BD intersect at P, find $\angle APB$.



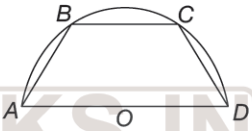
(A) 30° **(B)** 45°
(C) 60° **(D)** 90°

4. In the circle with centre 'O' as shown, chord AB and CD intersect at P and are perpendicular to each other. If $AP = 4$, $PB = 6$ and $PC = 2$, then the area of the circle is



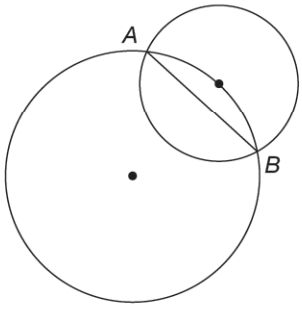
(A) 45π **(B)** 49π
(C) 50π **(D)** 41π

5. In the figure, the semi-circle centered at O has a diameter 6 cm. The chord BC is parallel to AD and $BC = \frac{1}{3}AD$. The area of the trapezium $ABCD$ in cm^2, is:



(A) 4 **(B)** $4\sqrt{2}$
(C) 8 **(D)** $8\sqrt{2}$

6. In the figure, the radius of the larger circle is 2 cm and the radius of the smaller circle is 1 cm and the larger circle passes through the center of the smaller circle. The length (in cm) of the chord AB is:



(A) $\frac{\sqrt{15}}{2}$ **(B)** 2
(C) $\frac{3}{2}$ **(D)** $\frac{5\sqrt{34}}{17}$

7. In an acute-angled triangle ABC, the altitudes from A, B, C when extended intersect the circumcircle again

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at points A_1, B_1, C_1, respectively. If $\angle ABC = 45°$ then $\angle A_1B_1C_1$ equals

(A) 45° **(B)** 60°
(C) 90° **(D)** 135°

8. In a right triangle ABC. the in circle touches the hypotenuse AC at D. If $AD = 10$ and $DC = 3$, the inradius of ABC is -

(A) 5 **(B)** 4
(C) 3 **(D)** 2

9. Three circles each of radius 1 touch one another externally and they lie between two parallel lines. The minimum possible distance between the lines is

(A) $2 + \sqrt{3}$ **(B)** $3 + \sqrt{3}$
(C) 4 **(D)** $2 + \frac{1}{\sqrt{3}}$

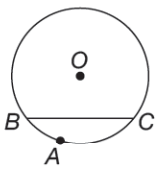
10. In ΔABC, angle B is obtuse. The smallest circle that covers the triangle is the

(A) Circumcircle
(B) Circle with AB as diameter
(C) Circle with BC as diameter
(D) Circle with AC as diameter

11. C_1 and C_2 are two circle in a plane. If N is the total number of common tangent then which of the following is wrong

(A) $N = 2$ when C_1 and C_2 intersect but do not touch
(B) $N = 4$ when C_1 and C_2 are disjoint
(C) when C_1 and C_2 touch then N must be 3
(D) N can never be more than 4

12. In the figure, BC is a chord of the circle with centre O and A is a point on the minor are BC. Then $\angle BAC - \angle OBC$ is equal to



(A) 30° **(B)** 60°
(C) 80° **(D)** 90°

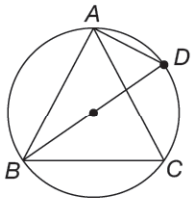
13. ABC is a triangle in which $AB = 4$ cm, $BC = 5$ cm and $AC = 6$ cm. A circle is drawn to touch side BC at P, side AB extended at Q and side AC extended at R. Then, AQ equals:

(A) 7.0 cm **(B)** 7.5 cm
(C) 6.5 cm **(D)** 15.0 cm

14. Let AP be a diameter of a circle of radius r and PT be the tangent to the circle at the point P such that the line AT intersects the circle at B. If $PT = 8$ units and $BT = 4$ units, then r is equal to

(A) $4\sqrt{3}$ units **(B)** 4 units
(C) $\frac{4}{\sqrt{3}}$ units **(D)** $2\sqrt{3}$ units

15. In figure, for ΔABC, chord $AB \cong$ chord BC, $\angle ABC = 72°$ and the angle bisector of $\angle ABC$ intersects the circle in Point D, then what is the measure of angle $\angle BDA$?



(A) 100° **(B)** 36°
(C) 18° **(D)** 54°

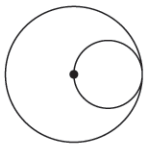
16. The product of the length of three sides of a triangle is 196 cm^3 and the radius of its circumcircle is 2.5 cm. The area of the triangle is:

(A) 39.2 cm^2 **(B)** 19.6 cm^2
(C) 32 cm^2 **(D)** 16.25 cm^2

17. The circle and the square have the same center and the same area. If the circle has radius 1, if the length of AB is, then find k

(A) 4 **(B)** 3
(C) 2 **(D)** 5

18. In the figure shown, the bigger circle has radius 1 unit. If the radius of smaller circle is $\frac{1}{k}$, then find k.

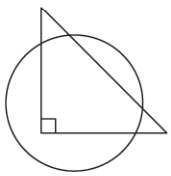


(A) 2 **(B)** 3
(C) 5 **(D)** 7

19. The common tangents to the non-intersecting circle I and II with centres P and Q meet the line joining P and Q at 'O'. Given that the length $OP = 28$ cm and the diameters of I and II are in the ratio 4: 3. If the length OQ is k cm of the circle II is k cm, then find some of digits of k.

(A) 1 **(B)** 2
(C) 3 **(D)** 4

20. Two legs of a right triangle are $8\sqrt{\pi}$, and $9\sqrt{\pi}$, as shown. A circle is drawn so that the area inside the circle but outside the triangle equals the area inside the triangle but outside the circle. The radius of the circle is (Use $\pi = 22/7$)



(A) $6\sqrt{\pi}$ **(B)** 6
(C) 5 **(D)** 8

21. Let P be a point on the circumference of a circle. Perpendiculars PA and PB are drawn to points A and B on two mutually perpendicular diameters, If $AB = 36$ cm, the diameter of the circle is:

(A) 16 cm **(B)** 24 cm
(C) 36 cm **(D)** 72 cm

22. A semi-circle is drawn with AB as its diameter. From C, a point on AB, a line perpendicular to AB is drawn meeting the circumference of the semi-circle at D,

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given that $AC = 2$ cm and $CD = 6$ cm, the area of the semi-circle is:

(A) $32\ \pi$ **(B)** $50\ \pi$
(C) $40\ \pi$ **(D)** $36\ \pi$

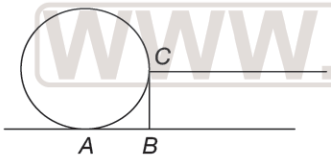
23. A triangle with side lengths in the ratio 3: 4: 5 is inscribed in a circle of radius 3. The area of the triangle is equal to:

(A) 8.64 **(B)** 12
(C) 6 **(D)** 10.28

24. Two circles, both of radii a, touch each other and each of them touches internally a circle of radius 2a. Then the radius of the circle which touches all the three circles is

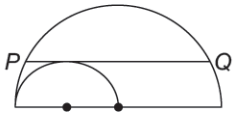
(A) $\frac{1}{2}$a **(B)** $\frac{2}{3}a$
(C) $\frac{3}{4}a$ **(D)** a

25. A hoop is resting vertically at stair case as shown in the diagram $AB = 12$ cm and $BC = 8$ cm. The radius of the hoop is



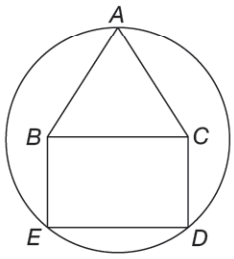
(A) 13 **(B)** $12\sqrt{2}$
(C) 14 **(D)** $13\sqrt{2}$

26. Two semi circles are constructed as shown in the figure. The chord PQ of the greater circle touches the smaller circle and is parallel to the diameter of larger circle. If the length of PQ is 10 units, then what is the area between the semi circles?



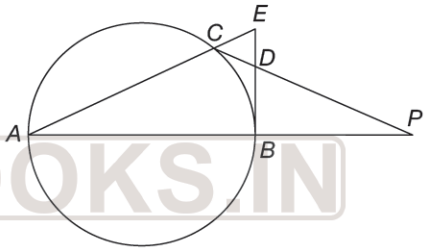
(A) 50 **(B)** 50π
(C) $\frac{1}{2}.25\pi$ **(D)** $\frac{1}{2}.25$

27. In the adjoining diagram ABC is an equilateral triangle and $BCDE$ is a square. The side of the equilateral triangles is 2010. The radius of the circle is:



(A) 2010 **(B)** 4020
(C) 6030 **(D)** 8040

28. In the adjoining figure AB is a diameter of a circle. AB is produced to P such that BP = radius of the circle. PC is a tangent to the circle. The tangent at B and AC produced cut at E. Then CDE is



(A) Isosceles with $EC = ED$
(B) isosceles with $EC = CD$
(C) Equilateral
(D) A scalene triangle

29. In a right triangle, a semi-circle is inscribed so that its diameter lies on the hypotenuse and its centre divides the hypotenuse in to two segments of length 15 cm and 20 cm. The length of the arc of the semi-circle between the points at which the legs touch the semi-circle is $K\ \pi$. Then the numerical value of K is _____.

(A) 4 **(B)** 5
(C) 6 **(D)** 7

30. In a quadrilateral, three consecutive sides are of lengths 2, 3, and 4. A circle of radius 1.2 is inscribed in the quadrilateral. The area of the quadrilateral is _____.

(A) 7.2 sq. units **(B)** 6.2 sq. units
(C) 7.5 sq. units **(D)** 6.5 sq. units

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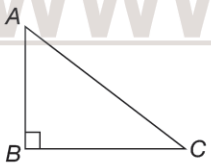
PART – III PRMO Level Problems

1. A ΔABC is inscribed in a circle of radius 4 cm. Suppose that $\angle A = 60°$, $AC - AB = 4$ cm, and the area of ΔABC is x cm^2. Find the value of $\left(\frac{x}{2}\right)^2$.

2. A ΔABC is inscribed in a circle of radius 1, with $\Delta BAC = 60°$. Altitudes AD and BE of ΔABC intersect at H. Find the smallest possible value of the length of the segment AH.

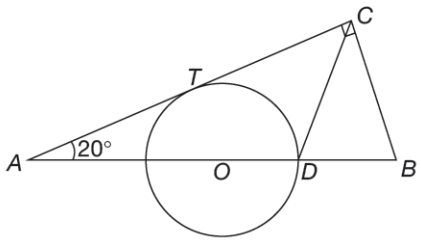
3. Four points in the order A, B, C, D lie on a circle with the extension of AB meeting the extension of DC at E and the extension of AD meeting the extension of BC at F. Let EP and FQ be tangents to this circle with points of tangency P and Q, respectively. Suppose $EP = 60$ and $FQ = 63$. Determine the length of EF.

4. ΔABC is a right-angled triangle with $\angle ABC = 90°$. A circle C_1 is drawn with AB as diameter, and another circle C_2 is drawn with BC as diameter. The circles C_1 and C_2 meet at the points B and P. If $AB = 5$ cm, $BC = 12$ cm and $BP = x$ cm, find the value of $\frac{120}{x}$.

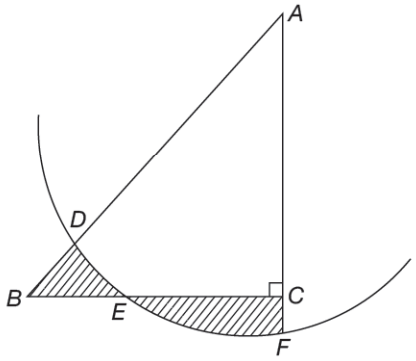


5. A triangle ΔABC has its vertices lying on a circle C of radius 1, with $\angle BAC = 60°$. A circle with I is inscribed in ΔABC. The line AI meets circle C again at D. Find the length of the segment ID.

6. In the triangle ABC, $\angle A = 20°$, $\angle C = 90°$, O is a point on AB and D is the midpoint of OB. Suppose the circle centered at O with radius OD touches the side AC at T. Determine the size of $\angle BCD$ in degrees.



7. In the following right-angled triangle ABC, $AC = BC = 1$, and DEF is an arc of a circle with center A. Suppose the shaded areas BDE and CEF are equal and $AD = \frac{x}{\sqrt{\pi}}$. What is the value of x?



8. Let O be the centre of the circumcircle of ΔABC, P and Q the midpoints of AO and BC, respectively. Suppose $\angle CBA = 4\angle OPQ$ and $\angle ACB = 6\angle OPQ$. Find the size of $\angle OPQ$ in degrees.

9. In the triangle ABC, $AC = 2BC$, $\angle C = 90°$ and D is the foot of the altitude from C onto AB. A circle with diameter AD intersects the segment AC at E. Find $AE : EC$.

10. A ΔABC is inscribed in a circle of radius 1, with $\angle BAC = 60°$. Altitudes AD and BE of ΔABC intersect at H. Find the smallest possible value of the length of the segment AH.

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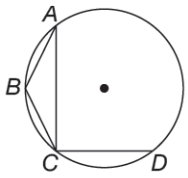
11. Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of ΔABC traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

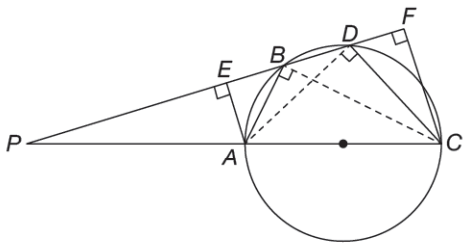
12. In the following figure, $AB = BC$, $AC = CD$, and $\angle ACD = 90°$. If the radius of the circle is "r" units, then find the length of the chord BC.



(A) $r\sqrt{\sqrt{2}-1}$ (B) $r\sqrt{\sqrt{3}-1}$

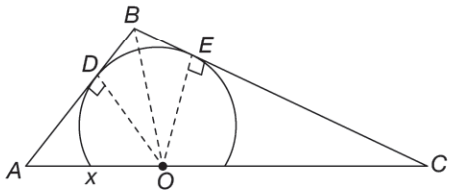
(C) $r\sqrt{2-\sqrt{2}}$ (D) $r\sqrt{2-\sqrt{3}}$

13. A circle with diameter $\overline{AC}$ is intersected by a secant at point B and D. The secant and the diameter intersect at point P outside the circle. Perpendiculars $\overline{AE}$ and $\overline{CF}$ are drawn from the extremities of the diameter to the secant. If $EB = 2$, and $BD = 6$, find DF.



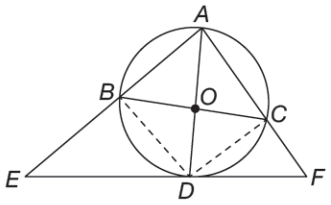
(A) 2 **(B)** 3
(C) 4 **(D)** 5

14. In ΔABC, in which $AB = 12$, $BC = 18$, and $AC = 25$, a semi-circle is drawn so that diameter lies on $\overline{AC}$, and so that it is tangent to $\overline{AB}$ and $\overline{BC}$. If O is the center of the circle, find the measure of $\overline{AO}$.



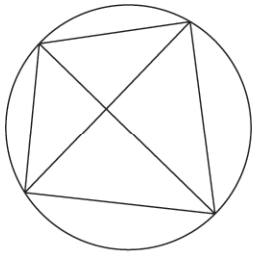
(A) 10 **(B)** 20
(C) 30 **(D)** 40

15. Triangle ABC is inscribed in a circle with diameter $\overline{AD}$. A tangent to the circle at D cuts $\overline{AB}$ extended at E and $\overline{AC}$ extended at F. If $AB = 4$, $AC = 6$, and $BE = 8$, find CF.



(A) 1 **(B)** 2
(C) 4 **(D)** 5

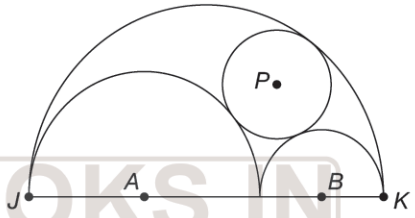
16. n dots are drawn on the circumference of a circle. By joining all the dots to one another by straight lines, the maximum number of regions that can be formed in the circle is counted. For example, when $n = 4$ the maximum number of regions is 8.



What is the maximum number of regions that can be formed when $n = 7$?

17. ΔABC is a triangle such that $\angle C = 90°$. Suppose $AC = 156$ cm, $AB = 169$ cm, and the perpendicular distance from C to AB is x cm. Find the value of x.

18. In the figure below, semi-circles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semi-circle with diameter $\overline{JK}$. The two smaller semi-circles are externally tangent to each other and internally tangent to the largest semi-circle. A circle centered at P is drawn externally tangent to the two smaller semi-circles and internally tangent to the largest semi-circle. Radius of the circle centered at P is $\frac{a}{b}$ where a and b are co-prime. What is $5a + 6b$?



19. The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and the line ED is perpendicular to the line AD. Segment $\overline{AE}$ intersects the circle at point C between A and E. The area of ΔABC is $\frac{P}{S}$ where P and S are co-prime. What is $P - 2S$?

20. In ΔABC, $AB = 86$ and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

21. Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. The radius of circle C_1 is $\sqrt{R}$. What is R?

22. Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2\,AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. The ratio of the area of ΔDCE to the area of ΔABD is $\frac{21}{\gamma}$. What is γ?

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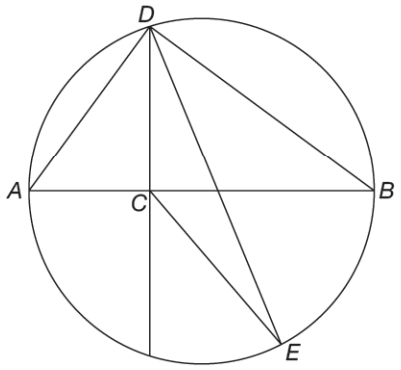
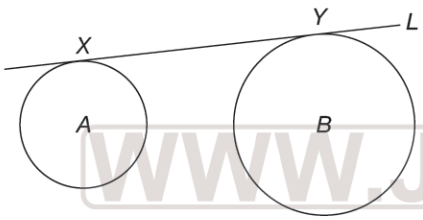
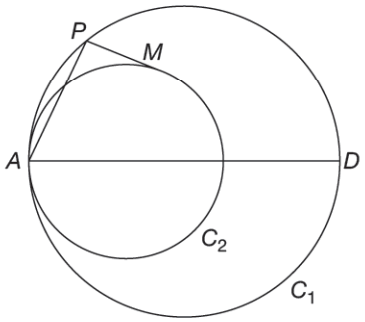


23. The figure below shows two circles with centres A and B, and a line L which is a tangent to the circles at X and Y. Suppose that $XY = 40$ cm, $AB = 41$ cm and the area of the quadrilateral $ABYX$ is 300 cm^2. If a and b denote the areas of the circles with centre A and centre B, respectively, find the value of $\frac{b}{a}$.



24. Two circles C_1 and C_2 of radii 10 cm and 8 cm respectively are tangent to each other internally at a point A. AD is the diameter of C_1 and P and M are points on C_1 and C_2 respectively such that PM is tangent to C_2, as shown in the figure below. If $PM = \sqrt{20}$ cm and $\angle PAD = X^\circ$, find the value of X.



25. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1, perpendicular to CD at N with $AN > NB$. A circle Γ_2 centred at C with radius CN intersects Γ_1 at points P and Q, and the segments PQ and CD intersect at M. Given that the radii of Γ_1 and Γ_2 are 61 and 60, respectively, find the length of AM.

26. Let B be a point on the circle centred at O with diameter AC and let D and E be the circumcentres of the triangles OAB and OBC, respectively. Given that $\sin \angle BOC = \frac{4}{5}$ and $AC = 24$, find the area of the triangle BDE.

27. Circle A has radius 100. Circle B has an integer radius $r < 100$ and remains internally tangent to circle A as it rolls once around the circumference of circle A. The two circles have the same points of tangency at the beginning and end of circle B's trip. How many possible values can r have?

28. Many cathedrals have windows with portions containing a ring of congruent circles that are circumscribed by a larger circle. In the figure shown, the number of smaller circles is four. If the ratio of the sum of the areas of the four smaller circles to the area of the larger circle is $2[x + 4y\sqrt{2}]$ then find the value of $x^2 + y^3$.

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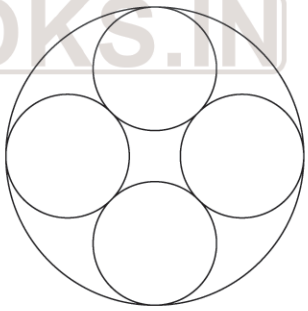
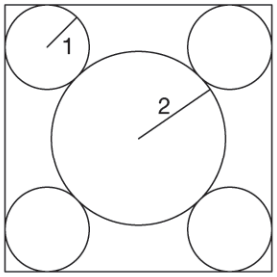
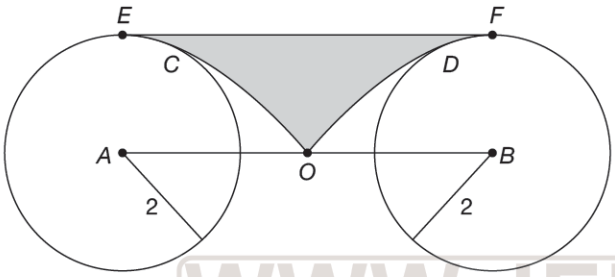


29. Points A and B lie on a circle centered at O, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. If the ratio of the area of the smaller circle to that of the larger circle is $\frac{x}{y}$ then find the value of $x^2 + y^2$.

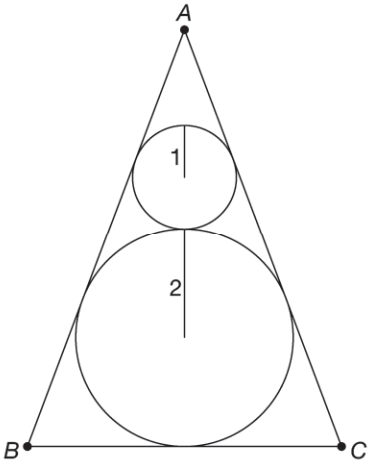
30. Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2, as shown. If the area of the square is $x + y\sqrt{2}$, then find the value of $\frac{5x}{x - y}$?



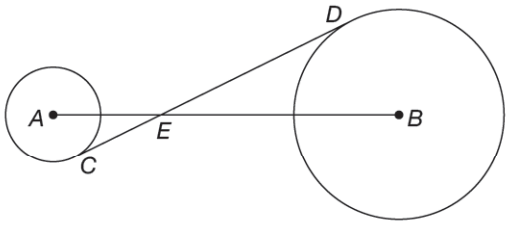
31. Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and EF is a common tangent. What is the area of the shaded region $ECODF$ to the nearest integer? (Take $\pi = 3.14$)



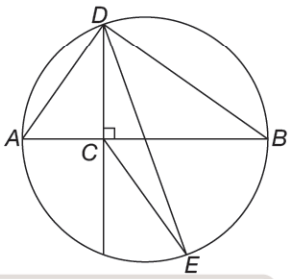
32. A circle of radius 1 is tangent to a circle of radius 2. The sides of ΔABC are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of ΔABC to the nearest integer?



33. Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent intersect at C and D, respectively. Lines AB and CD intersect at E, and $AE = 5$. What is $3CD$?



34. Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \cdot AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. If the ratio of the area of ΔDCE to the area of ΔABD is $\frac{x}{y}$, then find $\sqrt{\frac{1}{x} + y}$.



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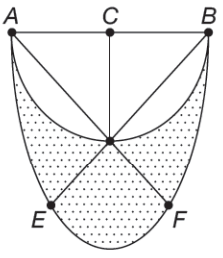
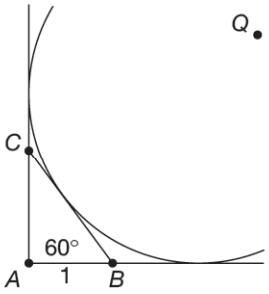
35. Semi-circle $\overset{\frown}{AB}$ has center C and radius 1. Point D is on $\overset{\frown}{AB}$ and $\overline{CD} \perp \overline{AB}$. Extend $\overset{\frown}{BD}$ and $\overset{\frown}{AD}$ to E and F, respectively, so that circular arcs $\overset{\frown}{AE}$ and $\overset{\frown}{BF}$ have B and A as their respective centers and circular arc EF has center D. The area of the shaded "smile", $AEFBDA$ is $(2\pi - \pi\sqrt{2} - k)$, find k.



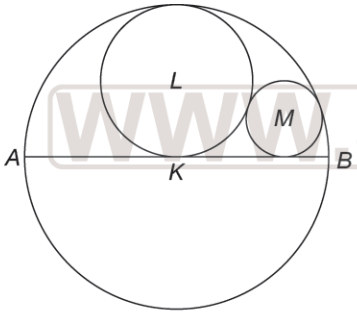
36. A circle of radius r has chord $\overline{AB}$ of length 10 and $\overline{CD}$ of length 7. When $\overline{AB}$ and $\overline{CD}$ are extended through B and C, respectively, they intersect at P, which is outside the circle. If $\angle APD = 60°$ and $BP = 8$, then $r^2 =$

37. A circle with centre O is tangent to the coordinate axes and to the hypotenuse of the $30° - 60° - 90°$ triangle ABC as shown, where $AB = 1$. What is the radius of the circle (round off to nearest natural number)?



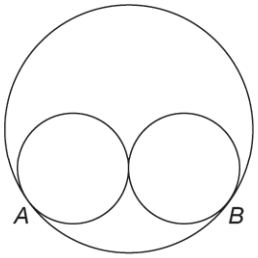
38. A circle is circumscribed about a triangle with sides 3, 4, and 5, thus dividing the interior of the circle into four regions. Let A, B, and C be the areas of the non-triangular regions, with C being the largest. Then $-(A + B - C) = ?$

39. In the adjoining figure, circle K has diameter AB; circle L is tangent to circle K and to AB at the center of circle K; and circle M is tangent to circle K, to circle L and to AB. The ratio of the area of circle K to the area of circle M is



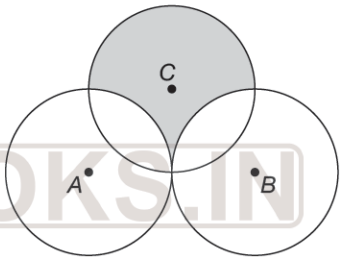
40. A circle with area A_1 is contained in the interior of a larger circle with area $A_1 + A_2$. If the radius of the larger circle is 3, and if A_1, A_2, $A_1 + A_2$ is an arithmetic progression, and the radius of the smaller circle is given by r. Find r^2.

41. Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

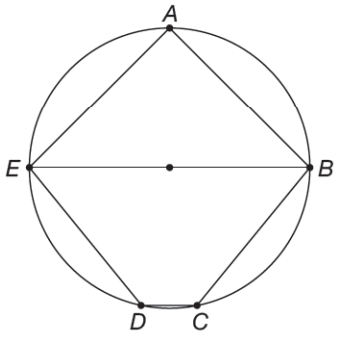


42. Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. If the area of this triangle is A, find $\frac{4A}{\sqrt{3}}$.

43. Circles A, B, and C each have radius 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}$. What is the area inside Circle C but outside circle A and circle B?



44. In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio 4 : 5. If the degree measure of angle BCD is k, find $\frac{k}{5}$.



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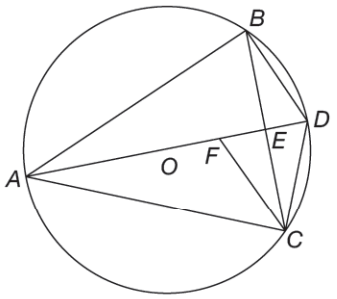
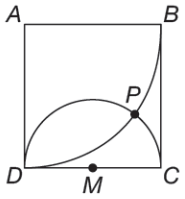
45. A circle with center O has area 156π. Triangle ABC is equilateral, $\overline{BC}$ is a chord on the circle, $OA = 4\sqrt{3}$, and point O is outside ΔABC. What is the side length of ΔABC?

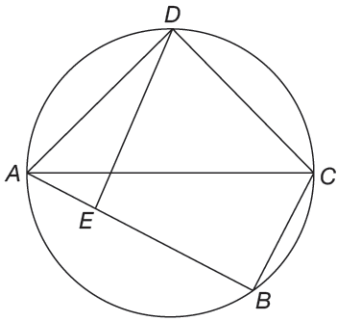
46. In the figure below, ABC is an isosceles triangle inscribed in a circle with centre O and diameter AD, with $AB = AC$. AD intersects BC at E, and F is the midpoint of OE. Given that BD is parallel to FC and $BC = 2\sqrt{5}$ cm, find square of the length CD (in cm).



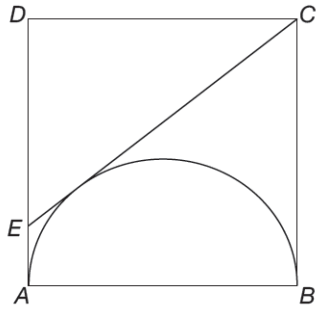
47. Square $ABCD$ has sides of length 4, and M is the midpoint of $\overline{CD}$. A circle with radius 2 and center M intersects a circle with radius 4 and center A at points P and D. If the distance from P to $\overline{AD}$ is $\frac{a}{b}$, then $a + b = ?$



48. Let $ABCD$ be a quadrilateral inscribed in a circle with diameter AC, and let E be the foot of perpendicular from D onto AB, as shown in the figure below. If $AD = DC$ and the area of quadrilateral $ABCD$ is 24 cm^2, find the square of Length of DE (in cm).



49. Square $ABCD$ has side length 2. A semi-circle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semi-circle from C intersects side $\overline{AD}$ at E. Find 2 $\overline{CE}$.



50. A right circular cylinder with its diameter equal to its height is inscribed in a right circular cone. The cone has diameter 10 and altitude 12, and the axes of the cylinder and cone coincide. Then the radius of the cylinder is R. Find the value of (11R).

51. A quadrilateral is inscribed in a circle of radius $20\sqrt{2}$. Three of the sides of this quadrilateral have length 20. What is the length of the fourth side?

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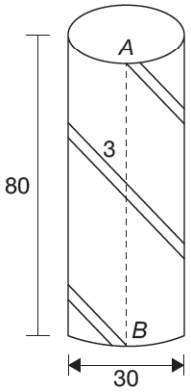
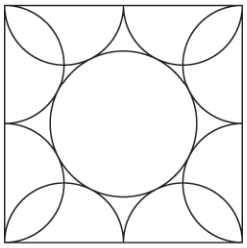
52. A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

53. A white cylindrical silo has a diameter of 30 feet and a height of 80 feet. A red stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. Find $\frac{1}{3}$rd of the area of the stripe in square feet.



54. An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. If the ratio of the cone's height to its radius is given by a/b. Find sum of a & b.

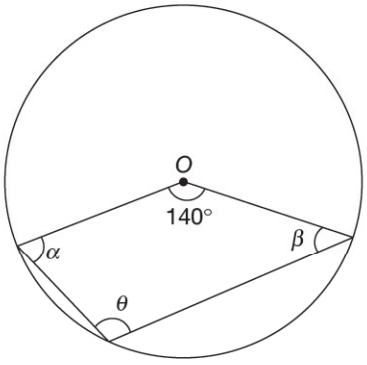
55. Eight semi-circles line the inside of a square with side length 2 as shown. If the radius of the circle tangent to all of these semi-circles is given by $\frac{\sqrt{a}-1}{b}$, where a and b are prime number. Find the product of a and b.



EXERCISE – II

PART – I Previous Years' NSEJS Problems

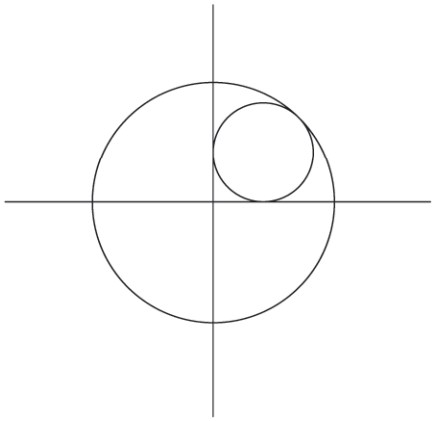
1. In the figure shown: **[NSEJS-2008]**



(A) $\alpha+\beta=110°$ **(B)** $\alpha+\beta=55°$

(C) $\theta=110°$ **(D)** $\theta=140°$

2. In the figure shown, the bigger circle has radius 1 unit. Therefore, the radius of smaller circle must be **[NSEJS-2009]**



(A) $\sqrt{2}+1$ **(B)** $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ **(D)** $\frac{1}{\sqrt{2}+1}$

3. Two circles of radii 2 and 3 cm touch each other externally. The length of direct common tangent to the two circles will be: **[NSEJS-2009]**

(A) $2\sqrt{6}$ cm **(B)** $\sqrt{26}$ cm

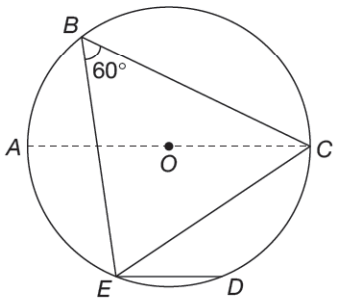
(C) 5 cm **(D)** 2.4 cm

4. In case of a right circular cylinder, the radius of base and height are in the ratio 2: 3. Therefore, the ratio of lateral surface area to the total surface area is: **[NSEJS-2010]**

(A) 5: 3 **(B)** 3: 5

(C) 2: 5 **(D)** 2: 3

5. In the figure shown, chord ED is parallel to diameter AC of a circle. If $\angle CBE = 60°$, then $\angle DEC$ must be: **[NSEJS-2010]**



(A) 15° **(B)** 30°

(C) 10° **(D)** 20°

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6. The hypotenuse of a right-angled triangle is 10 cm and the radius of its inscribed circle is 1 cm. Therefore, perimeter of the triangle is: **[NSEJS-2010]**

(A) 22 cm **(B)** 24 cm
(C) 26 cm **(D)** 30 cm

7. The area of the circle is double when its radius r is increased by a. Therefore, radius r equals. **[NSEJS-2010]**

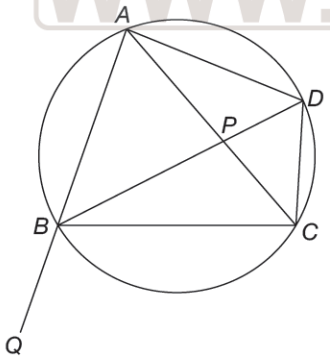
(A) $\left(\sqrt{2}+1\right)a$ **(B)** $\left(\sqrt{2}-1\right)a$

(C) a **(D)** $\left(2-\sqrt{2}\right)a$

8. A point P is outside a circle at a distance of 13 cm from its centre. A secant from P cuts the circle in Q and R such that $QR = 7$ cm and the segment PQ of the secant exterior to the circle is 9 cm. Therefore, the radius of circle is: **[NSEJS-2010]**

(A) 3 cm **(B)** 4 cm
(C) 5 cm **(D)** 6 cm

9. In the figure shown below, $m\angle ABD = 2x$, $m\angle ADB = 3x$, $m\angle APB = 4x$ and $m\angle CBQ = 7x$. Therefore, $m\angle BCD$ is **[NSEJS-2011]**



(A) 90° **(B)** 85°
(C) 72° **(D)** 96°

10. A conical vessel of radius 6 cm and height 8 cm is completely filled with water. A meter sphere is now lowered into the water. The size of the sphere is such that when it touches the inner surface, it just gets immersed. The fraction of water that overflows from the conical vessel is **[NSEJS-2011]**

(A) $\frac{3}{8}$ **(B)** $\frac{5}{8}$

(C) $\frac{7}{8}$ **(D)** $\frac{5}{16}$

11. A sphere and cube have equal surface areas. The ratio of their volumes is **[NSEJS-2011]**

(A) $\frac{\pi}{6}$ **(B)** $\frac{6}{\pi}$

(C) $\sqrt{\frac{6}{\pi}}$ **(D)** $\sqrt{\frac{\pi}{3}}$

12. An equilateral triangle has area A $\sqrt{3}$. Three circles are drawn with their centers at the vertices of the triangle. Diameter of each circle is equal to the length of each side of the triangle. The area of the triangle *NOT* included in any of the three circles is **[NSEJS-2011]**

(A) $A\left(\sqrt{3}-\frac{\pi}{6}\right)$ **(B)** $A\left(\pi-\sqrt{3}\right)$

(C) $A\left(3\pi-\sqrt{3}\right)$ **(D)** $A\left(\sqrt{3}-\frac{\pi}{2}\right)$

13. What is the radius of the circumcircle of a triangle whose sides are 30 cm, 36 cm, and 30 cm in length? **[NSEJS-2012]**

(A) 18 cm **(B)** 18.25 cm
(C) 18.50 cm **(D)** 18.75 cm

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14. A circle is inscribed in an isosceles trapezium $ABCD$ in which AB is parallel to DC. If $AB = 10$ and $DC = 30$, Find the area of the circle. **[NSEJS-2012]**

(A) 45 π **(B)** 50 π
(C) 60 π **(D)** 75 π

15. If $ABCD$ is a cyclic quadrilateral. $AB = 204$, $BC = 104$, $CD = 195$, $DA = 85$ and $BD = 221$, then find AC **[NSEJS-2012]**

(A) 205 **(B)** 210
(C) 220 **(D)** 225

16. Two circles each of radius 3 touch each other externally in the plane. In how many ways can a circle of radius 8 be placed in the plane touching each of these two circles? **[NSEJS-2013]**

(A) 2 **(B)** 4
(C) 6 **(D)** 8

17. Around a lawn which is of semicircular shape a pavement of uniform width is laid. If the ratio of the area of the lawn to the area of the pavement is 25: 24, then the ratio of the outer and inner perimeters of the pavement is **[NSEJS-2013]**

(A) $\frac{7}{5}$ **(B)** $\frac{6}{5}$

(C) $\frac{5}{4}$ **(D)** $\frac{5}{2\sqrt{6}}$

18. If the radius of the base of a cone is doubled then the slant area becomes 3 times the original slant area. Suppose when the radius of the base of the cone is quadrupled (that is increased to 4 times), the slant area becomes k times the original slant area. Then the integer closest to k is **[NSEJS-2013]**

(A) 6 **(B)** 8
(C) 10 **(D)** 11

19. If set of marbles, of radius 5 cm, is poured into a cube of side 1 m. The maximum number of marbles that can be filled into the box are **[NSEJS-2014]**

(A) 2000 **(B)** 1000
(C) 1500 **(D)** 3000

20. A round table cover has six equal designs as shown in the adjacent figure. If the radius of the cover is 4 cm, then cost of making the designs at the rate of ₹10.00 per cm^2 (round off your answer to a nearest rupee) is **[NSEJS-2014]**

(A) ₹85 **(B)** ₹86
(C) ₹90 **(D)** ₹87

21. Two parallel chords 96 cm and 28 cm long are on the opposite side of the centre of the circle with radius 50 cm. Find the area of the quadrilateral whose vertices are the end points of the chords. **[NSEJS-2015]**

(A) 3488 **(B)** 3848
(C) 3844 **(D)** 3484

22. Total surface area of a sphere S with radius $\sqrt{2}+\sqrt{3}$ cm is: **[NSEJS-2016]**

(A) $400\,\pi\,(5+2\sqrt{6})$ sq mm
(B) $\pi(\sqrt{2}+\sqrt{3})^2$ sq cm
(C) $2\pi(\sqrt{2}+\sqrt{3})^2$ sq cm
(D) $40\pi(5+2\sqrt{6})$ sq mm

23. A craft teacher reshapes the wax from a cylinder of candle with section diameter 6 cm and the height 6 cm into a sphere. The radius of this sphere will be: **[NSEJS-2016]**

(A) $r = 6\sqrt{3/2}$ cm **(B)** $r = 6$ cm
(C) $r = 3\sqrt[3]{3/2}$ cm **(D)** $r = 6\sqrt[3]{2}$ cm

24. In a n-sided regular polygon, the radius of the circum-circle is equal in length to the shortest diagonal. The number of values of $n > 60$ for which this can happen is: **[NSEJS-2016]**

(A) 0 **(B)** 1
(C) 10 **(D)** 2

25. If $\square ABCD$ is a cyclic quadrilateral, $AB = 204$, $BC = 104$, $CD = 195$, $DA = 85$ and $BD = 221$, then $AC =$ _____. **[NSEJS-2017]**

(A) 210 **(B)** 220
(C) 225 **(D)** 240

26. What is the radius of the circumcircle of a triangle whose sides are 30 cm, 36 cm, and 30 cm. **[NSEJS-2017]**

(A) 15 cm **(B)** 16 cm
(C) 17 cm **(D)** 18 cm

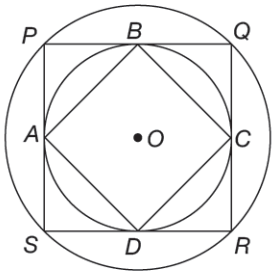
27. Let AB be diameter of a circle C_1 of radius 30 cm with center O. Two circles C_2 and C_3 of radii 15 cm and 10 cm touch C_1 internally at A and B, respectively. A fourth circle C_4 touches C_1, C_2, and C_3. What is the largest possible radius of C_4? **[NSEJS-2018]**

(A) 12 cm **(B)** 15 cm
(C) 20 cm **(D)** 30 cm

28. Let ABC be an equilateral triangle. The bisector of $\angle BAC$ meets the circumcircle of ABC in D. Suppose $DB + DC = 4$. The diameter of the circumcircle of ABC is **[NSEJS-2018]**

(A) 4 **(B)** $3\sqrt{3}$
(C) $2\sqrt{3}$ **(D)** 2

29. In the given figure, two concentric circles are shown with centre O. $PQRS$ is a square inscribed in the outer circle. It also circumscribes the inner circle, touching it at points B, C, D, and A. What is the ratio of the perimeter of the outer circle to that of quadrilateral $ABCD$. **[NSEJS-2019]**



(A) $\frac{\pi}{4}$ **(B)** $\frac{3\pi}{2}$
(C) $\frac{\pi}{2}$ **(D)** π

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30. In an equilateral triangle, three coins of radii 1 unit each are kept so that they touch each other and also sides of the triangle. The area of triangle ABC (in sq. units) is **[NSEJS-2019]**

(A) $4+2\sqrt{3}$ **(B)** $4\sqrt{3}+6$

(C) $12+\frac{7\sqrt{3}}{4}$ **(D)** $3+\frac{7\sqrt{3}}{4}$

PART – II Previous Years' KVPY Problems

1. If the volume of a sphere increases by 72.8%, then its surface area increases by **[KVPY-2009]**

(A) 20% **(B)** 44%

(C) 24.3% **(D)** 48.6%

2. A line segment l of length a cm is rotated about a vertical line L keeping the line l in one of the following three positions (I) l is parallel to L and is at a distance of r cm from L, (II) l is perpendicular to L and its mid-point is at a distance r cm from L, (III) l and L are in the same plane and l is inclined to L at an angle 30° with its midpoint at a distance r cm from L. Let A_1, A_2, A_3 be the areas so generated. If $r > (a/2)$, then **[KVPY-2009]**

(A) $A_1 < A_3 < A_2$ **(B)** $A_2 < A_1 < A_3$

(C) $A_1 = A_3 < A_2$ **(D)** $A_1 = A_2 = A_3$

3. The three different face diagonals of a cuboid (rectangular parallelepiped) have lengths 39, 40, 41. The length of the main diagonal of the cuboid that joins a pair of opposite corners is **[KVPY-2010]**

(A) 49 **(B)** $49\sqrt{2}$

(C) 60 **(D)** $60\sqrt{2}$

4. A cow is tied to a corner (vertex) of a regular hexagonal fenced area of side a m by a rope of length $\frac{5a}{2}$ m in a grass field (the cow cannot graze inside the fenced area). What is the maximum possible area of the grass field to which the cow has access to graze? **[KVPY-2010]**

(A) $5\pi a^2$ **(B)** $\frac{5}{2}\pi a^2$

(C) $6\pi a^2$ **(D)** $3\pi a^2$

5. A closed conical vessel is filled with water fully and is placed with its vertex down. The water is let out at a constant speed. After 21 min, it was found that the height of the water column is half of the original height. How much more time in minutes does it require to empty the vessel? **[KVPY-2010]**

(A) 21 **(B)** 14

(C) 7 **(D)** 3

6. A piece of paper in the shape of a sector of a circle (see Fig. 1) is rolled up to form a right-circular cone (see Fig. 2). The value of the angle θ is: **[KVPY-2011]**

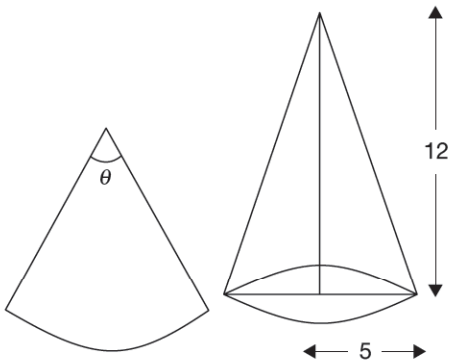

Fig. 1 Fig. 2

(A) $\frac{10\pi}{13}$ **(B)** $\frac{9\pi}{13}$

(C) $\frac{5\pi}{13}$ **(D)** $\frac{6\pi}{13}$

7. Three circles each of radius 1 touch one another externally and they lie between two parallel lines. The minimum possible distance between the lines is **[KVPY-2011]**

(A) $2+\sqrt{3}$ **(B)** $3+\sqrt{3}$

(C) 4 **(D)** $2+\frac{1}{\sqrt{3}}$

8. Consider an incomplete pyramid of balls on a square base having 18 layers, and having 13 balls on each side of the top layer. Then, the total number N of balls in that pyramid satisfies **[KVPY-2011]**

(A) $9000 < N < 10000$

(B) $8000 < N < 9000$

(C) $7000 < N < 8000$

(D) $10000 < N < 12000$

9. A frog is presently located at the origin (0, 0) in the XY-plane. It always jumps from a point with integer coordinates to a point with integer coordinates moving a distance of 5 units in each jump. What is the minimum number of jumps required for the frog to go from (0, 0) to (0, 1)? **[KVPY-2011]**

(A) 2 **(B)** 3

(C) 4 **(D)** 9

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10. The diameter of one of the bases of a truncated cone is 100 mm. If the diameter of this base is increased by 21% such that it still remains a truncated cone with the height and the other base unchanged, the volume also increases by 21%. The radius of the other base (in mm) is **[KVPY-2011]**

(A) 65 **(B)** 55
(C) 45 **(D)** 35

11. In ΔABC, let AD, BE, and CF be the internal angle bisectors with D, E, and F on the sides BC, CA, and AB, respectively. Suppose AD, BE, and CF concur at I and B, D, I, F are concyclic, then $\angle IFD$ has measure **[KVPY-2012]**

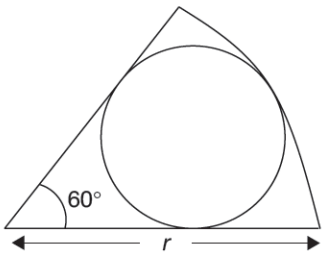
(A) 15° **(B)** 30°
(C) 45° **(D)** any value $\leq 90°$

12. A regular octagon is formed by cutting congruent isosceles right-angled triangles from the corners of a square. If the square has side length 1, the side length of the octagon is **[KVPY-2012]**

(A) $\frac{\sqrt{2}-1}{2}$ **(B)** $\sqrt{2}-1$

(C) $\frac{\sqrt{5}-1}{4}$ **(D)** $\frac{\sqrt{5}-1}{3}$

13. A circle is drawn in a sector of a larger circle of radius r, as shown in the figure given below.

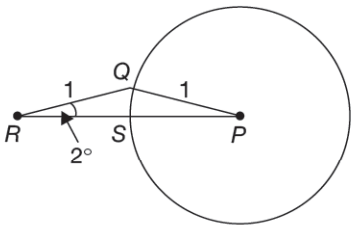


The smaller circle is tangent to the two bounding radii and the arc of the sector. The radius of the small circle is **[KVPY-2012]**

(A) $\frac{r}{2}$ **(B)** $\frac{r}{3}$

(C) $\frac{2\sqrt{3}r}{5}$ **(D)** $\frac{r}{\sqrt{2}}$

14. Suppose Q is a point on the circle with centre P and radius 1, as shown in the figure, R is a point outside the circle such that $QR = 1$ and $\angle QRP = 2°$. Let S be the point where the segment RP intersects the given circle. Then, measure of $\angle RQS$ equals **[KVPY-2012]**



(A) 86° **(B)** 87°
(C) 88° **(D)** 89°

15. Tangents to a circle at points P and Q on the circle intersect at a point R. If $PQ = 6$ and $PR = 5$, then the radius of the circle is **[KVPY-2013]**

(A) $\frac{13}{3}$ **(B)** 4

(C) $\frac{15}{4}$ **(D)** $\frac{16}{5}$

16. In an acute-angled ΔABC, the altitudes from A, B, C when extended intersect the circum-circle again at points A_1, B_2, C_1, respectively. If $\angle ABC = 45°$, then $\angle A_1 B_1 C_1$ equals **[KVPY-2013]**

(A) 45° **(B)** 60°
(C) 90° **(D)** 135°

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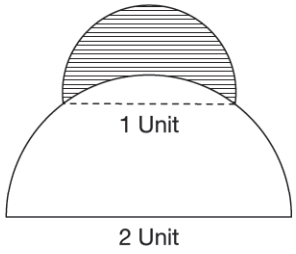
17. On each face of a cuboid, the sum of its perimeter and its area is written. Among the six numbers so written, there are three distinct numbers and they are 16, 24 and 31. The volume of the cuboid lies between **[KVPY-2013]**

(A) 7 and 14 **(B)** 14 and 21
(C) 21 and 28 **(D)** 28 and 35

18. Suppose the height of a pyramid with a square base is decreased by $p\%$ and the lengths of the sides of its square base are increased by $p\%$ (where, $p > 0$). If the volume remains the same, then **[KVPY-2013]**

(A) $50 < p < 55$ **(B)** $55 < p < 60$
(C) $60 < p < 65$ **(D)** $65 < p < 70$

19. A semi-circle of diameter 1 unit sits at the top of a semi-circle of diameter 2 units.

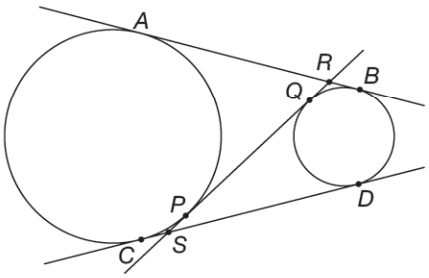


The shaded region inside the smaller semi-circle but outside the larger semi-circle is called a lune. The area of the lune is **[KVPY-2014]**

(A) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ **(B)** $\frac{\sqrt{3}}{4}-\frac{\pi}{24}$

(C) $\frac{\sqrt{3}}{4}-\frac{\pi}{12}$ **(D)** $\frac{\sqrt{3}}{4}-\frac{\pi}{8}$

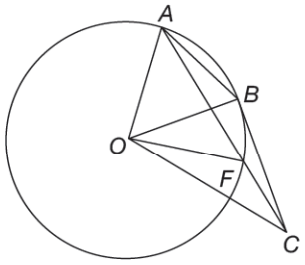
20. Suppose S_1 and S_2 are two unequal circles, AB and CD are the direct common tangents to these circles. A transverse common tangent PQ cuts AB in R and CD in S. If $AB = 10$, then RS is **[KVPY-2014]**



(A) 8 **(B)** 9

(C) 10 **(D)** 11

21. On the circle with center O, points A and B are such that $OA = AB$. A point C is located on the tangent at B to the circle such that A and C are on the opposite sides of the line OB and $AB = BC$. The line segment AC intersects the circle again at F. Then, the ratio $\angle BOF : \angle BOC$ is equal to **[KVPY-2014]**



(A) 1: 2 **(B)** 2: 3

(C) 3: 4 **(D)** 4: 5

22. Let $ABCD$ be a square of side length 1, and Γ a circle passing through B and C, and touching AD. The radius of Γ is **[KVPY-2015]**

(A) $\frac{3}{8}$ **(B)** $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ **(D)** $\frac{5}{8}$

23. Given are three cylindrical buckets X, Y, Z whose circular bases are of radii 1, 2, 3 units, respectively. Initially water is filled in these buckets upto the same height. Some water is then transferred from Z to X so that they both have the same volume of water. Some water is then transferred between X and Y so that they both have the same volume of water. If h_y, h_z denote the heights of water at this stage in the buckets Y, Z, respectively, then the ratio $\frac{h_y}{h_z}$ equals **[KVPY-2015]**

(A) $\frac{4}{9}$ **(B)** 1

(C) $\frac{9}{4}$ **(D)** $\frac{81}{40}$

24. The points A, B, C, D, E are marked on the circumference of a circle in clockwise direction such that $\angle ABC = 130°$ and $\angle CDE = 110°$. The measure of $\angle ACE$ in degree is **[KVPY-2016]**

(A) 50° **(B)** 60°

(C) 70° **(D)** 80°

25. Three circles of radii 1, 2, and 3 units, respectively touch each other externally in the plane. The circum-radius of the triangle formed by joining the centers of the circles is **[KVPY-2016]**

(A) 1.5 **(B)** 2

(C) 2.5 **(D)** 3

26. Let P be a point inside a ΔABC with $\angle ABC = 90°$. Let P_1 and P_2 be the images of P under reflection in AB and BC, respectively. The distance between the circum-centers of ΔABC and P_1PP_2 is **[KVPY-2016]**

(A) $\frac{AB}{2}$ **(B)** $\frac{AP+BP+CP}{3}$

(C) $\frac{AC}{2}$ **(D)** $\frac{AB+BC+AC}{2}$

27. Let a and b be two positive real numbers such that $a + 2b \le 1$. Let A_1 and A_2 be, respectively, the areas of circles with radii ab^3 and b^2. Then, the maximum possible value of $\frac{A_1}{A_2}$ is **[KVPY-2016]**

(A) $\frac{1}{16}$ **(B)** $\frac{1}{64}$

(C) $\frac{1}{16\sqrt{2}}$ **(D)** $\frac{1}{32}$

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28. Consider a cuboid all of whose edges are integers and whose base is a square. Suppose the sum of all its edges is numerically equal to the sum of the areas of all its six faces. Then, the sum of all its edges is **[KVPY-2016]**

(A) 12 **(B)** 18
(C) 24 **(D)** 36

29. Consider a semi-circle of radius 1 unit constructed on the diameter AB and let O be its centre. Let C be a point on AO such that $AC: CO = 2: 1$. Draw CD perpendicular to AO with D on the semi-circle. Draw OE perpendicular to AD with E on AD. Let OE and CD intersect at H. Then, DH equals **[KVPY-2016]**

(A) $\frac{1}{\sqrt{5}}$ **(B)** $\frac{1}{\sqrt{3}}$
(C) $\frac{1}{\sqrt{2}}$ **(D)** $\frac{\sqrt{5}-1}{2}$

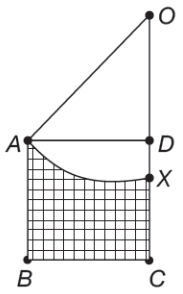
30. Let $A_1\ A_2\ A_3 \ldots\ A_9$ be a nine-sided regular polygon with side length 2 units. The difference between the lengths of the diagonals A_1A_5 and A_2A_4 equals **[KVPY-2017]**

(A) $2 + \sqrt{12}$ **(B)** $\sqrt{12} - 2$
(C) 6 **(D)** 2

31. Let AB be a line segment of length 2. Construct a semi-circle S with AB as diameter. Let C be the mid-point of the arc AB. Construct another semi-circle T external to the ΔABC with chord AC as diameter. The area of the region inside the semi-circle T but outside S is **[KVPY-2017]**

(A) $\frac{\pi}{2}$ **(B)** $\frac{1}{2}$
(C) $\frac{\pi}{\sqrt{2}}$ **(D)** $\frac{1}{\sqrt{2}}$

32. In the figure, $ABCD$ is a unit square. A circle is drawn with centre O on the extended line CD and passing through A. If the diagonal AC is tangent to the circle, then the area of the shaded region is **[KVPY-2017]**



(A) $\frac{9-\pi}{6}$ **(B)** $\frac{8-\pi}{6}$
(C) $\frac{7-\pi}{4}$ **(D)** $\frac{6-\pi}{4}$

33. Let X, Y, Z be, respectively, the areas of a regular pentagon, regular hexagon and regular heptagon which are inscribed in a circle of radius 1. Then, **[KVPY-2017]**

(A) $\frac{X}{5} < \frac{Y}{6} < \frac{Z}{7}$ and $X < Y < Z$

(B) $\frac{X}{5} < \frac{Y}{6} < \frac{Z}{7}$ and $X > Y > Z$

(C) $\frac{X}{5} > \frac{Y}{6} > \frac{Z}{7}$ and $X > Y > Z$

(D) $\frac{X}{5} > \frac{Y}{6} > \frac{Z}{7}$ and $X < Y < Z$

34. Suppose we have two circles of radius 2 each in the plane such that the distance between their centers is $2\sqrt{3}$. The area of the region common to both circles lies between **[KVPY-2017]**

(A) 0.5 and 0.6 **(B)** 0.65 and 0.7
(C) 0.7 and 0.75 **(D)** 0.8 and 0.9

35. Let C_1, C_2 be two circles touching each other externally at the point A and let AB be the diameter of circle C_1. Draw a secant BA_3 to circle C_2, intersecting circle C_1 at a point A_1 ($\neq A$), and circle C_2 at points A_2 and A_3. If $BA_1 = 2$, $BA_2 = 3$ and $BA_3 = 4$, then the radii of circles C_1 and C_2 are, respectively **[KVPY-2017]**

(A) $\frac{\sqrt{30}}{5}, \frac{3\sqrt{30}}{10}$ **(B)** $\frac{\sqrt{5}}{2}, \frac{7\sqrt{5}}{10}$
(C) $\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}$ **(D)** $\frac{\sqrt{10}}{3}, \frac{17\sqrt{10}}{30}$

36. A solid hemisphere is attached to the top of a cylinder, having the same radius as that of the cylinder. If the height of the cylinder were doubled (keeping both radii fixed), the volume of the entire system would have increased by 50%. By what percentage would the volume have increased if the radii of the hemisphere and the cylinder were doubled (keeping the height fixed)? **[KVPY-2018]**

(A) 300% **(B)** 400%
(C) 500% **(D)** 600%

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37. The number of solid cones with integer radius and integer height each having its volume numerically equal to its total surface area is **[KVPY-2018]**

(A) 0 **(B)** 1
(C) 2 **(D)** infinite

38. Let $ABCD$ be a square. An arc of a circle with A as centre and AB as radius is drawn inside the square joining the points B and D. Points P on AB, S on AD, Q, and R on arc BD are taken such that $PQRS$ is a square. Further suppose that PQ and RS are parallel to AC. Then $\frac{\text{area } PQRS}{\text{area } ABCD}$ is **[KVPY-2018]**

(A) $\frac{1}{8}$ **(B)** $\frac{1}{5}$
(C) $\frac{1}{4}$ **(D)** $\frac{2}{5}$

39. Let ABC be an equilateral triangle with side length a. Let R and r denote the radii of the circumcircle and the incircle of triangle ABC, respectively. Then, as a function of a, the ratio $\frac{R}{r}$. **[KVPY – 2019]**

(A) strictly increases
(B) strictly decreases
(C) remains constant
(D) strictly increases for $a < 1$ and strictly decreases for $a > 1$

40. Let $l > 0$ be a real number, C denote a circle with circumference l, and T denote a triangle with perimeter l. Then **[KVPY – 2019]**

(A) Given any positive real number a, we can choose C and T as above such that the ratio $\frac{\text{Area } (C)}{\text{Area } (T)}$ is greater than a

(B) Given any positive real number a, we can choose C and T as above such that the ratio $\frac{\text{Area } (C)}{\text{Area } (T)}$ is less than a

(C) Given any C and T as above, the ratio $\frac{\text{Area } (C)}{\text{Area } (T)}$ is independent of C and T

(D) there exist real numbers a and b such that for any circle C and triangle T as above, we must have $a < \frac{\text{Area } (C)}{\text{Area } (T)} < b$

PART – III Previous Years' PRMO Problems

1. O and I are the circumcentre and incentre of ΔABC respectively. Suppose O lies in the interior of ΔABC and I lies on the circle passing through B, O, and C. What is the magnitude of $\angle BAC$ in degrees? **[PRE-RMO-2012]**

2. PS is line segment of length 4 and O is the midpoint of PS. A semi-circular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semi-circular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXROQ$ bounded by the two semi-circular arcs. **[PRE-RMO-2012]**

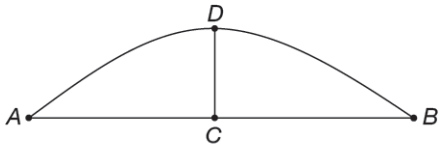
3. In a triangle ABC, let H, I, and O be the orthocentre, incentre, and circumcentre, respectively. If the points B, H, I, C lie on a circle, what is the magnitude of $\angle BOC$ in degrees? **[PRE-RMO-2013]**

4. Let S be a circle with centre O. A chord AB, not a diameter, divides S into two regions R_1 and R_2 such that O belongs to R_2. Let S_1 be a circle with centre in R_1, touching AB at X and S internally. Let S_2 be a circle with centre in R_2, touching AB at Y, the circle S internally and passing through the centre of S. The point X lies of the diameter passing through the center of S_2 and $\angle YXO = 30°$. If the radius of S_2 is 100, then what is the radius of S_1? **[PRE-RMO-2013]**

5. Let $ABCD$ be a convex quadrilateral with $\angle DA = \angle BDC = 90°$. Let the incircles of triangles ABD and BCD touch BD at P and Q, respectively, with P lying in between B and Q. If $AD = 999$ and $PQ = 200$ then what is the sum of the radii of the incircles of triangles ABD and BDC? **[PRE-RMO-2014]**

6. In a triangle ABC, let I denote the incenter. Let the lines AI, BI, and CI intersect the incircle at P, Q, and R, respectively. If $\angle BAC = 40°$, what is the value of $\angle QPR$ in degrees? **[PRE-RMO-2014]**

7. The figure below shows a broken piece of a circular plate made of glass.



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C is the midpoint of AB, and D is the midpoint of arc AB. Given that $AB = 24$ cm and $CD = 6$ cm, what is the radius of the plate in centimetres? (the figure is not drawn to scale.) **[PRE-RMO-2015]**

8. In rectangle $ABCD$, $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\angle BPC = 90°$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC, and CPD, what is the value of $r_1 + r_2 + r_3$? **[PRE-RMO-2015]**

9. The circle ω touches the circle Ω internally at P. The centre O of Ω lies outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume $PV > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\angle PYX$ in degrees? **[PRE-RMO-2015]**

10. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose $AB = 6$, $CD = 8$. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n)\ \lambda k$, where m, n, k are positive integers with $GCD\ (m, n, k) = 1$. What is the value of $m + n + k$? **[PRE-RMO-2017]**

11. Let Ω_1 be a circle with centre O and let AB be a diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1, respectively, such that A_1 and B_1 are on the opposite sides of AB. Given that $AB_1 = 15$ and $OP = 10$, find the radius of Ω_1. **[PRE-RMO-2017]**

12. In a quadrilateral $ABCD$, it is given that $AB = AD = 13$, $BC = CD = 20$, $BD = 24$. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r? **[PRE-RMO-2018]**

13. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AD \perp AB$. Suppose $ABCD$ has an incircle which touches AB at Q and CD at P. Given that $PC = 36$ and $QB = 49$, Find PQ. **[PRE-RMO-2018]**

14. A point P in the interior of a regular hexagon is at distance 8,8,16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r? **[PRE-RMO-2018]**

15. Let AB be a chord of circle with centre O. Let C be a point on the circle such that $\angle ABC = 30°$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20°$. Find the measure of $\angle CDO$ in degrees. **[PRE-RMO-2018]**

16. Triangles ABC and DEF are such that $\angle A = \angle D$, $AB = DE = 17$, $BC = EF = 10$ and $AC - DF = 12$. What is $AC + DF$? **[PRE-RMO-2018]**

17. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD, respectively. Let AI_1 and AI_2 meet BC in E and F, respectively. If $\angle BI_1E = 60°$. What is the measure of $\angle CI_2F$ in degrees? **[PRE-RMO-2018]**

18. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked? **[PRE-RMO-2019]**

19. Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B, and C intersect Ω at A_1, B_1 and C_1 respectively and the internal bisectors of angles A_1, B_1, and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2, and C_2, respectively. If the smallest angle of triangle ABC is 40°, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees? **[PRE-RMO-2019]**

20. Let AB be a diameter of a circle and let C be a point on the segment AB such that $AC : CB = 6 : 7$. Let D be a point on the circle such that DC is perpendicular to AB. Let DE be the diameter through D. If $[XYZ]$ denotes the area of the triangle XYZ. Find $[ABD]/[CDE]$ to the nearest integer. **[PRE-RMO-2019]**

21. Let $ABCD$ be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of $ABCD$ is the least. If $\{PA, PB, PC, PD\} = \{3, 4, 6, 8\}$. What is the maximum possible area of $ABCD$? **[PRE-RMO-2019]**

22. A village has a circular wall around it, and the wall has four gates pointing north, south east and west. A tree stands outside the village, 16m north of the north gate, and it can be just seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter in meters, of the wall that surrounds the village? **[PRE-RMO-2019]**

23. Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of ΔABC. Draw tangents to Ω that are parallel to the sides of ABC. Let r_1, r_2, r_3 be the

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inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$. **[PRE-RMO-2019]**

24. Let ABC be a triangle such that $AB = AC$. Suppose the tangent to the circumcircle of ABC at B is perpendicular of AC. Find $\angle ABC$ measured in degrees. **[PRE-RMO-2019]**

25. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is α and the smaller one $\beta°$ then what is the value of $\alpha - \beta$? **[PRE-RMO-2019]**

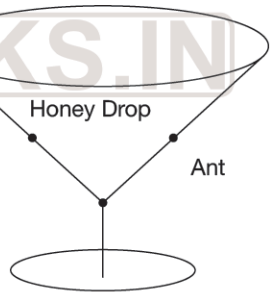
26. Let R denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ divide R into four regions , $R_i = 1, 2, 3, 4$. If denotes the area of the region R_i and if $|R_1| > |R_2| > |R_3| > |R_4|$, determine $|R_1| - |R_2| - |R_3| - |R_4|$.
[Here $|\Omega|$ denotes the area of the region Ω in the plane.] **[PRE-RMO-2019]**

27. In parallelogram $ABCD$, $AC = 10$ and $BD = 28$. The points K and L in the plane of $ABCD$ move in such a way that $AK = BD$ and $BL = AC$. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$? **[PRE-RMO-2019]**

28. Let t be the area of a regular pentagon with each side equal to 1. Let $P(x) = 0$ be the polynomial equation with least degree, having integer coefficients, satisfied by $x = t$ and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of $P(x)$. What is the integer closest to $\sqrt{M}$? $(\sin 18° = (\sqrt{5} - 1)/2)$ **[PRE-RMO-2019]**

29. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equal the angle of reflection. The distance travelled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N? **[PRE-RMO-2019]**

30. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the midpoint of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass. (see the figure). If d is the shortest distance it should crawl to reach the honey drop, what is the integer part of d? (Ignore the thickness of the glass) **[PRE-RMO-2019]**



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ANSWER KEYS

Know Your Progress - 1

1. $20\sqrt{3}$ **2.** 13 cm **3.** 9.6 cm **8.** $a = 40°$, $b = 90°$

Know Your Progress - 2

3. $AB = 13$ cm, $AC = 15$ cm **4.** 7 cm **9.** D **10.** B **11.** (D) 70°
12. (D) PT and RS are parallel

Know Your Progress - 3

1. 154 cm^2 **2.** 14 m **3.** 21/2 m **4.** 300° **5.** 1910.8 cm **6.** 77/8 cm^2 **7.** 44.8 cm^2
8. 16.8 cm **9.** 105 cm^2 **10.** 10 cm

Know Your Progress - 4

1. 1: 2 **2.** 52 m **3.** 972 × 10.6 metres **4.** 126 **5.** 1437.33 cubic cm **6.** 1814 cm^2
7. 414.48 sq. cm, 753.6 c.c **8.** 2520 cc **9.** V = 256.67 cubic centimeters
10. 1351 cm^3 (approx)

Practice Exercise

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EXERCISE – I

PART – I NSEJS Level Problems

1. A	**2.** C	**3.** D	**4.** B	**5.** C	**6.** B	**7.** A	**8.** B	**9.** C	**10.** A
11. D	**12.** D	**13.** B	**14.** B	**15.** D	**16.** B	**17.** C	**18.** D	**19.** C	**20.** C
21. D	**22.** B	**23.** A	**24.** A	**25.** B	**26.** C	**27.** C	**28.** D	**29.** B	**30.** A
31. D	**32.** D	**33.** A	**34.** B	**35.** A					

PART – II KVPY Level Problems

1. D	**2.** D	**3.** C	**4.** C	**5.** D	**6.** A	**7.** C	**8.** D	**9.** A	**10.** D
11. C	**12.** D	**13.** B	**14.** A	**15.** D	**16.** B	**17.** A	**18.** A	**19.** C	**20.** B
21. D	**22.** B	**23.** A	**24.** B	**25.** A	**26.** C	**27.** A	**28.** C	**29.** C	**30.** A

PART – III PRMO Level Problems

1. 48	**2.** 1	**3.** 87	**4.** 26	**5.** 1	**6.** 35	**7.** 2	**8.** 12	**9.** 4	**10.** 1
11. 50	**12.** C	**13.** A	**14.** A	**15.** B	**16.** 57	**17.** 60	**18.** 72	**19.** 66	**20.** 61
21. 30	**22.** 63	**23.** 16	**24.** 60	**25.** 78	**26.** 45	**27.** 8	**28.** 35	**29.** 82	**30.** 11
31. 4	**32.** 23	**33.** 44	**34.** 2	**35.** 1	**36.** 73	**37.** 2	**38.** 6	**39.** 16	**40.** 3
41. 69	**42.** 2	**43.** 2	**44.** 26	**45.** 6	**46.** 6	**47.** 21	**48.** 24	**49.** 5	**50.** 30
51. 50	**52.** 8	**53.** 80	**54.** 4	**55.** 10					

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. A	**2.** D	**3.** A	**4.** B	**5.** B	**6.** A	**7.** A	**8.** C	**9.** A	**10.** A
11. C	**12.** D	**13.** D	**14.** D	**15.** C	**16.** C	**17.** A	**18.** D	**19.** B	**20.** D
21. C	**22.** A	**23.** C	**24.** B	**25.** B	**26.** Bonus	**27.** C	**28.** A	**29.** C	**30.** B

PART – II Previous Years' KVPY Problems

1. B	**2.** D	**3.** A	**4.** A	**5.** D	**6.** A	**7.** A	**8.** B	**9.** B	**10.** B
11. B	**12.** B	**13.** B	**14.** B	**15.** C	**16.** C	**17.** D	**18.** C	**19.** B	**20.** C
21. B	**22.** D	**23.** D	**24.** B	**25.** C	**26.** C	**27.** B	**28.** C	**29.** C	**30.** D
31. B	**32.** D	**33.** D	**34.** C	**35.** A	**36.** C	**37.** B	**38.** D	**39.** C	**40.** A

PART – III Previous Years' PRMO Problems

1. 60	**2.** 2π-2	**3.** 120°	**4.** 60	**5.** 799	**6.** 55	**7.** 15	**8.** 8	**9.** 15	**10.** 75
11. 20	**12.** 8	**13.** 84	**14.** 14	**15.** 80	**16.** 30	**17.** 30	**18.** 36	**19.** 55	**20.** 13
21. 55	**22.** 48	**23.** 15	**24.** 30	**25.** 90	**26.** 48	**27.** 2	**28.** 16	**29.** 31	**30.** 36

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Chapter

8 Trigonometry

Learning Outcomes

After reading this chapter, you will be able to understand

- All Trigonometric Ratios
- Various Measures of Angles
- Trigonometric Identities
- Sine and Cosine Rule

Introduction to Trigonometry and Measurement of Angle

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers, and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analyzing a musical tone and in many other areas.

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(a) Measurement of angles: Commonly two systems of measurement of angles are used.

(i) Sexagesimal or English System: Here 1 right angle = 90° (degrees)

$1° = 60'$ (minutes)

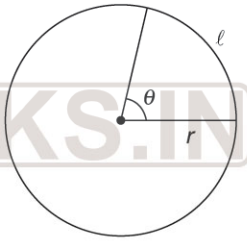
$1' = 60''$ (seconds)

(ii) Circular system: Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

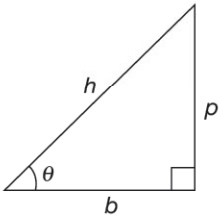
(b) Relation between the these systems: $\frac{D}{90} = \frac{R}{\pi/2}$

(c) If θ is the angle subtended at the centre of a circle of radius 'r', by an arc of length 'ℓ' then $\frac{\ell}{r} = \theta$.

Note that here ℓ, r are in the same units and θ is always in radians.



T-Ratios (or Trigonometric Functions)



In a right-angled triangle

$$\sin\theta = \frac{p}{h};\ \cos\theta = \frac{b}{h};\ \tan\theta = \frac{p}{b};$$

$$\operatorname{cosec}\theta = \frac{h}{p}\ \sec\theta = \frac{h}{b} \text{ and } \cot\theta = \frac{b}{p}$$

'p' is perpendicular, 'b' is base, and 'h' is hypotenuse.

Note

The quantity by which the cosine falls short of unity i.e., $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e., $1 - \sin\theta$ is called the conversed sine of θ.

Basic Trigonometric Identities

1. $\sin\theta \cdot \operatorname{cosec}\theta = 1$
2. $\cos\theta \cdot \sec\theta = 1$
3. $\tan\theta \cdot \cot\theta = 1$
4. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
5. $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$
6. $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$
7. $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$
8. $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$
9. $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$
10. Expressing trigonometric ratios in terms of each other:

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\dfrac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\dfrac{1}{\sqrt{1+\tan^2\theta}}$	$\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\dfrac{1}{\sec\theta}$	$\dfrac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\dfrac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\dfrac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\dfrac{1}{\tan\theta}$	$\cot\theta$	$\dfrac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2-1}$
$\sec\theta$	$\dfrac{1}{1-\sin^2\theta}$	$\dfrac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\dfrac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{1-\cos^2\theta}}$	$\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

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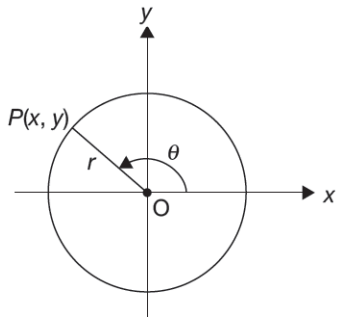
New Definition of *T*-Ratios

By using rectangular coordinates, the definitions of trigonometric functions can be extended to angles of any size in the following way(see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.



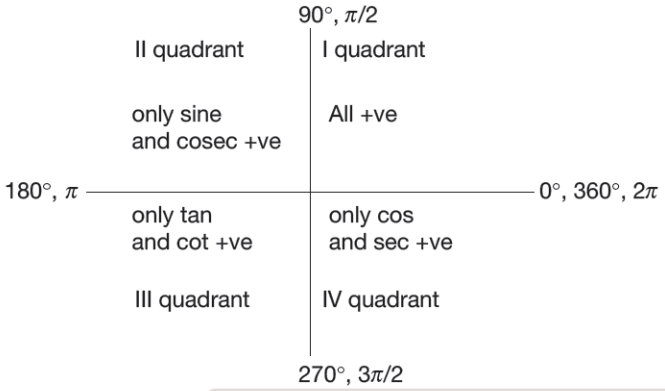
$$\sin\theta = y/r,$$

$$\cos\theta = x/r$$

$\tan\theta = y/x$,(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

Signs of Trigonometric Functions in Different Quadrants



Trigonometric Functions of Allied Angles

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b)

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$
$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$
$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$
$\sin(360° + \theta) = \sin\theta$	$\cos(360° + \theta) = \cos\theta$

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Values of *T*-Ratios of Some Standard Angles

Angles / *T*-ratio	0° / 0	30° / $\pi/6$	45° / $\pi/4$	60° / $\pi/3$	90° / $\pi/2$	180° / π	270° / $3\pi/2$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	−1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	−1	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
$\cot\theta$	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	−1	N.D.
$\operatorname{cosec}\theta$	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	−1

N.D. → Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$

Trigonometric Ratios of the Sum and Difference of Two Angles

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B \cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \dfrac{\cot B \cot A + 1}{\cot B - \cot A}$

Some more results:

(i) $\sin^2 A - \sin^2 B = \sin(A + B) \cdot \sin(A - B) = \cos^2 B - \cos^2 A$.

(ii) $\cos^2 A - \sin^2 B = \cos(A+B) \cdot \cos(A - B)$.

Example 1

If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.

Solution

Let r_1 and r_2 be the radii of the given circles and let their arcs of same length 's' subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and

$$75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$$

$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}$ and $\frac{5\pi}{12} = \frac{s}{r_2}$

$\Rightarrow \quad \frac{\pi}{3} r_1 = s$ and $\frac{5\pi}{12} r_2 = s$

$\Rightarrow \quad \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2$

$\Rightarrow \quad 4r_1 = 5r_2$

$\Rightarrow \quad r_1 : r_2 = 5 : 4$ **Ans.**

Example 2

If $\sin \theta + \sin^2\theta = 1$, then prove that

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$$

Solution

Given that $\sin \theta = 1 - \sin^2\theta = \cos^2\theta$

$$\text{L.H.S.} = \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1$$

$$= (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$$

Example 3

$4(\sin^6\theta + \cos^6\theta) - 6(\sin^4\theta + \cos^4\theta)$ is equal to

(A) 0 **(B)** 1

(C) –2 **(D)** none of these

Solution

$$4[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)]$$

$$- 6[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$$

$$= 4[1 - 3\sin^2\theta\cos^2\theta] - 6[1 - 2\sin^2\theta\cos^2\theta]$$

$$= 4 - 12\sin^2\theta\cos^2\theta - 6 + 12\sin^2\theta\cos^2\theta = -2$$

Ans. (C)

Example 4

If $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ then θ is equal to:

(A) 30° **(B)** 150°

(C) 210° **(D)** none of these

Solution

Let us first find out θ lying between 0 and 360°.

Since $\sin\theta = -\frac{1}{2}$

$\Rightarrow \quad \theta = 210°$ or $330°$ and $\tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = 30°$ or $210°$

Hence, $\theta = 210°$ or $\frac{7\pi}{6}$ is the value satisfying both.

Ans. (C)

Example 5

Prove that $\sqrt{3}$ cosec 20° – sec 20° = 4.

Solution

$$\text{L.H.S.} = \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20° \cdot \cos 20°}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20°\cos 20°}$$

$$= \frac{4(\sin 60° \cdot \cos 20° - \cos 60° \cdot \sin 20°)}{\sin 40°}$$

$$= 4 \cdot \frac{\sin(60° - 20°)}{\sin 40°} = 4 \cdot \frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Example 6

Prove that tan 70° = cot 70° + 2 cot 40° .

Solution

$$\text{L.H.S.} = \tan 70° = \tan(20° + 50°) = \frac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$$

or $\tan 70° - \tan 20° \tan 50° \tan 70° = \tan 20° + \tan 50°$

or $\tan 70° = \tan 70° \tan 50° \tan 20° + \tan 20° + \tan 50°$

$= 2 \tan 50° + \tan 20° \{\tan 70° \times \tan 20° = 1\}$

$= \cot 70° + 2\cot 40° = \text{R.H.S}$

Example 7

If $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha \cdot \cos\alpha$ (1)

and $x\sin\alpha - y\cos\alpha = 0$, (2)

then $x^2 + y^2$ is equal to (where x, y are non-zero real numbers)

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(A) -1 **(B)** ± 1 **(C)** 1 **(D)** 2

Solution

$$\tan\alpha = \frac{y}{x} \qquad \text{(from 2)}$$

and $(1) \div \cos^3\alpha$

$$x\tan^3\alpha + y = \tan\alpha \cdot \sec\alpha$$

$$\Rightarrow \quad x\frac{y^3}{x^3} + y = \frac{y}{x}\cdot\sqrt{1+\frac{y^2}{x^2}}$$

$$\Rightarrow \quad y\frac{\left(y^2+x^2\right)}{x^2} = \frac{y}{x}\frac{\sqrt{x^2+y^2}}{x}$$

$$\Rightarrow \quad y^2 + x^2 = 1$$

Example 8

The value of $\dfrac{\cos^3 10^\circ + \sin^3 20^\circ}{\cos 10^\circ + \sin 20^\circ}$ is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{1}{4}$

Solution

We have

$$\frac{\cos^3 10^\circ + \sin^3 20^\circ}{\cos 10^\circ + \sin 20^\circ}$$

$$= \frac{1}{4}\left(\frac{4\cos^3 10^\circ + 4\sin^3 20^\circ}{\cos 10^\circ + \sin 20^\circ}\right)$$

$$= \frac{1}{4}\left(\frac{\cos 30^\circ + 3\cos 10^\circ + 3\sin 20^\circ - \sin 60^\circ}{\cos 10^\circ + \sin 20^\circ}\right)$$

$$= \frac{3}{4}\frac{(\cos 10^\circ + \sin 20^\circ)}{(\cos 10^\circ + \sin 20^\circ)} = \frac{3}{4}$$ **Ans. (B)**

Example 9

If $\sin 6^\circ \sin 42^\circ \sin 54^\circ \sin 66^\circ \sin 78^\circ = \dfrac{\sqrt{a}+b}{c}$, where a, b and c are relatively prime numbers, then value of $(a+b+c)$ is:

(A) 10 **(B)** 22 **(C)** 20 **(D)** 70

Solution

$$(\sin 6^\circ \sin 54^\circ \sin 66^\circ)\sin 42^\circ \sin 78^\circ$$

$$\Rightarrow \quad \frac{1}{4}(\sin 18^\circ \sin 42^\circ \sin 78^\circ)$$

$$\Rightarrow \quad \frac{1}{16}\sin 54^\circ = \frac{\sqrt{5}+1}{64}$$ **Ans. (D)**

Note: $\sin\theta \sin(60-\theta)\sin(60+\theta) = \dfrac{1}{4}\sin 3\theta$

$$\sin 54^\circ = \frac{\sqrt{5}+1}{4}$$

Example 10

If $\dfrac{\sin A}{\sin B} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{\cos A}{\cos B} = \dfrac{\sqrt{5}}{2}$, $O < A, B < \dfrac{\pi}{2}$ then $\tan A + \tan B$ is equal to:

(A) $\sqrt{\frac{3}{5}}$ **(B)** $\sqrt{\frac{5}{3}}$

(C) $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{5}}$ **(D)** $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}}$

Solution

$$\frac{\tan A}{\sqrt{3}} = \frac{\tan B}{\sqrt{5}} = K. \qquad \{\text{divide both}\}$$

$$2\sin A = \sqrt{3}\sin B$$

$$2\frac{\tan A}{\sec A} = \sqrt{3}\frac{\tan B}{\sec B} \Rightarrow \frac{2\sqrt{3}K}{\sqrt{1+3K^2}} = \frac{\sqrt{3}\sqrt{5}K}{\sqrt{1+5K^2}}$$

$$\Rightarrow \quad K = \frac{1}{\sqrt{5}}$$

$$\therefore \quad \tan A + \tan B = \sqrt{3}K + \sqrt{5}K$$ **Ans. (C)**

Example 11

If $\tan 25^\circ + \tan 85^\circ + \tan 145^\circ = a + b\sqrt{c}$, where $a, b, c \in I$ and c is a prime number, then value of $(a+b+c)$ is:

(A) 12 **(B)** 5 **(C)** 6 **(D)** 10

Solution

$$\tan 25^\circ + \tan 85^\circ + \tan 145^\circ$$

$$= \tan 25^\circ + \frac{\sin 85^\circ}{\cos 85^\circ} - \frac{\sin 35^\circ}{\cos 35^\circ}$$

$$= \frac{\sin 25^\circ}{\cos 25^\circ} + \frac{\sin 85^\circ\cos 35^\circ - \sin 35^\circ\cos 85^\circ}{\cos 85^\circ\cos 35^\circ}$$

$$= \frac{\sin 25^\circ\cos 85^\circ\cos 35^\circ + \sin 50^\circ\cos 25^\circ}{\cos 25^\circ\cos 85^\circ\cos 35^\circ}$$

$$= \frac{\sin 25^\circ\left[\dfrac{\cos 120^\circ + \cos 50^\circ}{2} + 2\cos^2 25^\circ\right]}{\left(\dfrac{1}{4}\cos 75^\circ\right)}$$

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$$= \frac{\sin 25^\circ\left[-\frac{1}{4}+\frac{1}{2}-\sin^2 25^\circ+2-2\sin^2 25^\circ\right]}{\cos 75^\circ}$$

$= 3\tan 75^\circ = 6+3\sqrt{3}$ **Ans. (A)**

Example 12

Given positive angles (in degree) A, B, and C with $A + B + C = 90^\circ$.

Find the characteristics of x to base 10, if $100\tan A = 200\tan B = 300\tan C = x$.

Solution

We have

$$A + B = 90^\circ - C$$

$$\Rightarrow \quad \tan(A+B) = \tan(90^\circ - C) = \cot C$$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B} = \frac{1}{\tan C}$$

$$\therefore \quad \sum \tan A\ \tan B = 1 \qquad (1)$$

Now substituting the value of $\tan A = \frac{x}{100}$; $\tan B = \frac{x}{200}$; $\tan C = \frac{x}{300}$ in (1), we get

$$\frac{x^2}{2\cdot 10^4}+\frac{x^2}{6\cdot 10^4}+\frac{x^2}{3\cdot 10^4} = 1 \Rightarrow \frac{x^2}{2}+\frac{x^2}{6}+\frac{x^2}{3} = 10^4$$

$$\Rightarrow \quad \left(\frac{3+1+2}{6}\right)x^2 = 10^4$$

$$\Rightarrow \quad x^2 = 10^4$$

$$\Rightarrow \quad x = 100$$ **Ans. 2**

Example 13

$\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ$ is equal to

(A) $\sin 36^\circ$ **(B)** $\cos 36^\circ$ **(C)** $\sin 7^\circ$ **(D)** $\cos 7^\circ$

Solution

The given expression is equal to

$$(\sin 47^\circ + \sin 61^\circ) - (\sin 11^\circ + \sin 25^\circ)$$

$$= 2\sin 54^\circ\cos 7^\circ - 2\sin 18^\circ\cos 7^\circ$$

$$= 2\cos 7^\circ(\sin 54^\circ - \sin 18^\circ)$$

$$= 2\cos 7^\circ\left[\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right] = \cos 7^\circ.$$ **Ans. (D)**

Example 14

Let $n = \sum_{r=1}^{89}\tan^2 r^\circ$ and $m = \sum_{r=1}^{89}\operatorname{cosec}^2 r^\circ$ then $m - n$ is equal to

(A) 89 **(B)** $\frac{89}{2}$ **(C)** 178 **(D)** 0

Solution

$$n = \sum_{r=1}^{89}\tan^2 r^\circ$$

$$m = \sum_{r=1}^{89}\operatorname{cosec}^2 r^\circ = \sum_{r=1}^{89}\sec^2 r^\circ$$

$$m - n = \sum \sec^2 r^\circ - \tan^2 r = 89$$ **Ans. (A)**

Example 15

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}+\frac{\sin\theta}{\sqrt{1+\tan^2\theta}}-2\tan\theta\sqrt{\operatorname{cosec}^2\theta-1} = 3$$

if $\theta \in$

(A) $\left(0, \frac{\pi}{2}\right)$ **(B)** $\left(\frac{\pi}{2}, \pi\right)$

(C) $\left(\pi, \frac{3\pi}{2}\right)$ **(D)** $\left(\frac{3\pi}{2}, 2\pi\right)$

Solution

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}+\frac{\sin\theta}{\sqrt{1+\tan^2\theta}}-2\tan\theta\sqrt{\operatorname{cosec}^2\theta-1}$$

$$\equiv \sin^2\theta+\cos^2\theta-\sin\theta\cos\theta+\frac{\sin\theta}{|\sec\theta|}-2\tan\theta|\cot\theta|$$

$$= 1-\sin\theta\cos\theta+\sin\theta|\cos\theta|-2\tan\theta|\cot\theta|$$

$= 1 - 2\tan\theta|\cot\theta|$ if $\cos\theta > 0$

$= 3$ if $\cos\theta > 0$ and $\cot\theta < 0$

$\therefore$ θ is an angle of 4th quadrant.

Hence $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$ **Ans. (D)**

Example 16

The value of

$$\frac{\sin^2(-212^\circ)\cos 32^\circ+\cos^3(-148^\circ)}{\sin(-82^\circ)\cos(-8^\circ)+\sin 368^\circ\sin(-172^\circ)-\sin 58^\circ\sin 148^\circ}$$

is:

(A) $\cos 32^\circ - \sin 32^\circ$ **(B)** $\sin 32^\circ - \cos 32^\circ$

(C) $\cos 32^\circ + \sin 32^\circ$ **(D)** $\sqrt{1-\cos 24^\circ}$

Solution

$$\frac{\sin^3 32^\circ-\cos^3 32^\circ}{-\cos^2 8^\circ-\sin^2 8^\circ-\sin 58^\circ\cos 58^\circ}$$

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$$= \frac{(\sin 32° - \cos 32°)(\sin^2 32° + \cos^2 32° + \sin 32° \cos 32°)}{-1 - \sin 32° \cos 32°}$$

$= \cos 32° - \sin 32°$ **Ans. (A)**

Example 17

If $3 \sin \beta = \sin(2\alpha + \beta)$, then $\tan(\alpha + \beta) - 2 \tan \alpha$ is

(A) independent of α

(B) independent of β

(C) dependent on both α and β

(D) independent of α but dependent on β

Solution

$$\frac{\sin(2\alpha + \beta)}{\sin \beta} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{\sin(2\alpha + \beta) + \sin \beta}{\sin(2\alpha + \beta) - \sin \beta} = \frac{3+1}{3-1}$$

$$\Rightarrow \quad \frac{2\sin(\alpha + \beta)\cos \alpha}{2\sin \alpha \cos(\alpha + \beta)} = 2$$

$$\Rightarrow \quad \tan(\alpha + \beta) = 2 \tan \alpha$$

$$\Rightarrow \quad \tan(\alpha + \beta) - 2 \tan \alpha = 0$$ **Ans. (A, B)**

Example 18

Let $\cos x + \cos y = 3/2$ and $\sin x - \sin y = 1/2$, then

(A) $\cos(x + y) = \frac{1}{4}$

(B) $\cos\left(\frac{x+y}{2}\right) = \pm\frac{\sqrt{5}}{2\sqrt{2}}$

(C) $\sin\left(\frac{x+y}{2}\right) = \pm\frac{\sqrt{3}}{2\sqrt{2}}$

(D) $\cos 2x + \cos 2y + \cos(x - y) = 2$

Solution

$\cos x + \cos y = \frac{3}{2}$ and $\sin x - \sin y = \frac{1}{2}$

Squaring and Adding

$$\Rightarrow \quad 2 + 2\cos(x + y) = \frac{9}{4} + \frac{1}{4}$$

$$\Rightarrow \quad \cos(x + y) = \frac{1}{4}$$

$$\Rightarrow \quad 2\cos^2\left(\frac{x+y}{2}\right) - 1 = \frac{1}{4}$$

$$\Rightarrow \quad \cos\left(\frac{x+y}{2}\right) = \pm\frac{\sqrt{5}}{2\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(\frac{x+y}{2}\right) = \pm\frac{\sqrt{3}}{2\sqrt{2}}$$ **Ans. (B, C)**

Example 19

If $L = \cos^2 84° + \cos^2 36° + \cos 36° \cos 84°$

$M = \cot 73° \cot 47° \cot 13°$

$N = 4 \sin 156° \sin 84° \sin 36°$ then which of the following option is(are) correct?

(A) $L < 1$ **(B)** $M > \tan 2$

(C) $N > \sin\frac{\pi}{4}$ **(D)** $LMN > 0$

Solution

$L = \frac{3}{4}$, $M = \tan 51°$, $N = \sin 72°$ **Ans. (A, B, C, D)**

Example 20

The value of $\cos^2 \theta - \cos \theta \cos(60° - \theta) + \cos^2(60° - \theta)$ is $\frac{p}{q}$, where p and q are relatively prime numbers then $p + q$ is equal to

Solution

$$\cos^2 \theta - \cos \theta \cos(60 - \theta) + \cos^2(60 - \theta)$$

$$= \frac{1}{2}\left[1 + \cos 2\theta - (\cos 60 + \cos(2\theta - 60°)) + 1 + \cos(120° - 2\theta\right]$$

$$= \frac{1}{2}\left[1 + \cos 2\theta - \frac{1}{2} - \cos(2\theta - 60°) + 1 + \cos(120° - 2\theta)\right]$$

$$= \frac{1}{2}\left(\frac{3}{2} + \cos 2\theta + \cos(120° - 2\theta) - \cos(2\theta - 60°)\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} + 2\cos 60° \cos(2\theta - 60°) - \cos(2\theta - 60°)\right)$$

$$= \frac{3}{4}$$ **Ans. 7**

Example 21

$\frac{\tan 2\alpha + \tan 3\alpha - \tan 5\alpha}{\tan 2\alpha \cdot \tan 3\alpha \cdot \tan 5\alpha}$ is equal to

(A) $\tan\alpha$ **(B)** $\cot \alpha$ **(C)** -1 **(D)** 1

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Solution

$$5\alpha = 2\alpha + 3\alpha$$

take tangent both side

$$\Rightarrow \quad \tan 5\alpha = \frac{\tan 2\alpha + \tan 3\alpha}{1 - \tan 2\alpha \tan 3\alpha}$$

$$\Rightarrow \quad \tan 5\alpha - \tan 2\alpha \cdot \tan 3\alpha \cdot \tan 5\alpha = \tan 2\alpha + \tan 3\alpha$$

$$\therefore \quad -1 = \frac{\tan 2\alpha + \tan 3\alpha - \tan 5\alpha}{\tan 2\alpha \cdot \tan 3\alpha \cdot \tan 5\alpha}$$ **Ans. (C)**

Example 22

The value of $\frac{3 + \cot 70^\circ \cot 10^\circ}{\cot 70^\circ + \cot 10^\circ}$ is

(A) $\cot 50^\circ$ **(B)** $\cot 40^\circ$
(C) $\cot 20^\circ$ **(D)** $\cot 70^\circ$

Solution

$$\frac{3 + \cot 70^\circ \cot 10^\circ}{\cot 70^\circ + \cot 10^\circ} = \frac{3\sin 70^\circ \sin 10^\circ + \cos 70^\circ \cos 10^\circ}{\sin 10^\circ \cos 70^\circ + \cos 10^\circ \sin 70^\circ}$$

$$= \frac{2\sin 70^\circ \sin 10^\circ + \cos(70^\circ - 10^\circ)}{\sin 80^\circ}$$

$$= \frac{(\cos 60^\circ - \cos 80^\circ) + \frac{1}{2}}{\sin 80^\circ}$$

$$= \frac{1 - \cos 80^\circ}{\sin 80^\circ} = \tan 40^\circ = \cot 50^\circ$$

Ans. (A)

Example 23

Value of $\tan 399^\circ + \tan 6^\circ + \tan 399^\circ \cdot \tan 6^\circ$ is

(A) 0 **(B)** 1 **(C)** -1 **(D)** 2

Solution

$\tan 399^\circ + \tan 6^\circ + \tan 399^\circ \cdot \tan 6^\circ$

$= \tan 39^\circ + \tan 6^\circ + \tan 39^\circ \tan 6^\circ$

$\tan 45^\circ = \tan(39^\circ + 6^\circ)$

$$\Rightarrow \quad 1 = \frac{\tan 39^\circ + \tan 6^\circ}{1 - \tan 39^\circ \tan 6^\circ}$$

$$\Rightarrow \quad 1 - \tan 39^\circ.\tan 6^\circ = \tan 39^\circ + \tan 6^\circ$$

$$\Rightarrow \quad 1 = \tan 39^\circ + \tan 6^\circ + \tan 39^\circ \tan 6^\circ$$ **Ans. (B)**

Example 24

$$32\left(1 + \cos\frac{\pi}{10}\right)\left(1 + \cos\frac{3\pi}{10}\right)\left(1 + \cos\frac{7\pi}{10}\right)\left(1 + \cos\frac{9\pi}{10}\right)$$

is co-prime with

(A) 1 **(B)** 4 **(C)** 3 **(D)** 9

Solution

$$32\left(1 + \cos\frac{\pi}{10}\right)\left(1 + \cos\frac{3\pi}{10}\right)\left(1 - \cos\frac{3\pi}{10}\right)\left(1 - \cos\frac{\pi}{10}\right)$$

$$= 32\left(1 - \cos^2\frac{\pi}{10}\right)\left(1 - \cos^2\frac{3\pi}{10}\right)$$

$$= 32\sin^2 18^\circ \cdot \sin^2 54^\circ$$

$$= 32\left(\left(\frac{\sqrt{5} - 1}{4}\right)\left(\frac{\sqrt{5} + 1}{4}\right)\right)^2$$

$$= 32\left(\frac{5 - 1}{16}\right)^2 = 32 \times \frac{1}{16} = 2$$

Which is co-prime with 1, 3, and 9 from the option

Ans. (A, C, D)

Example 25

The tangents of two acute angles of a ΔABC are given by $\tan A = \frac{5}{12}$ and $\tan B = \frac{3}{4}$, respectively. Select the correct option (s)?

(A) $\tan C = \frac{-56}{33}$

(B) ΔABC is acute-angled triangle

(C) $\tan C = \frac{56}{33}$

(D) ΔABC is obtuse-angled triangle

Solution

In ΔABC

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

$$\Rightarrow \quad \frac{5}{12} + \frac{3}{4} + \tan C = \frac{5}{12} \cdot \frac{3}{4} \cdot \tan C$$

$$\Rightarrow \quad \tan C = -\frac{56}{33}$$

[as $\tan C < 0 \Rightarrow$ obtuse-angled triangle] **Ans. (A, D)**

Example 26

If $\cos x + \cos^2 x = 1$ then value of $\sin^2 x + \sin^4 x$ will be divisible by

(A) 1 **(B)** 2 **(C)** 4 **(D)** 3

Solution

$\cos x + \cos^2 x = 1 \Rightarrow \cos x = 1 - \cos^2 x \Rightarrow \cos x = \sin^2 x$

$\sin^2 x + \sin^4 x = \cos x + \cos^2 x = 1$ **Ans. (A)**

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Example 27

Let $\cos\alpha\cot\beta - \cos\alpha - \cot\beta + 1 = 0$ then which is /are correct option (s) (wherever defined)?

(A) $\sin^2\alpha + \cos^2\beta = 1$

(B) $\sin^2\alpha + \sec^2\beta = 2$

(C) $\tan^2\alpha + \cos^2\beta = \frac{1}{2}$

(D) $\operatorname{cosec}^2\beta - (1+\sin^2\alpha)^2 = 1$

Solution

$$\cos\alpha\cdot\cot\beta - \cos\alpha - \cot\beta + 1 = 0$$

$$\Rightarrow \quad (\cos\alpha - 1)(\cot\beta - 1) = 0$$

$$\cos\alpha = 1 \quad \text{or} \quad \cot\beta = 1$$

$$\Rightarrow \quad \sin\alpha = 0 \quad \text{or} \quad \sin\beta = \pm\frac{1}{\sqrt{2}};\ \cos\beta = \pm\frac{1}{\sqrt{2}}$$

$$\tan\alpha = 0$$

Ans. (B, C, D)

Example 28

If $\tan(\cos^2\theta)\cdot\tan(\sin^2\theta)\cdot\tan(\cos 2\theta) = A\tan(\cos^2\theta) + B\tan(\sin^2\theta) + C\tan(\cos 2\theta)\ \forall\ \theta \in R$, where A, B, C are real constants, then

(A) $A + B + C = 0$ **(B)** $A - B + C = 1$

(C) $A + B + C = -1$ **(D)** $A + B + C = 3$

Solution

$$\cos^2\theta = \sin^2\theta + \cos 2\theta$$

If $\gamma = \alpha + \beta$

$$\tan\gamma\cdot\tan\alpha\cdot\tan\beta = \tan\gamma - \tan\alpha - \tan\beta.$$

$$\therefore \quad A = 1;\ B = -1;\ C = -1$$

Know Your Progress - 1

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1. Reduce $63°\ 14'\ 51''$ to centesimal Measure.
2. The diameter of a sheel is 28 cm, through what distance does its centre move during one revolution of the wheel along the ground?
3. What must be the radius of a circular running path, round which an athlete must run 5 times in order to describe 1760 meters?
4. The angles of a triangle are in A.P. and the number of grades in the least is to the number of radians in the greatest as $40 : \pi$. find the angle in degrees.
5. Express in the three systems of angular measurement the magnitude of the angle of a regular decagon.
6. In the right triangle $AB = 3$ cm and $AC = 5$ cm. Find all trigonometric ratios.
7. If $\tan\theta = \frac{m}{n}$, then find $\sin\theta$.
8. If $\operatorname{cosec} A = \frac{13}{5}$, then prove that $\tan^2 A - \sin^2 A = \sin^4 A\sec^2 A$.
9. In ΔABC, right-angled at B, $AC + AB = 9$ cm and $BC = 3$ cm. Determine the value of $\cot C$, $\operatorname{cosec} C$, $\sec C$.
10. If $\cot\theta = \frac{3}{4}$, prove that $\sqrt{\frac{\sec\theta - \operatorname{cosec}\theta}{\sec\theta + \operatorname{cosec}\theta}} = \frac{1}{\sqrt{7}}$.
11. **Evaluate:** $\frac{2}{3}(\cos^4 30° - \sin^4 45°) - 3(\sin^2 60° - \sec^2 45°) + \frac{1}{4}\cot^2 30°$
12. Find the value of $2\tan^2 45° + 3\cos^2 30° - \sin^2 60°$.
13. Find the value of $\frac{\cos 45°}{\sec 30° + \operatorname{cosec} 30°}$.
14. A Rhombus of side of 10 cm has two angles of $60°$ each. Find the length of diagonals and also find its area.
15. Find the value of $\tan 60°$, geometrically.
16. If $A + B = 90°$, prove that:
$$\sqrt{\frac{\tan A\tan B + \tan A\cot B}{\sin A\sec B} - \frac{\sin^2 B}{\cos^2 A}} = \tan A.$$
17. If A, B, C are the interior angles of a ΔABC, show that:
(i) $\sin\frac{B+C}{2} = \cos\frac{A}{2}$ **(ii)** $\cos\frac{B+C}{2} = \sin\frac{A}{2}$
18. **Evaluate:**
$$\frac{\sin\theta\cos\theta\sin(90°-\theta)}{\cos(90°-\theta)} + \frac{\cos\theta\sin\theta\cos(90°-\theta)}{\sin(90°-\theta)} + \frac{\sin^2 27° + \sin^2 63°}{\cos^2 40° + \cos^2 50°}$$

19. Without using trigonometric tables, evaluate the following:

$$\frac{\cos 70^\circ}{3\sin 20^\circ} + \frac{4(\sec^2 59^\circ - \cot^2 31^\circ)}{3} - \frac{2}{3}\sin 90^\circ$$

20. If $\sin 3\theta = \cos(\theta - 6^\circ)$ and 3θ and $\theta - 6^\circ$ are acute, find the value of θ.

21. Prove that: $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$

22. Prove that: $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = \sec\theta\operatorname{cosec}\theta + 1$

23. If $\sec^2\theta(1+\sin\theta)(1-\sin\theta) = k$, then find the value of k.

24. Prove that: $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1+\sin\theta}{\cos\theta}$

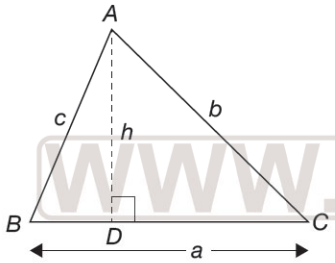
25. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then Prove that: $x^2 + y^2 + z^2 = r^2$.

Sine Formulae

In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.



Cosine Formulae

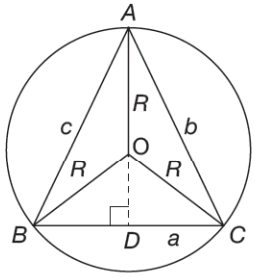
(a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

or $a^2 = b^2 + c^2 - 2bc\cos A$

Radius of the Circumcircle '*R*'



Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre and vertex of triangle is called circumradius '*R*'.

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

Radius of the Incircle '*r*'

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius '*r*'.

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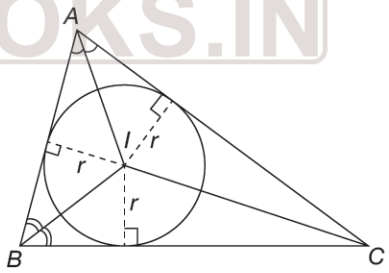


$$r = \frac{\Delta}{s}$$

Example 29

Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\dfrac{3}{2+\sqrt{3}}$ **(B)** $\dfrac{\sqrt{3}}{2+\sqrt{3}}$

(C) $\dfrac{\sqrt{3}}{2-\sqrt{3}}$ **(D)** $\dfrac{1}{2+\sqrt{3}}$

Solution

Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If sides opposite to these angles are a, b, c, respectively, then a will be the greatest side.

Now from sine formula $\frac{a}{\sin 120°} = \frac{b}{\sin 30°} = \frac{c}{\sin 30°}$

$\Rightarrow \quad \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$

$\Rightarrow \quad \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = k$ (say)

then $a = \sqrt{3}\,k$, perimeter $= (2 + \sqrt{3})k$

$\therefore$ required ratio $= \frac{\sqrt{3}k}{(2+\sqrt{3})k} = \frac{\sqrt{3}}{2+\sqrt{3}}$ **Ans. (B)**

Example 30

In triangle ABC, if $b = 3$, $c = 4$ and $\angle B = \pi/3$, then number of such triangles is:

(A) 1 **(B)** 2
(C) 0 **(D)** infinite

Solution

Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$\Rightarrow \quad \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6} = \frac{\sin C}{4}$

$\Rightarrow \quad \sin C = \frac{2}{\sqrt{3}} > 1$

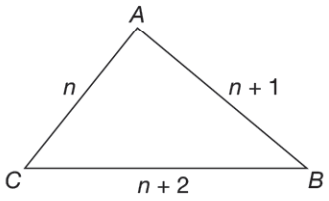
which is not possible.

Hence there exists no triangle with given elements.

Ans. (C)

Example 31

The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.



Solution

Let the sides be n, $n + 1$, $n + 2$.

i.e., $AC = n$, $AB = n + 1$, $BC = n + 2$

Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad 3\angle B + \angle C = 180° \Rightarrow \angle C = 180° - 3\angle B$

We have, sine law as,

$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1}$

$\Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180 - 3B)}{n+1}$

$\Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin 3B}{n+1}$

from (i) and (ii);

$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n}$

$\Rightarrow \quad \cos B = \frac{n+2}{2n}$ (iv)

and from (ii) and (iii);

$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1}$

$\Rightarrow \quad \frac{\sin B}{n} = \frac{\sin B(3 - 4\sin^2 B)}{n+1}$

$\Rightarrow \quad \frac{n+1}{n} = 3 - 4\,(1 - \cos^2 B)$ (v)

from (iv) and (v), we get

$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2$

$\Rightarrow \quad \frac{n+1}{n} + 1 = \left(\frac{n^2 + 4n + 4}{n^2}\right)$

$\Rightarrow \quad \frac{2n+1}{n} = \frac{n^2 + 4n + 4}{n^2}$

$\Rightarrow \quad 2n^2 + n = n^2 + 4n + 4$

$\Rightarrow \quad n^2 - 3n - 4 = 0$

$\Rightarrow \quad (n - 4)(n + 1) = 0$

$n = 4$ or -1

where $n \neq -1$

$\therefore \quad n = 4.$

Hence the sides are 4, 5, 6 **Ans.**

Example 32

In a triangle ABC, if $B = 30°$ and $c = \sqrt{3}\ b$, then A can be equal to:

(A) 45° **(B)** 60° **(C)** 90° **(D)** 120°

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Solution

We have $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca} \Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{3b^2 + a^2 - b^2}{2 \times \sqrt{3}b \times a}$

$\Rightarrow \quad a^2 - 3ab + 2b^2 = 0 \Rightarrow (a - 2b)\,(a - b) = 0$

$\Rightarrow \quad$ Either $a = b \Rightarrow A = 30°$

or $\quad a = 2b$

$\Rightarrow \quad a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 90°.$ **Ans. (C)**

Example 33

In a triangle ABC, $(a^2 - b^2 - c^2)\tan A + (a^2 - b^2 + c^2)\tan B$ is equal to:

(A) $(a^2 + b^2 - c^2)\tan C$ **(B)** $(a^2 + b^2 + c^2)\tan C$

(C) $(b^2 + c^2 - a^2)\tan C$ **(D)** none of these

Solution

Using cosine law:

The given expression is equal to $-2bc \cos A \tan A + 2ac \cos B \tan B$

$$= 2abc\left(-\frac{\sin A}{a} + \frac{\sin B}{b}\right) = 0$$ **Ans. (D)**

Example 34

In a triangle ABC, if $a : b : c = 4 : 5 : 6$, then ratio between its circumradius and inradius is:

(A) $\dfrac{16}{7}$ **(B)** $\dfrac{16}{9}$ **(C)** $\dfrac{7}{16}$ **(D)** $\dfrac{11}{7}$

Solution

$$\frac{R}{r} = \frac{abc}{4\Delta}\Big/\frac{\Delta}{s} = \frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)} \quad \text{(i)}$$

$\because \quad a : b : c = 4 : 5 : 6 \Rightarrow \dfrac{a}{4} = \dfrac{b}{5} = \dfrac{c}{6} = k$ (say)

$\Rightarrow \quad a = 4k,\ b = 5k,\ c = 6k$

$\therefore \quad s = \dfrac{a+b+c}{2} = \dfrac{15k}{2},\ s - A = \dfrac{7k}{2},$

$$s - b = \frac{5k}{2},\ s - c = \frac{3k}{2}$$

using (i) in these values $\dfrac{R}{r} = \dfrac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)} = \dfrac{16}{7}$

Ans. (A)

Example 35

If A, B, C are the angles of a triangle, prove that: $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$.

Solution

$$\cos A + \cos B + \cos C$$

$$= 2\cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2\sin\frac{C}{2} \cdot \cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$\left\{\because \frac{C}{2} = 90° - \left(\frac{A+B}{2}\right)\right\}$$

$$= 1 + 2\sin\frac{C}{2} \cdot 2\sin\frac{A}{2} \cdot \sin\frac{B}{2}$$

$$= 1 + 4\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$$

$$= 1 + \frac{r}{R}$$

{as, $r = 4R \sin A/2 \cdot \sin B/2 \cdot \sin C/2$}

$\Rightarrow \quad \cos A + \cos B + \cos C = 1 + \dfrac{r}{R}.$

Hence proved.

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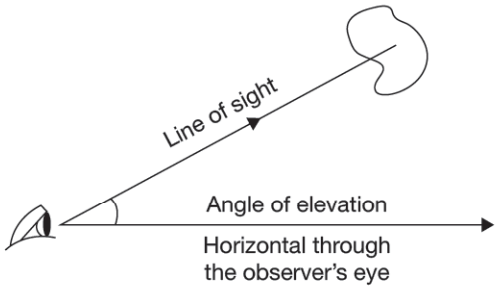
Know Your Progress - 2

1. If in a ΔABC, $\angle A = \dfrac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.
2. Show that, in any ΔABC : $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$.
3. If in a ΔABC, $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A - B)}{\sin(B - C)}$, show that a^2, b^2, c^2 are in A.P.
4. If $a : b : c = 4 : 5 : 6$, then show that $\angle C = 2\angle A$.
5. In any ΔABC, prove that
   (a) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$
   (b) $\dfrac{b^2}{a}\cos A + \dfrac{c^2}{b}\cos B + \dfrac{a^2}{c}\cos C = \dfrac{a^4 + b^4 + c^4}{2abc}$
6. If in ΔABC, $A = 3$, $b = 4$ and $c = 5$, find
   (a) Δ **(b)** R **(c)** r

Angle of Elevation

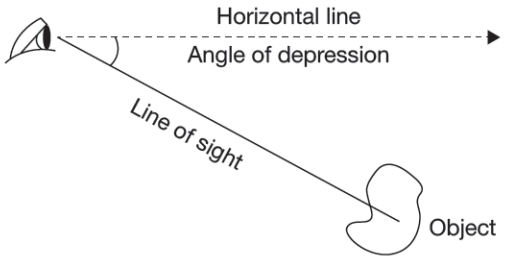
In order to see an object that is at a higher level compared to the ground level, we have to look up. The line joining the object and the eye of the observer is known as the line of sight and the angle that this line of sight makes with the horizontal drawn through the eye of the observer is known as the **angle of elevation**.

Therefore, the angle of elevation of an object helps in finding out its height (Figure).



Angle of Depression

When the object is at a lower level than the observer's eyes, he has to look downwards to have a view of the object. In that case, the angle that the line of sight makes with the horizontal through the observer's eye is known as the **angle of depression (Figure)**.



Example 36

A man is standing on the deck of a ship, which is 8 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill.

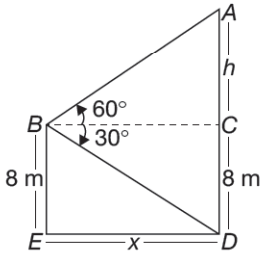
Solution

Let x be distance of hill from man and $h + 8$ be height of hill which is required.

In right triangle ACB,

$$\Rightarrow \quad \tan 60° = \frac{AC}{BC} = \frac{h}{x}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x}$$



In right triangle BCD,

$$\tan 30° = \frac{CD}{BC} = \frac{8}{x}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{8}{x} \Rightarrow x = 8\sqrt{3}$$

$$\therefore \text{ Height of hill} = h + 8 = \sqrt{3}\,.x + 8\,(\sqrt{3})(8\sqrt{3}) + 8$$

$$= 32 \text{ m}.$$

Distance of ship from hill $= x = 8\sqrt{3}$ m.

Example 37

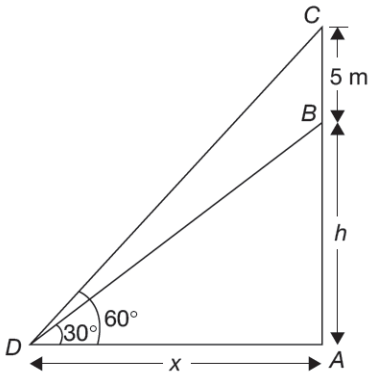
A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height 5 meters. At a point on the plane, the angle of elevation of the bottom and the top of the flag staff are, respectively, 30° and 60°. Find the height of tower.

Solution

Let AB be the tower of height h metre and BC be the height of flag staff surmounted on the tower.

Let the point on the plane be D at a distance x meter from the foot of the tower

In ΔABD



$$\tan 30° = \frac{AB}{AD}$$

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$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\Rightarrow \quad x = \sqrt{3}\, h \qquad \text{(i)}$$

In ΔADC

$$\tan 60° = \frac{AC}{AD}$$

$$\Rightarrow \quad \sqrt{3} = \frac{5+h}{x}$$

$$\Rightarrow \quad x = \frac{5+h}{\sqrt{3}} \qquad \text{(ii)}$$

From (i) and (ii)

$$\Rightarrow \quad \sqrt{3}\, h = \frac{5+h}{\sqrt{3}}$$

$$\Rightarrow \quad 3h = 5 + h$$

$$\Rightarrow \quad 2h = 5$$

$$\Rightarrow \quad h = \frac{5}{2} = 2.5 \text{ m}$$

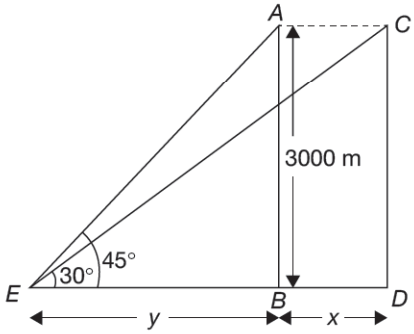
So, the height of tower = 2.5 m

Example 38

The angle of elevation of an aeroplane from a point on the ground is 45°. After a flight of 15 sec, the elevation changes to 30°. If the aeroplane is flying at a height of 3000 metres, find the speed of the aeroplane.

Solution

Let the point on the ground is E which is y metres from point B and let after 15 sec flight it covers x metres distance.



In ΔAEB

$$\tan 45° = \frac{AB}{EB}$$

$$\Rightarrow \quad 1 = \frac{3000}{y}$$

$$\Rightarrow \quad y = 3000 \text{ m} \qquad \text{(i)}$$

In ΔCED

$$\Rightarrow \quad \tan 30° = \frac{CD}{ED}.$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{3000}{x+y} \qquad (\because AB = CD)$$

$$\Rightarrow \quad x + y = 3000\sqrt{3} \qquad \text{(ii)}$$

From equation (i) and (ii)

$$\Rightarrow \quad x + 3000 = 3000\sqrt{3}$$

$$\Rightarrow \quad x = 3000\sqrt{3} - 3000$$

$$\Rightarrow \quad x = 3000\,(\sqrt{3} - 1)$$

$$\Rightarrow \quad x = 3000 \times (1.732 - 1)$$

$$\Rightarrow \quad x = 3000 \times 0.732$$

$$\Rightarrow \quad x = 2196 \text{ m.}$$

$$\text{Speed of Aeroplane} = \frac{\text{Distance covered}}{\text{Time taken}} = \frac{2196}{15} \text{ m/sec}$$

$$= \frac{2196}{15} \times \frac{18}{5} \text{ km/hr} = 527.04 \text{ km/hr.}$$

Hence, the speed of aeroplane is 527.04 km/hr.

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Example 39

If the angle of elevation of a cloud from a point h metres above a lake is α and the angle of depression of its reflection in the lake is β, prove that the distance of the cloud from the point of observation is $\dfrac{2h\sec\alpha}{\tan\beta - \tan\alpha}$.

Solution

Let AB be the surface of the lake and let C be a point of observation such that $AC = h$ metres. Let D be the position of the cloud and D' be its reflection in the lake. Then $BD = BD$.

In ΔDCE

$$\tan\alpha = \frac{DE}{CE}$$

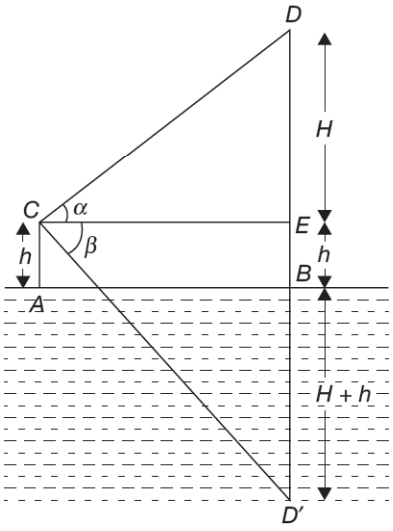
$$\Rightarrow \quad CE = \frac{H}{\tan\alpha} \qquad \text{(i)}$$

In $\Delta CED'$

$$\tan\beta = \frac{ED'}{EC}$$

$$\Rightarrow \quad CE = \frac{h + H + h}{\tan\beta}$$

$$\Rightarrow \quad CE = \frac{2h + H}{\tan\beta} \qquad \text{(ii)}$$



From (i) and (ii)

$$\Rightarrow \quad \frac{H}{\tan\alpha} = \frac{2h + H}{\tan\beta}$$

$$\Rightarrow \quad H\tan\beta = 2h\tan\alpha + H\tan\alpha$$

$$\Rightarrow \quad H\tan\beta - H\tan\alpha = 2h\tan\alpha$$

$$\Rightarrow \quad H(\tan\beta - \tan\alpha) = 2h\tan\alpha$$

$$\Rightarrow \quad H = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha} \qquad \text{(iii)}$$

In ΔDCE

$$\sin A = \frac{DE}{CD}$$

$$\Rightarrow \quad CD = \frac{DE}{\sin\alpha}$$

$$\Rightarrow \quad CD = \frac{H}{\sin\alpha}$$

Substituting the value of H from (iii)

$$CD = \frac{2h\tan\alpha}{(\tan\beta - \tan\alpha)\sin\alpha}$$

$$\Rightarrow \quad CD = \frac{2h\dfrac{\sin\alpha}{\cos\alpha}}{(\tan\beta - \tan\alpha)\sin\alpha}$$

$$CD = \frac{2h\sec\alpha}{\tan\beta - \tan\alpha}$$

Hence, the distance of the cloud from the point of observation is $\dfrac{2h\sec\alpha}{\tan\beta - \tan\alpha}$.

Hence, proved.

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Know Your Progress - 3

1. Two pillars of equal height stand on either side of a roadway that is 60 m wide. At a point in the roadway between the pillars, the angle of elevation of the top of pillars are 60° and 30°. Find the height of the pillars.

2. If the angles of elevation of the top of a tower from two points distant a and b from the base and in the same straight line with it are complementary, then find the height of the tower.

3. From the top of a cliff 25 m high the angle of elevation of a top of tower is found to be equal to the angle of depression of the foot of the tower. Find the height of the tower.

4. A tree 12 m high is broken by the wind in such a way that its top touches the ground and makes an angle 60° with the ground. Find the height from the bottom of the tree from where it is broken by the wind is approximately.

5. AB is a vertical pole and C is the middle point. The end A is on the level ground and P is any point on the level ground other than A. The portion CB subtends an angle β at P. If $AP : AB = 2 : 1$, then find β.

6. The shadow of a tower is 30 meters when the sun's altitude is 30°. When the sun's altitude is 60°, then the length of shadow will be:
   (A) 60 m **(B)** 15 m
   (C) 10 m **(D)** 5 m

7. The angles of elevation of the top of a vertical tower from two points 30 meters apart, and on the same straight line passing through the base of tower, are 300 and 600, respectively. The height of the tower is:
   (A) 10 m **(B)** 15 m
   (C) $15\sqrt{3}$ m **(D)** 30 m

8. When a eucalyptus tree is broken by strong wind, its top strikes the ground at an angle of 30° to the ground and at a distance of 15 m from the foot. What is the height of the tree?
   (A) $15\sqrt{3}$ m **(B)** $10\sqrt{3}$ m
   (C) 20 m **(D)** 10 m

Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. The difference between two angles is 19° and their sum is $\frac{890}{9}$ grades. Find the greater angles.

(A) 63° **(B)** 35°
(C) 27° **(D)** 54°

2. If $\tan\theta = 4$, then $\left(\frac{\tan\theta}{\frac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta}\right)$ is equal to:

(A) 0 **(B)** $2\sqrt{2}$
(C) $\sqrt{2}$ **(D)** 1

3. If $\tan 15° = 2 - \sqrt{3}$, then the value of $\cot^2 75°$ is:

(A) $7 + \sqrt{3}$ **(B)** $7 - 2\sqrt{3}$
(C) $7 - 4\sqrt{3}$ **(D)** $7 + 4\sqrt{3}$

4. The angles of the triangles ABC and DEF are given as follows: $A = 90°$, $B = 30°$, $D = 90°$ and $E = 30°$. If the side BC is twice the side EF, which of the following statement is true?

(A) $\sin B = 2\sin E$ **(B)** $\sin E = 2\sin B$
(C) $\sin B = \sin E$ **(D)** $\sin A = \sin B$

5. Evaluate: $\frac{\sec^2 54° - \cot^2 36°}{\text{cosec}^2 57° - \tan^2 33°} + 2\sin^2 38°\sec^2 52° - \sin^2 45° + \frac{2}{\sqrt{3}}\tan 17°\tan 60°\tan 73°$.

(A) $\frac{9}{2}$ **(B)** $\frac{7}{2}$
(C) $\frac{3}{2}$ **(D)** $\frac{1}{2}$

6. If $\cos^4\theta + \sin^2\theta = m$ then

(A) $1 \le m \le 2$ **(B)** $\frac{3}{4} \le m \le 1$
(C) $\frac{1}{2} \le m \le 1$ **(D)** $\frac{3}{4} \le m \le \frac{13}{16}$

7. If $\sin\phi$ and $\cos\phi$ are the roots of the equation $ax^2 + bx + c = 0$, then

(A) $(a-c)^2 + c^2 = b^2$ **(B)** $(a-c)^2 - c^2 = b^2$
(C) $(a+c)^2 + c^2 = b^2$ **(D)** $(a+c)^2 - c^2 = b^2$

8. If $\sin\alpha + \text{cosec}\,\alpha = 2$, then $\sin^n\alpha + \text{cosec}^n\alpha =$

(A) $n(\sin\alpha + \text{cosec}\,\alpha)$ **(B)** 2
(C) $\frac{n}{\sin\alpha + \text{cosec}\,\alpha}$ **(D)** n

9. An aeroplane, when flying at a height 2500 m from the ground, passes vertically above another aeroplane. At an instant when the angles of elevation of the two aeroplanes from the same point on the ground are 45° and 30°, respectively, then the vertical distance between the two aeroplanes at that instant is:

(A) 1158 m **(B)** 1058 m
(C) 1008 m **(D)** none

10. If the angle of elevation of cloud from a point 200 metres above a lake is 30° and the angle of depression of its reflection in the lake is 60°, then the height of the cloud (in metres) above the lake is:

(A) 200 **(B)** 300
(C) 500 **(D)** none

11. The angle of elevation of the top of tower from the top and bottom of a building h meter high are α and β, then the height of tower is:

(A) $h\sin\alpha\cos\beta/\sin(\alpha+\beta)$
(B) $h\cos\alpha\cos\beta/\sin(\beta-\alpha)$
(C) $h\cos\alpha\sin\beta/\sin(\beta-\alpha)$
(D) None of these

12. A man at the top of vertical lighthouse, observes a boat coming directly towards it. If it takes 20 minutes for the angle of depression to change from 30° to 60°, the time taken by the boat to reach the lighthouse from the point when the angle of depression was 30°, is:

(A) 30 minutes **(B)** 20 minutes
(C) 10 minutes **(D)** 5 minutes

13. If each of α, β and γ is a positive acute angle such that $\sin(\alpha+\beta-\gamma) = \frac{1}{2}$. $\cos(\beta+\gamma-\alpha) = \frac{1}{2}$ and $\tan(\gamma+\alpha-\beta) = 1$, then the values of α, β and γ is:

(A) 45°, 45° and 90° **(B)** 60°, 45° and 75°
(C) $\left(37\frac{1}{2}\right)^\circ$, 45° and $\left(52\frac{1}{2}\right)^\circ$ **(D)** none

14. If $0 < x < \pi$, and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is:

(A) $(4 - \sqrt{7})/3$ **(B)** $-(4 + \sqrt{7})/3$
(C) $(1 + \sqrt{7})/4$ **(D)** $(1 - \sqrt{7})/4$

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15. Let $\cos(\alpha+\beta)=\frac{4}{5}$ and let $\sin(\alpha-\beta)=\frac{5}{13}$, where $0\le\alpha,\beta\le\frac{\pi}{4}$. Then $\tan 2\alpha=$

(A) $\frac{25}{16}$ **(B)** $\frac{56}{33}$

(C) $\frac{19}{12}$ **(D)** $\frac{20}{7}$

16. If $A=\sin^2 x+\cos^4 x$, then for all real x

(A) $1\le A\le 2$ **(B)** $\frac{3}{4}\le A\le\frac{13}{16}$

(C) $\frac{3}{4}\le A\le 1$ **(D)** $\frac{13}{16}\le A\le 1$

17. In a ΔPQR, if $3\sin P+4\cos Q=6$ and $4\sin Q+3\cos P=1$, then the angle R is equal to

(A) $\frac{3\pi}{4}$ **(B)** $\frac{5\pi}{6}$

(C) $\frac{\pi}{6}$ **(D)** $\frac{\pi}{4}$

18. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as

(A) $\sin A\cos A+1$ **(B)** $\sec A\operatorname{cosec} A+1$

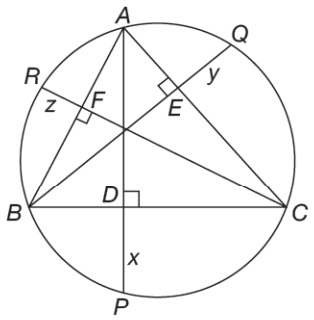
(C) $\tan A+\cot A$ **(D)** $\sec A+\operatorname{cosec} A$

19. $ABCD$ is a trapezium such that AB and CD are parallel and $BC\perp CD$. If $\angle ADB=\theta$, $BC=p$ and $CD=q$, then AB is equal to

(A) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ **(B)** $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

(C) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ **(D)** $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

20. As shown in the figure AD is the altitude on BC and AD produced meets the circumcircle of ΔABC at P where $DP=x$. Similarly $EQ=y$ and $FR=z$. If A, b, c, respectively, denotes the sides BC, CA and AB then $\frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}$ has the value equal to



(A) $\tan A+\tan B+\tan C$

(B) $\cot A+\cot B+\cot C$

(C) $\cos A+\cos B+\cos C$

(D) $\operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C$

21. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c}=\sqrt{\frac{3}{2}}$, then $\angle A$ is equal to

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$

(C) $\frac{5\pi}{12}$ **(D)** $\frac{\pi}{2}$

22. If K is a point on the side BC of an equilateral triangle ABC and if $\angle BAK=15°$, then the ratio of lengths $\frac{AK}{AB}$ is

(A) $\frac{3\sqrt{2}(3+\sqrt{3})}{2}$ **(B)** $\frac{\sqrt{2}(3+\sqrt{3})}{2}$

(C) $\frac{\sqrt{2}(3-\sqrt{3})}{2}$ **(D)** $\frac{3\sqrt{2}(3-\sqrt{3})}{2}$

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23. In a triangle ABC, $\angle A=60°$ and $b:c=(\sqrt{3}+1):2$ then $(\angle B-\angle C)$ has the value equal to

(A) 15° **(B)** 30°

(C) 22.5° **(D)** 45°

24. In an acute triangle ABC, $\angle ABC=45°$, $AB=3$ and $AC=\sqrt{6}$. The angle $\angle BAC$, is

(A) 60° **(B)** 65°

(C) 75° **(D)** 15° or 75°

25. In a triangle ABC, if $A=30°$ and $\frac{b}{c}=\frac{2+\sqrt{3}+\sqrt{2}-1}{2+\sqrt{3}-\sqrt{2}+1}$, then C, is equal to

(A) 75° **(B)** 15°

(C) $52\frac{1°}{2}$ **(D)** $97\frac{1°}{2}$

26. In a triangle ABC, if $a=6$, $b=3$ and $\cos(A-B)=\frac{4}{5}$, the area of the triangle is

(A) 8 **(B)** 9

(C) 12 **(D)** $\frac{15}{2}$

27. In ΔABC, if $a=2b$ and $A=3B$, then the value of $\frac{c}{b}$ is equal to

(A) 3 **(B)** 2

(C) 1 **(D)** $\sqrt{3}$

28. If the sides of a triangle are $\sin\alpha$, $\cos\alpha$, $\sqrt{1+\sin\alpha\cos\alpha}$, $0<\alpha<\frac{\pi}{2}$, the largest angle is

(A) 60° **(B)** 90°
(C) 120° **(D)** 150°

29. If the angle A, B, and C of a triangle are in an arithmetic progression and if a, b, and c denote the lengths of the sides opposite to A, B, and C, respectively, then the value of expression

$E = \left(\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A\right)$, is

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{3}}{2}$
(C) 1 **(D)** $\sqrt{3}$

30. If in a triangle $\sin A : \sin C = \sin(A-B) : \sin(B-C)$ then $a^2 : b^2 : c^2$
(A) are in A.P. **(B)** are in G.P.
(C) are in H.P. **(D)** none of these

PART – II KVPY Level Problems

1. If $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$ then $bc + \frac{1}{ck} + \frac{ak}{1+bk}$ is equal to

(A) $\frac{a}{k} + \frac{1}{k}$ **(B)** $\frac{a}{k} + \frac{1}{ak}$
(C) $\frac{1}{k^2}$ **(D)** $\frac{a}{k}$

2. The least value of $6\tan^2\phi + 54\cot^2\phi + 18$ is
I: 54 when A.M. ≥ G.M. is applicable for $6\tan^2\phi$, $54\cot^2\phi$, 18.
II: 54 when A.M. ≥ G.M. is applicable for $6\tan^2\phi$, $54\cot^2\phi$, and 18 added further.
III: 78 when $\tan^2\phi = \cot^2\phi$.
(A) I is correct
(B) I and II are correct
(C) III is correct
(D) None of the above is correct

3. If $\pi < a < \frac{3\pi}{2}$, then $\sqrt{\frac{1+\cos a}{1-\cos a}}$ is equal to:

(A) $\frac{2}{\sin a}$ **(B)** $-\frac{2}{\sin a}$
(C) $\frac{1}{\sin a}$ **(D)** $-\frac{1}{\sin a}$

4. The value of $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$ is:

(A) 1 **(B)** – 1
(C) 0 **(D)** None of these

5. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$, then $x + y + z$ is equal to:

(A) 1 **(B)** 0
(C) –1 **(D)** None of these

6. $\sin^{2n} x + \cos^{2n} x$ lies between:
(A) –1 and 1 **(B)** 0 and 1
(C) 1 and 2 **(D)** None of these

7. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is:
(A) 3/4 **(B)** $3\sqrt{3}$
(C) 3 **(D)** $3\sqrt{3}/2$

8. Which of the following is not the value of sin 27° – cos 27°?

(A) $-\frac{\sqrt{3-\sqrt{5}}}{2}$ **(B)** $-\frac{\sqrt{5-\sqrt{5}}}{2}$
(C) $-\frac{\sqrt{5}-1}{2\sqrt{2}}$ **(D)** None of these

9. If $x = \frac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then $\frac{1-\cos\theta+\sin\theta}{1+\sin\theta}$ is equal to:
(A) $1+x$ **(B)** $1-x$
(C) x **(D)** $1/x$

10. The value of the expression

$\frac{2(\sin 1° + \sin 2° + \sin 3° + \ldots + \sin 89°)}{2(\cos 1° + \cos 2° + \ldots + \cos 44°) + 1}$ equals:

(A) $\sqrt{2}$ **(B)** $1/\sqrt{2}$
(C) 1/2 **(D)** 1

11. $\frac{1}{4}\left[\sqrt{3}\cos 23° - \sin 23°\right]$ is equal to:
(A) cos 43° **(B)** cos 7°
(C) cos 53° **(D)** None of these

12. tan 100° + tan 125° + tan 100° tan 125° is equal to:
(A) 0 **(B)** 1/2
(C) –1 **(D)** 1

13. If $\sin\theta_1 - \sin\theta_2 = a$ and $\cos\theta_1 + \cos\theta_2 = b$, then
(A) $a^2 + b^2 \geq 4$ **(B)** $a^2 + b^2 \leq 4$
(C) $a^2 + b^2 \geq 3$ **(D)** $a^2 + b^2 \leq 2$

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14. If $\tan\alpha$ is equal to the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and $\cos\beta$ is equal to the slope of the bisector of the first quadrant, then $\sin(\alpha+\beta)\sin(\alpha-\beta)$ is equal to:

(A) $\frac{3}{5}$ **(B)** $\frac{3}{5}$

(C) $\frac{2}{\sqrt{5}}$ **(D)** $\frac{4}{5}$

15. Let $f(n) = 2\cos nx\ \forall\ n \in N$, then $f(1)\,f(n+1) - f(n)$ is equal to:

(A) $f(n+3)$ **(B)** $f(n+2)$

(C) $f(n+1)\,f(2)$ **(D)** $f(n+2)\,f(2)$

16. If $x_1, x_2, x_3, \ldots, x_n$ are in A.P. whose common difference is A, then the value of $\sin a(\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + \ldots + \sec x_{n-1}\sec x_n)$ is:

(A) $\frac{\sin(n-1)a}{\cos x_1 \cos x_n}$

(B) $\frac{\sin na}{\cos x_1 \cos x_n}$

(C) $\sin(n-1)a\cos x_1 \cos x_n$

(D) $\sin na \cos x_1 \cos x_n$

17. Let a and b be such that $\pi < (\alpha-\beta) < 3\pi$. If $\sin\alpha + \sin\beta = -\frac{21}{65}$ and $\cos\alpha + \cos\beta = -\frac{17}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ is:

(A) $-\frac{3}{\sqrt{130}}$ **(B)** $\frac{3}{\sqrt{130}}$

(C) $\frac{6}{65}$ **(D)** $-\frac{6}{65}$

18. If $\cos 25° + \sin 25° = p$, then $\cos 50°$ is:

(A) $\sqrt{2-p^2}$ **(B)** $-\sqrt{2-p^2}$

(C) $p\sqrt{2-p^2}$ **(D)** $-p\sqrt{2-p^2}$

19. If $\tan A = \frac{1-\cos B}{\sin B}$, then $\tan 2A$ is:

(A) $\tan 2A = \tan B$

(B) $\tan 2A = \tan^2 B$

(C) $\tan 2A = \tan^2 B + 2\tan B$

(D) None of these

20. If $\tan^2\theta = 2\tan^2\phi + 1$, then $\cos 2\theta + \sin^2\phi$ equals

(A) -1 **(B)** 0

(C) 1 **(D)** None of these

21. Given that $(1+\sqrt{1+x})\tan y = 1+\sqrt{1-x}$. Then $\sin 4y$ is equal to:

(A) $4x$ **(B)** $2x$

(C) x **(D)** None of these

22. If θ_1 and θ_2 are two values lying in $[0, 2\pi]$ for which $\tan\theta = \lambda$, then $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$ is equal to

(A) 0 **(B)** -1

(C) 2 **(D)** 1

23. If $\tan\theta = \sqrt{n}$ where $n \in N \geq 2$, then $\sec 2\theta$ is always:

(A) a rational number **(B)** an irrational number

(C) a positive integer **(D)** a negative integer

24. $\cos^3 x \sin 2x = \sum_{x=0}^{n} a_r \sin(rx)\ \forall\ x \in R$, then

(A) $n = 5,\ a_1 = 1/2$ **(B)** $n = 5,\ a_1 = 1/4$

(C) $n = 5,\ a_2 = 1/7$ **(D)** $n = 5,\ a_2 = 1/4$

25. If $\cos x = \frac{2\cos y - 1}{2 - \cos y}$, where $x, y \in (0, \pi)$, then $\tan\frac{x}{2}\cot\frac{y}{2}$ is equal to:

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$

(C) $\frac{1}{\sqrt{2}}$ **(D)** $\frac{1}{\sqrt{3}}$

26. The value of $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is:

(A) 2 **(B)** 3

(C) 4 **(D)** None of these

27. In triangle ABC, $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P., then the value of $\cot\frac{A}{2}\cdot\cot\frac{C}{2}$ is equal to:

(A) 1 **(B)** 2

(C) 3 **(D)** 4

28. If $\tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha + 2\tan^2\alpha\tan^2\beta\tan^2\gamma = 1$, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

(A) 3 **(B)** 2

(C) 1 **(D)** None of these

29. In triangle ABC, $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$, then the values of $\tan A$, $\tan B$, $\tan C$ are:

(A) 1, 2, 3 **(B)** 3, 2/3, 7/3

(C) 4, 1/2, 3/2 **(D)** None of these

30. If $0 < a < \frac{\pi}{6}$, then a (cosec A) is:

(A) less than $\pi/6$ **(B)** greater than $\pi/6$

(C) less than $\pi/3$ **(D)** greater than $\pi/3$

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31. If $\left|\cos\theta\left\{\sin\theta+\sqrt{\sin^2\theta+\sin^2 a}\right\}\right| \le k$, then the value of k is:

(A) $\sqrt{1+\cos^2 a}$ **(B)** $\sqrt{1+\sin^2 a}$

(C) $\sqrt{2+\sin^2 a}$ **(D)** $\sqrt{2+\cos^2 a}$

32. The numerical value of $\tan\frac{\pi}{3}+2\tan\frac{2\pi}{3}+4\tan\frac{4\pi}{3}+8\tan\frac{8\pi}{3}$ is equal to:

(A) $-5\sqrt{3}$ **(B)** $-5/\sqrt{3}$

(C) $5\sqrt{3}$ **(D)** $5/\sqrt{3}$

33. If $a\sin x+b\cos(x+\theta)+b\cos(x-\theta)=d$, then the minimum value of $|\cos\theta|$ is equal to:

(A) $\frac{1}{2|b|}\sqrt{d^2-a^2}$ **(B)** $\frac{1}{2|a|}\sqrt{d^2-a^2}$

(C) $\frac{1}{2|d|}\sqrt{d^2-a^2}$ **(D)** None of these

34. If $x \in (\pi, 2\pi)$ and $\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \cot\left(a+\frac{x}{2}\right)$, then a is equal to:

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ **(D)** None of these

35. If $\tan x = n\tan y$, $n \in R^+$, then the maximum value of $\sec^2(x-y)$ is equal to:

(A) $\frac{(n+1)^2}{2n}$ **(B)** $\frac{(n+1)^2}{n}$

(C) $\frac{(n+1)^2}{2}$ **(D)** $\frac{(n+1)^2}{4n}$

36. If A, B, C are acute positive angles such that $A+B+C=\pi$ and $\cot A\cot B\cot C=k$, then

(A) $K \le \frac{1}{3\sqrt{3}}$ **(B)** $K \ge \frac{1}{3\sqrt{3}}$

(C) $K < \frac{1}{9}$ **(D)** $K > \frac{1}{3}$

37. The range of k for which the inequality $k\cos^2 x - k\cos x\, 1 \ge 0\ \forall\ x \in (-\infty, \infty)$, is:

(A) $k < \frac{-1}{2}$ **(B)** $\frac{-1}{2} \le k \le 4$

(C) $k > 4$ **(D)** $\frac{1}{2} \le k \le 5$

38. If $\theta = 3a$ and $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$ The value of the expression $a\,\text{cosec}\,A - b\sec a$ is:

(A) $\frac{a}{\sqrt{a^2+b^2}}$ **(B)** $2\sqrt{a^2+b^2}$

(C) $a+b$ **(D)** None of these

39. In triangle ABC if angle C is 90° and area of triangle is 30 sq. units, then the minimum possible value of the hypotenuse c is equal to:

(A) $30\sqrt{2}$ **(B)** $60\sqrt{2}$

(C) $120\sqrt{2}$ **(D)** $2\sqrt{30}$

40. Given then a, b, c are the sides of a triangle ABC which is right angled at C, then the minimum value of $\left(\frac{c}{a}+\frac{c}{b}\right)^2$ is:

(A) 0 **(B)** 4

(C) 6 **(D)** 8

41. ABC is an equilateral triangle of side 4 cm. If R, r and h are the circum radius, in radius and altitude, respectively, then $\frac{R+r}{h}$ is equal to

(A) 4 **(B)** 2

(C) 1 **(D)** 3

42. A piece of paper is in the shape of a square of side 1 m long. It is cut at the four comers to make a regular polygon of eight sides (octagon). The area of the polygon is

(A) $2(\sqrt{2}-1)\text{m}^2$ **(B)** $(\sqrt{2}-1)\text{m}^2$

(C) $\frac{1}{\sqrt{2}}\text{m}^2$ **(D)** none of these

43. In ΔABC, $\angle B = \pi/3$. The range of values of x, where $x = \sin A\sin C$, is the interval

(A) $\left[-\frac{1}{4}, \frac{3}{4}\right]$ **(B)** $\left(0, \frac{3}{4}\right)$

(C) $\left[\frac{1}{4}, \frac{3}{4}\right]$ **(D)** $\left(0, \frac{3}{4}\right]$

44. If in ΔABC, AC is double of AB, then the value of $\cot\frac{A}{2}\cot\frac{B-C}{2}$ is equal to

(A) $\frac{1}{3}$ **(B)** $-\frac{1}{3}$

(C) 3 **(D)** $\frac{1}{2}$

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45. If in triangle ABC, $\angle B = 90°$, then $\tan^2(A/2)$ is

(A) $\frac{b-c}{b+c}$ **(B)** $\frac{b+c}{b-c}$

(C) $\frac{2bc}{b-c}$ **(D)** none of these

46. If in a triangle, $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, then the triangle is

(A) right angled **(B)** isosceles

(C) equilateral **(D)** none of these

47. In ΔABC, if $A = 30°$, $b = 2$, $c = \sqrt{3}+1$ then $\frac{C-B}{2}$ is equal to

(A) 15° **(B)** 30°

(C) 45° **(D)** none of these

48. In triangle ABC, if $\cos A + \cos B + \cos C = \frac{7}{4}$, then $\frac{R}{r}$ is equal to

(A) $\frac{3}{4}$ **(B)** $\frac{4}{3}$

(C) $\frac{2}{3}$ **(D)** $\frac{3}{2}$

49. In triangle ABC, $a^2 + c^2 = 2002\, b^2$, then $\frac{\cot A + \cot C}{\cot B}$ is equal to

(A) $\frac{1}{2001}$ **(B)** $\frac{2}{2001}$

(C) $\frac{3}{2001}$ **(D)** $\frac{4}{2001}$

50. If the hypotenuse of a right–angled triangle is four times the length of the perpendicular drawn from the opposite vertex to it, then the difference of the two acute angles will be

(A) 60° **(B)** 15°

(C) 75° **(D)** 30°

51. In triangle ABC, internal angle bisector $\angle A$ makes an angle θ with side BC. The value of sine is

(A) $\left|\sin\left(\frac{B-C}{2}\right)\right|$ **(B)** $\left|\sin\left(\frac{B}{2}-C\right)\right|$

(C) $\cos\left(\frac{B-C}{2}\right)$ **(D)** $\cos\left(\frac{B}{2}-C\right)$

52. In an acute-angled triangle ABC, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then

(A) $b + 2c < 2a < 2b + 2c$ **(B)** $b + 4c < 4a < 2b + 4c$

(C) $b + 4c < 4a < 4b + 4c$ **(D)** $b + 3c < 3a < 3b + 3c$

53. If I is the in center of a triangle ABC, then the ratio $IA : IB : IC$ is equal to

(A) $\operatorname{cosec}\frac{A}{2} : \operatorname{cosec}\frac{B}{2} : \operatorname{cosec}\frac{C}{2}$

(B) $\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2}$

(C) $\sec\frac{A}{2} : \sec\frac{B}{2} : \sec\frac{C}{2}$

(D) none of these

54. In ΔABC, the bisector of the angle A meets the side BC at D and the circumscribed circle at E, then DE equals

(A) $\frac{a^2 \sec\frac{A}{2}}{2(b+c)}$ **(B)** $\frac{a^2 \sin\frac{A}{2}}{2(b+c)}$

(C) $\frac{a^2 \cos\frac{A}{2}}{2(b+c)}$ **(D)** $\frac{a^2 \operatorname{cosec}\frac{A}{2}}{2(b+c)}$

55. Two medians drawn from the acute angles of a right–angled triangle intersect at an angle $\pi/6$. If the length of the hypotenuse of the triangle is 3 units, then the area of the triangle (in sq. units) is

(A) $\sqrt{3}$ **(B)** 3

(C) $\sqrt{2}$ **(D)** 9

56. In ΔABC, the median AD divides $\angle BAC$ such that $\angle BAD : \angle CAD = 2 : 1$. Then $\cos(A/3)$ is equal to

(A) $\frac{\sin B}{2\sin C}$ **(B)** $\frac{\sin C}{2\sin B}$

(C) $\frac{2\sin B}{\sin C}$ **(D)** none of these

57. Let area of triangle ABC is $(\sqrt{3}-1)/2$, $b = 2$ and $c = (\sqrt{3}-1)$ and $\angle A$ is acute. The measure of the angle C is

(A) 15° **(B)** 30°

(C) 60° **(D)** 75°

58. If H is the orthocentre of a acute–angled triangle ABC whose circum circle is $x^2 + y^2 = 16$, then circum diameter of the triangle HBC is

(A) 1 **(B)** 2

(C) 4 **(D)** 8

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59. In triangle ABC, line joining circum centre and in centre is parallel to side AC, then $\cos A + \cos C$ is equal to

(A) -1 **(B)** 1
(C) -2 **(D)** 2

60. If A, band A are given in a triangle and c_1, c_2 are the possible values of the third side, then $c_1^2 + c_2^2 - 2\, c_1 c_2 \cos A$ is equal to

(A) $4a^2 \sin 2A$ **(B)** $4a^2 \sin^2 A$
(C) $4a^2 \cos 2A$ **(D)** $4a^2 \cos^2 A$

61. In triangle ABC, if P, Q, R divides sides BC, AC and AB, respectively, in the ratio $k : 1$ (in order). If the ratio $\left(\frac{\text{area } PQR}{\text{area } ABC}\right)$ is $\frac{1}{3}$, then k is equal to

(A) 1/3 **(B)** 2
(C) 3 **(D)** none of these

62. The side of triangle ABC are in A.P. (order being A, b, c) and satisfy $\frac{2!}{1!9!} + \frac{2!}{3!7!} + \frac{1}{5!5!} = \frac{8^a}{(2b)!}$, then the value of $\cos A + \cos B$ is

(A) $\frac{12}{7}$ **(B)** $\frac{13}{7}$
(C) $\frac{11}{7}$ **(D)** $\frac{10}{7}$

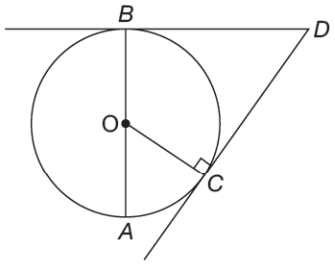
63. If a, b, and c are the sides of a triangle, then the minimum value of $\sum \frac{2a}{b+c-a}$ is

(A) 3 **(B)** 9
(C) 6 **(D)** 1

64. In triangle ABC, $\angle A = 60°$, $\angle B = 40°$, and $\angle C = 80°$. If P is the centre of the circum circle of triangle ABC with radius unity, then the radius of the circum circle of triangle BPC is

(A) 1 **(B)** $\sqrt{3}$
(C) 2 **(D)** $\sqrt{3}/2$

65. In the given figure, AB is the diameter of the circle, centered at 'O'. If $\angle COA = 60°$, $AB = 2r$, $AC = d$ and CD is equal to



(A) $d\sqrt{3}$ **(B)** $d/\sqrt{3}$
(C) 3d **(D)** $\sqrt{3}\, d/2$

66. The radii r_1, r_2, r_3 of the escribed circles of the triangle ABC are in H.P. If the area of the triangle is 24 cm^2 and its perimeter is 24 cm, then the length of its largest side is

(A) 10 **(B)** 9
(C) 8 **(D)** none of these

67. If 'O' is the circum centre of ΔABC and R_1, R_2, and R_3 are the radii of the circum circles of triangles OBC, OCA and OAB, respectively, then $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$ has the value equal to

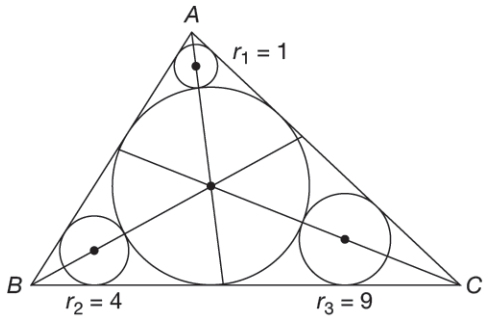
(A) $\frac{abc}{R^3}$ **(B)** $\frac{R^3}{abc}$
(C) $\frac{4\Delta}{R^2}$ **(D)** $\frac{\Delta}{4R^2}$

PART – III PRMO Level Problems

1. Let x and y be real numbers such that
$x^5 - 101x^3 - 999x^2 + 100900 = 0$.
What is the largest possible value of $x^2 + y^2 - \sqrt{32}$?

2. The ratios of the length of the sides BC and AC of ΔABC to the radius of circumscribed circle are equal to 2 and $\frac{3}{2}$, respectively. If the ratio of the lengths of the bisectors of the interior angles B and C is $\frac{\alpha\left(\sqrt{\alpha}-1\right)}{\beta\sqrt{\gamma}}$ where $\alpha, \beta, \gamma \in N$, then find the value of $(\alpha + \beta + \gamma)$.

3. ABC is a triangle. Circles with radii as shown are drawn inside the triangle each touching two sides and the incircle. Find the radius of the incircle of the ΔABC.



4. In triangle ABC, let $b = 6$, $c = 10$ and $r_1 = r_2 + r_3 + r$ then find area of triangle ABC.
[**Note:** All symbols used have usual meaning in a triangle.]

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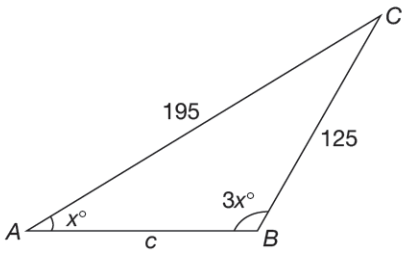
5. If in ΔABC, $\angle C = \frac{\pi}{8}$, $A = \sqrt{2}$ and $b = \sqrt{2+\sqrt{2}}$ then find the measure of angle A (in degrees).

6. In the figure as shown, find last two digits of the length of AB.



7. Let ABC be a right-angled triangle at C. If the inscribed circle touches the side AB at D and $(AD)(BD) = 11$, then find the area of triangle ABC.

8. Let T_n denotes then n^{th} term of a G.P. with common ratio 2 and $\left(\log_2\left(\log_3\left(\log_{512}\left(T_{100}\right)\right)\right)\right) T = 1$. If three sides of a triangle ABC are the values of $(T_1 + T_2)$, T_2 and T_3, then area of the triangle is $\frac{\sqrt{2160}}{N}$, where N is a positive integer. Find the remainder when N is divided by 2^{10}.

9. Let a, b, and c be the side lengths of a triangle ABC and assume that $a \le b$ and $a \le c$. If $x = \frac{b+c-a}{2}$, then find the minimum value of $\frac{ax}{rR}$, where r and R denote inradius and circumradius of triangle ABC.

10. Consider an obtuse-angled triangle with sides 8 cm, 15 cm and x cm (largest side being 15 cm). If x is an integer, then find the number of possible triangles.

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. The expression $(1 - \tan A + \sec A)(1 - \cot A + \operatorname{cosec} A)$ has value: [NSEJS-2008]
   (A) –1 **(B)** 0
   (C) +1 **(D)** +2

2. A person on the top of a tower observes a scooter moving with uniform velocity towards the base of the tower. He finds that the angle of depression changes from 30° to 60° in 18 minutes. The scooter will reach the base of the tower in next: [NSEJS-2008]
   (A) 9 minutes
   (B) $18/(\sqrt{3}-1)$ minutes
   (C) $6\sqrt{3}$ minutes
   (D) the time depends upon the height of the tower

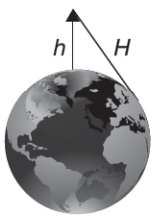
3. The trigonometric equation is:
   $\cot^2\theta\left[\frac{\sec\theta-1}{1+\sin\theta}\right]+\sec^2\theta\left[\frac{\sin\theta-1}{1+\sec\theta}\right]$ has the value: [NSEJS-2009]
   (A) –1 **(B)** 0
   (C) 1 **(D)** 2

4. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $[\sin^8\theta + \operatorname{cosec}^8\theta]$ will have the value [NSEJS-2010]
   (A) 2 **(B)** 2^4
   (C) 2^6 **(D)** 2^8

5. An aeroplane is flying horizontally at a height of 3150 m above a horizontal plane ground. At a particular instant it passes another aeroplane vertically below it. At this instant, the angles of elevation of the planes form a point on the ground are 30° and 60°. Hence, the distance between the planes at that instant is [NSEJS-2011]
   (A) 1050 m **(B)** 2100 m
   (C) 4200 m **(D)** 5250 m

6. If $\sin x + \sin y = a$ and $\cos x - \cos y = b$. Then find the value of $\frac{1}{2}(2 - a^2 - b^2)$ [NSEJS-2012]
   (A) $\cos(x + y)$ **(B)** $\cos(x - y)$
   (C) $\sin(x + y)$ **(D)** $\sin(x - y)$

7. If $\cot^2\theta(1 - 3\sec\theta + 2\sec^2\theta) = 1$ [NSEJS-2012]
   (A) 120° **(B)** 210°
   (C) 300° **(D)** 330°

8. We all know that the sky appears to touch the ground at a distance. The distance at which we perceive the sky to touch the ground is called horizon. The reason for the perception is due to the fact that the Earth is a sphere (almost) and not a flat surface. Which of these pictures below accurately depict the horizon for a person standing on a high rise building like Burj Khalifa in Dubai? (Here, 'h' represents the height of the building while line 'H' represents the horizon): [NSEJS-2012]
   (A)
   

   (B)

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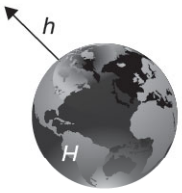
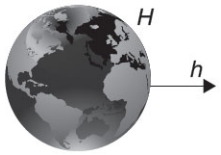
(C) 

(D) 

9. If $\sin\theta$ and $\cos\theta$ are roots of the equation $px^2 + qx + r = 0$, then: **[NSEJS-2018]**

(A) $p^2 - q^2 + 2pr = 0$

(B) $(p + r)^2 = q^2 - r^2$

(C) $p^2 + q^2 - 2pr = 0$

(D) $(p - r)^2 = q^2 + r^2$

10. An observer standing at the top of a tower, finds that the angle of elevation of a red bulb on the top of a light house of height H is α. Further, he finds that the angle of depression of reflection of the bulb in the ocean is β. Therefore, the height of the tower is **[NSEJS-2019]**

(A) $\dfrac{H(\tan\beta - \tan\alpha)}{(\tan\beta + \tan\alpha)}$ **(B)** $\dfrac{H\sin(\beta - \alpha)}{\cos(\alpha + \beta)}$

(C) $\dfrac{H(\cos\alpha - \cos\beta)}{(\cot\alpha + \cot\beta)}$ **(D)** H

11. In the convex quadrilateral $ABCD$, the diagonals AC and BD meet at 0 and the measure of angle AOB is 30°. If the areas of triangle AOB, BOC, COD, and AOD are 1, 2, 8 and 4 square units, respectively, what is the product of the lengths of the diagonals AC and BD in sq. units ? **[NSEJS-2019]**

(A) 60 **(B)** 56

(C) 54 **(D)** 64

12. If $\tan\theta + \sec\theta = 1.5$, then value of $\sin(\theta)$ is **[NSEJS-2019]**

(A) $\dfrac{5}{13}$ **(B)** $\dfrac{12}{13}$

(C) $\dfrac{3}{5}$ **(D)** $\dfrac{2}{3}$

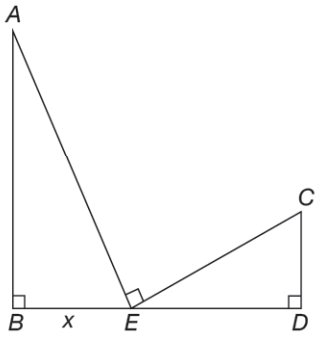
13. If $\sin^2 x + \sin^2 y + \sin^2 z = 0$, then which of the following is NOT a possible value of $\cos x + \cos y + \cos z$? **[NSEJS-2019]**

(A) 3 **(B)** –3

(C) –1 **(D)** –2

PART – II Previous Years' KVPY Problems

1. In given figure, $AB = 12$ cm, $CD = 8$ cm, $BD = 20$ cm, $\angle ABD = \angle AEC = \angle EDC = 90°$. If $BE = x$, then



[KVPY-2013]

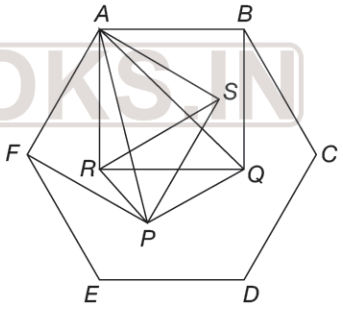
(A) x has two possible values whose difference is 4

(B) x has two possible values whose sum is 28

(C) x has only one value and $x \geq 12$

(D) x cannot be determined with the given information

2. In the figure given below, $ABCDEF$ is a regular hexagon of side length 1, $AFPS$ and $ABQR$ are squares. Then, the ratio ar(APQ)/ar(SRP) equals **[KVPY-2013]**



(A) $\dfrac{\sqrt{2}+1}{2}$ **(B)** $\sqrt{2}$

(C) $\dfrac{3\sqrt{3}}{4}$ **(D)** 2

3. The sides a, b, c of a triangle satisfy the relations $c^2 = 2ab$ and $a^2 + c^2 = 3b^2$. Then the measure of $\angle BAC$, in degrees, is **[KVPY-2019]**

(A) 30 **(B)** 45

(C) 60 **(D)** 90

4. Let ABC be a triangle in which $AB = BC$. Let X by a point on AB such that $AX : XB = AB : AX$. If $AC = AX$, then the measure of $\angle ABC$ equals **[KVPY-2019]**

(A) 18° **(B)** 36°

(C) 54° **(D)** 72°

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PART – III Previous Years' PRMO Problems

1. Let S be a circle with centre O. A chord AB_1 not a diameter divides S into two regions R_1 and R_2 such that O belongs to R_2. Let S_1 be a circle with centre in R_1, touching AB at X and S internally. Let S_2 be a circle with centre in R_2, touching AB at Y, the circle S internally and passing through the centre of S. The point X lies on the diameter passing through the centre of S_2 and $\angle YXO = 30°$. If the radius of S_2 is 100 then what is the radius of S_1? **[PRE-RMO 2013]**

2. Points G and O denote the centroid and the circumcenter of the ΔABC. Suppose that $\angle AGO = 90°$ and $AB = 17$ and $AC = 19$. Find the value of BC^2. **[PRE-RMO 2016]**

3. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lines between these chords. Suppose $AB = 6$, $CD = 8$. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n)/k$, where m, n, k are positive integers with $\gcd(m, n, k) = 1$. What is the value of $m + n + k$? **[PRE-RMO 2017]**

4. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A. **[PRE-RMO 2018]**

5. If $x = \cos 1° \cos 2° \cos 3° \ldots \cos 89°$ and $y = \cos 2° \cos 6° \cos 10° \ldots \cos 86°$, then what is the integer nearest to $\frac{2}{7}\log_2(y/x)$? **[PRE-RMO 2018]**

6. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the ant hill again, what is the distance in feet it would have walked? **[PRE-RMO 2019]**

7. How many distinct triangles ABC are there, up to similarity, such that the magnitudes of angles A, B and C in degrees are positive integers and satisfy $\cos A \cos B + \sin A \sin B \sin kC = 1$ for some positive integer k, where kC does not exceed 360°? **[PRE-RMO 2019]**

8. Let $ABCD$ be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of $ABCD$ is the least. If $\{PA, PB, PC, PD\} = \{3, 4, 6, 8\}$. What is the maximum possible area of $ABCD$? **[PRE-RMO 2019]**

9. Let ABC be a triangle such that $AB = AC$. Suppose the tangent to the circumcircle of ABC at B is perpendicular of AC. Find $\angle ABC$ measured in degrees. **[PRE-RMO 2019]**

10. In parallelogram $ABCD$, $AC = 10$ and $BD = 28$. The points K and L in the plane of $ABCD$ move in such a way that $AK = BD$ and $BL = AC$. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$? **[PRE-RMO 2019]**

11. Let t be the area of a regular pentagon with each side equal to 1. Let $P(x) = 0$ be the polynomial equation with least degree, having integer coefficients, satisfied by $x = t$ and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of $P(x)$. What is the integer closest to M? $(\sin 18° = \sqrt{5} - 1/2)$. **[PRE-RMO 2019]**

12. Let ABC be an isosceles triangle with $AB = BC$. A trisector of $\angle B$ meets AC at D. If AB, AC, and BD are integers and $AB - BD = 3$, find AC. **[PRE-RMO 2019]**

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13. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance travelled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N? **[PRE-RMO 2019]**

14. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid-point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass. If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d? (Ignore the thickness of the glass) **[PRE-RMO 2019]**

15. In a triangle ABC, it is known that $\angle A = 100°$ and $AB = AC$. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that $\sin 10° \approx 0.174$. **[PRE-RMO 2019]**

ANSWER KEYS

Know Your Progress - 1

1. 70 g 27′ 50″ **2.** 88 cm **3.** 56 metres **4.** 20°, 60° and 100° **5.** 144°, 160^g, $\frac{4\pi}{5}$ radians

6. $\sec\theta = \frac{5}{3}$, $\operatorname{cosec}\theta = \frac{5}{4}$, $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{4}$

7. $\sin\theta = \frac{n}{\sqrt{m^2+n^2}}$

9. $\cot C = \frac{3}{4}$, $\operatorname{cosec} C = \frac{5}{4}$, $\sec C = \frac{5}{3}$

11. $\frac{113}{24}$ **12.** $\frac{7}{2}$ **13.** $\frac{3-\sqrt{3}}{4\sqrt{2}}$ **14.** $50\sqrt{3}$ cm^2 **15.** $\tan 60°$ is $\sqrt{3}$

18. 2 **19.** 1 **20.** $\theta = 24°$ **23.** 1

Know Your Progress - 2

1. 90°

6. (a) 6 (b) $\frac{5}{2}$ (c) 1

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Know Your Progress - 3

1. $15\sqrt{3}$ m **2.** $h = \sqrt{ab}$ **3.** 50 meter **4.** 5.568 m

5. $\tan^{-1}\left(\frac{2}{9}\right)$ **6.** C **7.** C **8.** A

Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. D	**2.** D	**3.** C	**4.** C	**5.** A	**6.** B	**7.** B	**8.** B	**9.** C	**10.** D
11. C	**12.** A	**13.** C	**14.** B	**15.** B	**16.** C	**17.** C	**18.** B	**19.** A	**20.** A
21. C	**22.** C	**23.** B	**24.** C	**25.** C	**26.** B	**27.** D	**28.** C	**29.** D	**30.** A

PART – II KVPY Level Problems

1. B	**2.** B	**3.** B	**4.** B	**5.** B	**6.** B	**7.** C	**8.** C	**9.** C	**10.** A
11. D	**12.** D	**13.** B	**14.** D	**15.** B	**16.** A	**17.** A	**18.** A	**19.** A	**20.** B
21. C	**22.** B	**23.** A	**24.** B	**25.** B	**26.** C	**27.** C	**28.** C	**29.** A	**30.** C
31. B	**32.** A	**33.** A	**34.** A	**35.** D	**36.** A	**37.** B	**38.** B	**39.** D	**40.** D
41. C	**42.** A	**43.** D	**44.** C	**45.** A	**46.** A	**47.** B	**48.** B	**49.** B	**50.** A
51. C	**52.** D	**53.** A	**54.** A	**55.** A	**56.** A	**57.** A	**58.** D	**59.** B	**60.** D
61. B	**62.** A	**63.** C	**64.** A	**65.** A	**66.** A	**67.** A			

PART – III PRMO Level Problems

1. 6 **2.** 18 **3.** 11 **4.** 30 **5.** 45 **6.** 12 **7.** 11 **8.** 0 **9.** 3 **10.** 5

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. D **2.** A **3.** B **4.** A **5.** B **6.** A **7.** C **8.** A **9.** A **10.** A
11. A **12.** A **13.** D

PART – II Previous Years' KVPY Problems

1. A **2.** D **3.** B **4.** B

PART – III Previous Years' PRMO Problems

1. 6 **2.** 325 **3.** 75 **4.** 24 **5.** 19 **6.** 36 **7.** 6 **8.** 55 **9.** 30 **10.** 2
11. *BONUS **12.** 26 **13.** 31 **14.** 36 **15.** 27

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* Value of sin18° is wrong.

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Chapter

9 Coordinate Geometry

Learning Outcomes

After reading this chapter, you will be able to understand

- Section and Distance Formula and Application in Geometry
- Different Forms of Straight Line
- Locus of a Point
- Equation of Circle

Introduction and Application of Coordinate Geometry

Introduction of Coordinate Geometry

Coordinate geometry is the combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes. The resulting combination of analysis and geometry is referred as **analytical geometry**.

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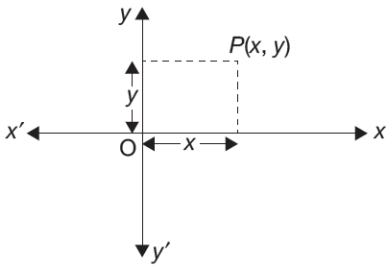
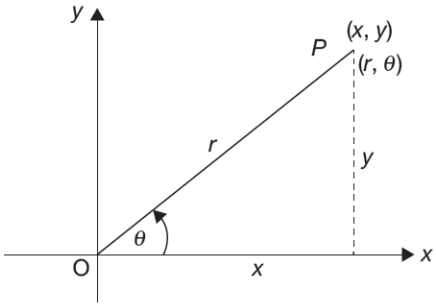
Cartesian Co-ordinates System



In 2-dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the ***x*-axis** and the vertical axis is ***y*-axis**. The point of intersection O is the origin of the coordinate system. Distances along the x-axis to the right of the origin are taken as positive, distances to the left as negative. Distances along the y-axis above the origin are positive; distances below are negative. The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x-coordinate (**or abscissa**) is the distance of the point from the y-axis in a direction parallel to the x-axis (i.e., horizontally). The y-coordinate (**or ordinate**) is the distance from the x-axis in a direction parallel to the y-axis (vertically). The origin O is the point $(0, 0)$.

Polar Co-ordinates System



A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r). The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anti-clockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).

If (x, y) are cartesian co-ordinates of a point P, then:

$$x = r\cos\theta,\ y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}\ ,\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Distance Formula and its Applications

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note

(i) Three given points A, B, and C are collinear, when sum of any two distances out of AB, BC, CA is equal to the remaining third, otherwise the points will be the vertices of a triangle.

(ii) Let A, B, C, and D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if $AB = BC = CD = DA$ and $AC = BD$; $AC \perp BD$

(b) Rhombus if $AB = BC = CD = DA$ and $AC \neq BD$; $AC \perp BD$

(c) Parallelogram if $AB = DC$, $BC = AD$; $AC \neq BD$; $AC \not\perp BD$

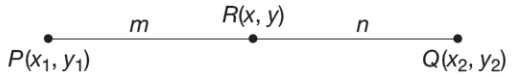
(d) Rectangle if $AB = CD$, $BC = DA$, $AC = BD$; $AC \not\perp BD$

Section Formula

The co-ordinates of a point dividing a line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m : n$ is given by:

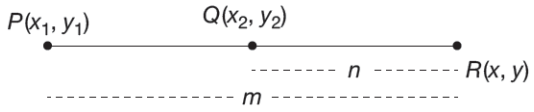
(a) For internal division: $P - R - Q \Rightarrow R$ divides line segment PQ, internally.

$$(x, y) \equiv \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$



(b) For external division: $R - P - Q$ or $P - Q - R$ $\Rightarrow R$ divides line segment PQ, externally.

$$(x, y) \equiv \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$



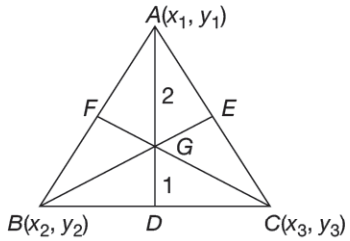
$\sqrt{17} < 1 \Rightarrow R$ lies on the left of P and $\frac{17}{\sqrt{15}} > 1$ $\Rightarrow R$ lies on the right of Q

(c) Harmonic conjugate: If P divides AB internally in the ratio $m : n$ and Q divides AB externally in the ratio $m : n$ then P and Q are said to be harmonic conjugate of each other w.r.t. AB. Mathematically $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e., AP, AB, and AQ are in H.P.

Co-ordinates of Some Particular Points

Let $A(x_1, y_1)$ $B(x_2, y_2)$, and $C(x_3, y_3)$ are vertices of any triangle ABC, then

Centroid



The centroid is the point of intersection of the medians (line joining the midpoint of sides and opposite vertices). Centroid divides each median in the ratio of 2 : 1.

Co-ordinates of centroid $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

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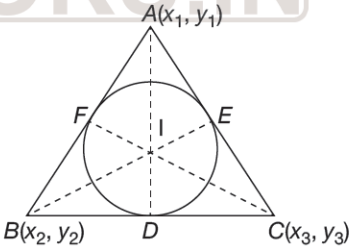
Incentre



The incentre is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle. Co-ordinates of incentre $I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$ where a, b, c are the sides of triangle ABC.

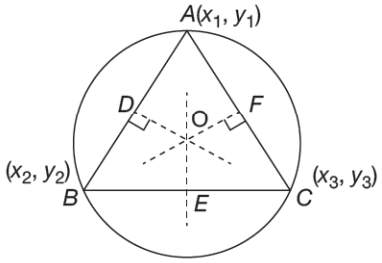
Note

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g.,

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$$

(ii) Incentre divides the angle bisectors in the ratio $(b + c) : a$, $(c + a) : b$, $(a + b) : c$.

Circumcentre



It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcentre of any triangle ABC, then $OA^2 = OB^2 = OC^2$. Also, it is a centre of a circle touching all the vertices of a triangle.

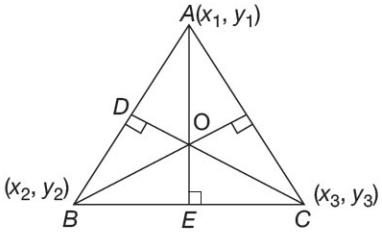
Note

(i) If the triangle is right angled, then its circumcentre is the mid-point of hypotenuse.

(ii) Co-ordinates of circumcentre

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Orthocentre



It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.

Note

(i) If a triangle is right angled, then orthocentre is the point where right angle is formed.

(ii) Co-ordinates of circumcentre

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

Remarks

(i) If the triangle is equilateral, then centroid, incentre, orthocentre, circumcentre, coincide.

(ii) Orthocentre, centroid, and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle, centroid, orthocentre, incentre, and circumcentre lie on the same line.

Example 1

The number of points on x-axis that are at a distance $c (c < 3)$ from the point (2, 3) is

(A) 2 **(B)** 1

(C) infinite **(D)** no point

Solution

Let a point on x-axis is $(x_1, 0)$. Then its distance from the point (2, 3)

$$= \sqrt{(x_1 - 2)^2 + 9} = c \quad \text{or} \quad (x_1 - 2)^2 = c^2 - 9$$

$$\therefore \quad x_1 - 2 = \pm\sqrt{c^2 - 9} \quad \text{since } c < 3 \Rightarrow c^2 - 9 < 0$$

$\therefore$ x_1 will be imaginary. **Ans. (D)**

Example 2

The distance between the point $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$, where $a > 0$ and $\alpha > \beta$, is:

(A) $4a\sin\dfrac{\alpha-\beta}{2}$ **(B)** $2a\sin\dfrac{\alpha+\beta}{2}$

(C) $2a\sin\dfrac{\alpha-\beta}{2}$ **(D)** $2a\cos\dfrac{\alpha-\beta}{2}$

Solution

$$d^2 = (a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2$$

$$= a^2(\cos\alpha - \cos\beta)^2 + a^2(\sin\alpha - \sin\beta)^2$$

$$= a^2\left\{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right\} + a^2\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right\}$$

$$= 4a^2\sin^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{\alpha-\beta}{2} + \cos^2\frac{\alpha+\beta}{2}\right\}$$

$$= 4a^2\sin^2\frac{\alpha-\beta}{2} \Rightarrow d = 2a\sin\frac{\alpha-\beta}{2}$$

Ans. (C)

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Example 3

Determine the ratio in which $y - x + 2 = 0$ divides the line joining $(3, -1)$ and $(8, 9)$.

Solution

Suppose the line $y - x + 2 = 0$ divides the line segment joining $A(3, -1)$ and $B(8, 9)$ in the ratio $\lambda : 1$ at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on $y - x + 2 = 0$ therefore $\left(\frac{9\lambda-1}{\lambda+1}\right) - \left(\frac{8\lambda+3}{\lambda+1}\right) + 2 = 0$

$$\Rightarrow \quad 9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$$

$$\Rightarrow \quad 3\lambda - 2 = 0 \text{ or } \lambda = \frac{2}{3}$$

So, the required ratio is $\frac{2}{3} : 1$, i.e., $2 : 3$ (internally) since here λ is positive.

Example 4

If $\left(\frac{5}{3}, 3\right)$ is the centroid of a triangle and its two vertices are $(0, 1)$ and $(2, 3)$, then find its third vertex, circumcentre, circumradius, and orthocentre.

Solution

Let the third vertex of triangle be (x, y), then

$$\frac{5}{3} = \frac{x+0+2}{3} \Rightarrow x = 3 \text{ and } 3 = \frac{y+1+3}{3} \Rightarrow y = 5.$$

So third vertex is $(3, 5)$

Now three vertices are $A(0, 1)$, $B(2, 3)$, and $C(3, 5)$

Let circumcentre be $P(h, k)$, then $AP = BP = CP = R$ (circumradius) $\Rightarrow AP^2 = BP^2 = CP^2 = R^2$

$$h^2 + (k-1)^2 = (h-2)^2 + (k-3)^2$$

$$= (h-3)^2 + (k-5)^2 = R^2 \qquad \text{(i)}$$

from the first two equations, we have

$$h + k = 3 \qquad \text{(ii)}$$

from the first and third equation, we obtain

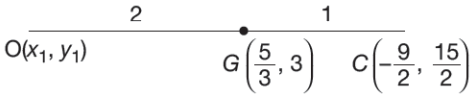
$$6h + 8k = 33 \qquad \text{(iii)}$$

On solving, (ii) and (iii), we get

$$h = -\frac{9}{2}, \; k = \frac{15}{2}$$

Substituting these values in (i), we have

$$R = \frac{5}{2}\sqrt{10}$$



Let $O(x_1, y_1)$ be the orthocentre, then $\frac{x_1 + 9\left(-\frac{9}{2}\right)}{3} = \frac{5}{3}$

$$\Rightarrow \quad x_1 = 14. \; \frac{y_1 + 2\left(\frac{15}{2}\right)}{3} = 3$$

$$\Rightarrow \quad y_1 = -6.$$

Hence, orthocentre of the triangle is $(14, -6)$.

Example 5

The vertices of a triangle are $A(0, -6)$, $B(-6, 0)$, and $C(1,1)$, respectively, then co-ordinates of the ex-centre opposite to vertex A are:

(A) $\left(\frac{-3}{2}, \frac{-3}{2}\right)$ **(B)** $\left(-4, -\frac{3}{2}\right)$

(C) $\left(\frac{-3}{2}, \frac{3}{2}\right)$ **(D)** $(-4, 6)$

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Solution

$$a = BC = \sqrt{(-6-1)^2 + (0-1)^2} = \sqrt{50} = 5\sqrt{2}$$

$$b = CA = \sqrt{(1-0)^2 + (1+6)^2} = \sqrt{50} = 5\sqrt{2}$$

$$c = AB = \sqrt{(0+6)^2 + (-6-0)^2} = \sqrt{72} = 6\sqrt{2}$$

co-ordinates of excentre opposite to vertex A will be:

$$x = \frac{-ax_1 + bx_2 + cx_3}{-a+b+c} = \frac{-5\sqrt{2}.0 + 5\sqrt{2}(-6) + 6\sqrt{2}(1)}{-5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}}$$

$$= \frac{-24\sqrt{2}}{6\sqrt{2}} = -4$$

$$y = \frac{-ay_1 + by_2 + cy_3}{-a+b+c} = \frac{-5\sqrt{2}(-6) + 5\sqrt{2.0} + 6\sqrt{2}(1)}{-5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}}$$

$$= \frac{36\sqrt{2}}{6\sqrt{2}} = 6$$

Hence co-ordinates of ex-centre are $(-4, 6)$ **Ans. (D)**

Know Your Progress - 1

1. (i) Find the distance between the points $P(-3, 2)$ and $Q(2, -1)$.
   (ii) If the distance between the points $P(-3, 5)$ and $Q(-x, -2)$ is $\sqrt{58}$, then find the value(s) of x.
   (iii) A line segment is of the length 15 units and one end is at the point (3, 2). If the abscissa of the other end is 15, then find possible ordinates.

2. (i) Find the co-ordinates of the point dividing the join of $A(1, -2)$ and $B(4, 7)$:
   (a) Internally in the ratio 1 : 2
   (b) Externally in the ratio of 2 : 1
   (ii) In what ratio is the line joining $A(8, 9)$ and $B(-7, 4)$ divided by
   (a) the point (2, 7)
   (b) the x-axis
   (c) the y-axis.

3. (i) The co-ordinates of the vertices of a triangle are (0, 1), (2, 3), and (3, 5):
   (a) Find centroid of the triangle.
   (b) Find circumcentre and the circumradius.
   (c) Find orthocentre of the triangle.

Area of Triangle and Quadrilaterals

Area of Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

To remember the above formula, take the help of the following method:

$$= \frac{1}{2}\left[\begin{matrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{matrix}\right]$$

$$= \frac{1}{2}\left|[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle:** If altitude of any equilateral triangle is P, then its area $= \frac{P^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area $= \left(\frac{a^2\sqrt{3}}{4}\right)$.

(iii) Area of quadrilateral with given vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$

Area of quad. $ABCD =$

$$\frac{1}{2}\left[\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{matrix}\right]$$

Note

Area of a polygon can be obtained by dividing the polygon into disjoined triangles and then adding their areas.

Example 6

If the vertices of a triangle are (1, 2), (4, –6), and (3, 5) then its area is:

(A) $\frac{25}{2}$ sq. units **(B)** 12 sq. units

(C) 5 sq. units **(D)** 25 sq. units

Solution

$$\Delta = \frac{1}{2}[1(-6 - 5) + 4(5 - 2) + 3(2 + 6)]$$

$$= \frac{1}{2}[-11 + 12 + 24] = \frac{25}{2} \text{ square units}$$ **Ans. (A)**

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Example 7

The point A divides the join of the points $(-5, 1)$ and $(3, 5)$ in the ratio $k : 1$ and coordinates of points B and C are $(1, 5)$ and $(7, -2)$, respectively. If the area of ΔABC be 2 units, then k equals:

(A) 7, 9 **(B)** 6, 7

(C) $7, \frac{31}{9}$ **(D)** $9, \frac{31}{9}$

Solution

$$A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$$

Area of $\Delta ABC = 2$ units

$$\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2)+1\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right]$$

$$= \pm 2$$

$$\Rightarrow \quad 14k - 66 = \pm 4(k+1) \Rightarrow k = 7 \text{ or } \frac{31}{9}$$ **Ans. (C)**

Example 8

Prove that the co-ordinates of the vertices of an equilateral triangle cannot all be rational.

Solution

Let $A(x_1\ y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible, let $x_1, y_1, x_2, y_2, x_3, y_3$ be all rational.

Now area of $\Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| =$ Rational (i)

Since ΔABC is equilateral

$$\therefore \text{ Area of } \Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 = \frac{\sqrt{3}}{4}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\} = \text{Irrational} \quad \text{(ii)}$$

From (i) and (ii),

Rational = Irrational

which is a contradiction.

Hence $x_1, y_1, x_2, y_2, x_3, y_3$ cannot all be rational.

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Know Your Progress - 2

1. Find the area of the triangle whose vertices are $A(1, 1)$, $B(7, -3)$, and $C(12, 2)$
2. Find the area of the quadrilateral whose vertices are $A(1, 1)$ $B(7, -3)$, $C(12, 2)$, and $D(7, 21)$
3. Prove that the points $A(a, b, c)$, $B(b, c + a)$, and $C(c, a + b)$ are collinear (by determinant method)

Equation of Straight Line

Straight Line

Introduction: A relation between x and y that is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one-degree equation in variables x and y always represents a straight line i.e., $\boldsymbol{ax + by + c = 0}$; $\boldsymbol{a}$ **and** $\boldsymbol{b \neq 0}$ simultaneously.

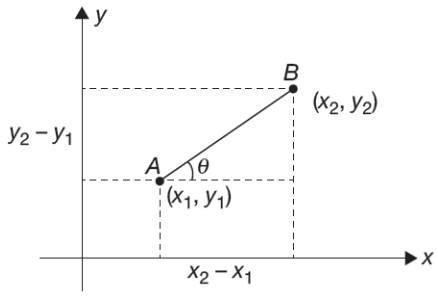
(a) Equation of a line parallel to x-axis at a distance 'a' is $\boldsymbol{y = a}$ or $\boldsymbol{y = -a}$.

(b) Equation of x-axis is $\boldsymbol{y = 0}$.

(c) Equation of a line parallel to y-axis at a distance 'b' is $\boldsymbol{x = b}$ or $\boldsymbol{x = -b}$.

(d) Equation of y-axis is $\boldsymbol{x = 0}$.

Slope of Line



If a given line makes an angle θ **(0° ≤ θ < 180°,** $\theta \neq$ **90°)** with the positive direction of x-axis, then slope of this line will be tan θ and is usually denoted by the letter $\boldsymbol{m}$, i.e., $\boldsymbol{m = \tan\theta}$ If $A(x_1, y_1)$ and $B(x_2, y_2)$ and $x_1 \neq x_2$, then slope of line $AB = \frac{y_2 - y_1}{x_2 - x_1}$

Remarks

(i) If $\theta = 90°$, m **does not exist** and line is parallel to y**-axis**.
(ii) If $\theta = 0°$, $m = 0$ and the line is parallel to x**-axis**.
(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)
(a) If lines are parallel, $m_1 = m_2$ and vice-versa.
(b) If lines are perpendicular, $m_1 m_2 = -1$ and vice-versa

Conditions for Collinearity of Three Given Points

Three given points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear if any one of the following conditions are satisfied.

(a) Area of triangle ABC is zero i.e., $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Slope of AB = slope of BC = slope of AC. i.e., $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

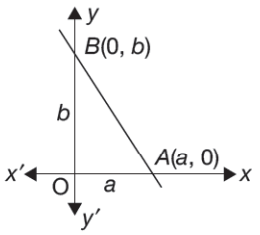
Standard Forms of Equations of a Straight Line

(a) Slope Intercept form: Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as: $y = mx + c$
If the line passes through origin, its equation is written as $y = mx$

(b) Point Slope form: If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as: $y - y_1 = m(x - x_1)$

(c) Two point form: Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as: $y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$ or $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

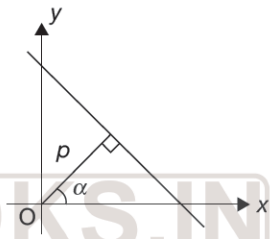
(d) Intercept form: If a and b are the intercepts made by a line on the axes of x and y, its equation is written as: $\frac{x}{a} + \frac{y}{b} = 1 = 1$



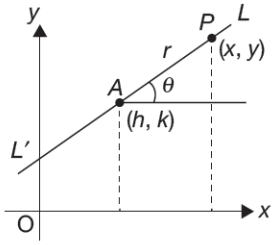
(i) Length of intercept of line between the coordinate axes = $\sqrt{a^2 + b^2}$

(ii) Area of triangle $AOB = \frac{1}{2} OA.OB = \left|\frac{1}{2} ab\right|$

(e) Normal form: If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as:
$x\cos\alpha + y\sin\alpha = p$($p$ is always positive) where $0 \le \alpha < 2\pi$.



(f) Parametric form: To find the equation of a straight line which passes through a given point $A(h, k)$ and makes a given angle θ with the positive direction of the x-axis. $P(x, y)$ is any point on the line LAL'.



Let $AP = r$, then $x - h = r\cos\theta$, $y - k = r\sin\theta$ and $\frac{x - h}{\cos\theta} = \frac{y - k}{\sin\theta} = r$ is the equation of the straight line LAL'.
Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

(g) General form: We know that a first-degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

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(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and intercept by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

Equation of Lines Parallel and Perpendicular to a Given Line

(a) Equation of line parallel to line $ax + by + c = 0$

$$\boldsymbol{ax + by + \lambda = 0}$$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$$\boldsymbol{bx - ay + k = 0}$$

Here λ, k are parameters and their values are obtained with the help of additional information given in the problem.

Angle between Two Lines

(a) If θ be the angle between two lines: $y = m_1x + c_1$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$.

Note

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of $\tan\theta$.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC found by these formulas are given by,

$\tan A = \frac{m_1 - m_2}{1 + m_1m_2}$; $\tan B = \frac{m_2 - m_3}{1 + m_2m_3}$ and

$\tan C = \frac{m_3 - m_1}{1 + m_3m_1}$

(b) If equation of lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then these lines are:

(i) Parallel $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2 + b_1b_2 = 0$

(iii) Coincident $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Example 9

Prove that every first-degree equation in x, y represents a straight line.

Solution

Let $ax + by + c = 0$ be a first degree equation in x, y where a, b, c are constants.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$.

Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$

Let R be any point on the line segment joining P and Q

Suppose R divides PQ in the ratio $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

We have $a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = \lambda\,0 + 0 = 0$

$\therefore R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by $ax + by + c = 0$. Thus, every point on the line segment joining P and Q lies on $ax + by + c = 0$.

Hence $ax + by + c = 0$ represents a straight line.

Example 10

The equation of the lines that passes through the point (3, 4) and the sum of its intercepts on the axes is 14 is:

(A) $4x - 3y = 24$, $x - y = 7$

(B) $4x + 3y = 24$, $x + y = 7$

(C) $4x + 3y + 24 = 0$, $x + y + 7 = 0$

(D) $4x - 3y + 24 = 0$, $x - y + 7 = 0$

Solution

Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ (i)

This passes through (3, 4), therefore $\frac{3}{a} + \frac{4}{b} =$ (ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$.

Putting $b = 14 - a$ in (ii), we get

$$\frac{3}{a} + \frac{4}{14 - a} = 1$$

$\Rightarrow$ $a^2 - 13a + 42 = 0 \Rightarrow (a - 7)(a - 6) = 0 \Rightarrow a = 7, 6$

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For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i), we get the equations of the lines

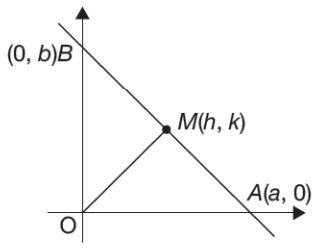
$$\frac{x}{7}+\frac{y}{7}=1 \text{ and } \frac{x}{6}+\frac{y}{8}=1 \text{ or } x+y=7 \text{ and } 4x+3y=24$$

Ans. (B)

Example 11

Two points A and B move on the positive direction of x-axis and y-axis respectively, such that $OA + OB = K$. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x + y)(x^2 + y^2) = Kxy$.

Solution



Let the equation of AB be

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{(i)}$$

given, $a + b = K$ (ii)

now, $m_{AB} \times m_{OM} = -1 \Rightarrow ah = bk$ (iii)

from (ii) and (iii),

$$a=\frac{kK}{h+k} \text{ and } b=\frac{kK}{h+k}$$

$\therefore$ from (i)

$$\frac{x(h+k)}{k\cdot K}+\frac{y(h+k)}{h\cdot K}=1$$

as it passes through (h, k)

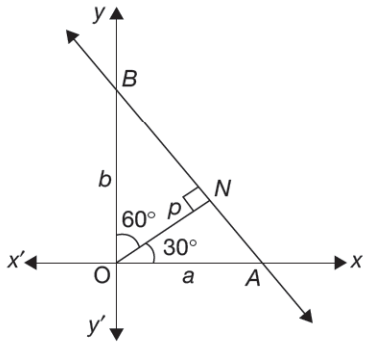
$$\frac{h(h+k)}{k\cdot K}+\frac{k(h+k)}{h\cdot K}=1 \Rightarrow (h+k)(h^2+k^2)=Khk$$

$\therefore$ locus of (h, k) is $(x + y)(x^2 + y^2) = Kxy$.

Example 12

Find the equation of the straight line on which the perpendicular from origin makes an angle 30° with positive x-axis and which forms a triangle of area $\left(\frac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinates axes.

Solution



$$\angle NOA = 30°$$

Let $ON = p > 0$, $OA = a$, $OB = b$

In ΔONA, $\cos 30° = \frac{ON}{OA} = \frac{p}{a} \Rightarrow \frac{\sqrt{3}}{2} = \frac{p}{a}$

or $a = \frac{2p}{\sqrt{3}}$

and in ΔONB, $\cos 60° = \frac{ON}{OB} = \frac{p}{b} \Rightarrow \frac{1}{2} = \frac{p}{b}$

or $b = 2p$

$\because$ Area of $\Delta OAB = \frac{1}{2}ab = \frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)(2p) = \frac{2p^2}{\sqrt{3}}$

$\therefore$ $\frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \Rightarrow p^2 = 25$

or $p = 5$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is $x\cos 30° + y\sin 30° = 5$

or $x\sqrt{3} + y = 10$

Example 13

Equation of a line that passes through point $A(2, 3)$ and makes an angle of 45° with x axis. If this line meets the line $x + y + 1 = 0$ at point P, then distance AP is:

(A) $2\sqrt{3}$ **(B)** $3\sqrt{2}$ **(C)** $5\sqrt{2}$ **(D)** $2\sqrt{5}$

Solution

Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45°$

hence $\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$

from first two parts $\Rightarrow x - 2 = y - 3 \Rightarrow x - y + 1 = 0$

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Co-ordinate of point P on this line is $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$

If this point is on line $x + y + 1 = 0$ then

$$\left(2+\frac{r}{\sqrt{2}}\right)+\left(3+\frac{r}{\sqrt{2}}\right)+1=0$$

$$\Rightarrow \quad r=-3\sqrt{2}\ ;\ |r|=3\sqrt{2}$$

Ans. (B)

Example 14

A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2, respectively. A point A is taken on the variable line such that $\frac{m+n}{OA}=\frac{m}{OA_1}+\frac{n}{OA_2}$. Show that the locus of A is a straight line passing through the point of intersection of L_1 and L_2 where O is being the origin.

Solution

Let the variable line passing through the origin is

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=r_i \qquad \text{(i)}$$

Let the equation of the line L_1 is

$$p_1x+q_1y=1 \qquad \text{(ii)}$$

Equation of the line L_2 is

$$p_2x+q_2y=1 \qquad \text{(iii)}$$

the variable line intersects the line (ii) at A_1 and (iii) at A_2.

Let $OA_1 = r_1$.

Then $A_1 = (r_1\cos\theta, r_1\sin\theta) \Rightarrow A_1$ lies on L_1

$$\Rightarrow \quad r_1=OA_1=\frac{1}{p_1\cos\theta+q_2\sin\theta}$$

Similarly, $$r_2=OA_2=\frac{1}{p_2\cos\theta+q_2\sin\theta}$$

Let $OA = r$

Let co-ordinate of A are $(h, k) \Rightarrow (h, k) \equiv (r\cos\theta, r\sin\theta)$

Now $\frac{m+n}{r}=\frac{m}{OA_1}+\frac{n}{OA_2} \Rightarrow \frac{m+n}{r}=\frac{m}{r_1}+\frac{n}{r_2}$

$$\Rightarrow \quad m+n=m(p_1r\cos\theta+q_1r\sin\theta)+n(p_2r\cos\theta+q_2r\sin\theta)$$

$$\Rightarrow \quad (p_1h+q_1k-1)+\frac{n}{m}(p_2h+q_2k-1)=0$$

Therefore, locus of A is

$$(p_1x+q_1y-1)+\frac{n}{m}(p_2x+q_2y-1)=0$$

$$\Rightarrow \quad L_1+\lambda L_2=0 \text{ where } \lambda=\frac{n}{m}.$$

This is the equation of the line passing through the intersection of L_1 and L_2.

Example 15

$8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if $PQ.\ PR = 20$.

Solution

Let line be $\frac{x+2}{\cos\theta}=\frac{y+3}{\sin\theta}=r$

$$\Rightarrow \quad x=r\cos\theta-2,\ y=r\sin\theta-3 \qquad \text{(i)}$$

Now, $$x^2+3y^2+4xy-8x-6y-9=0 \qquad \text{(ii)}$$

Taking intersection of (i) with (ii) and considering terms of r^2 and constant (as we need $PQ \cdot PR = r_1 \cdot r_2$ = product of the roots)

$$r^2(\cos^2\theta+3\sin^2\theta+4\sin\theta\cos\theta)+(\text{some terms})r+80=0$$

$$\therefore \quad r_1 \cdot r_2 = PQ \cdot PR = \frac{80}{\cos^2\theta+4\sin\theta\cos\theta+3\sin^2\theta}$$

$$\therefore \quad \cos^2\theta+4\sin\theta\cos\theta+3\sin^2\theta=4 \qquad (\because\ PQ \cdot PR = 20)$$

$$\therefore \quad \sin^2\theta-4\sin\theta\cos\theta+3\cos^2\theta=0$$

$$\Rightarrow \quad (\sin\theta-\cos\theta)(\sin\theta-3\cos\theta)=0$$

$\therefore \tan\theta = 1, \tan\theta = 3$

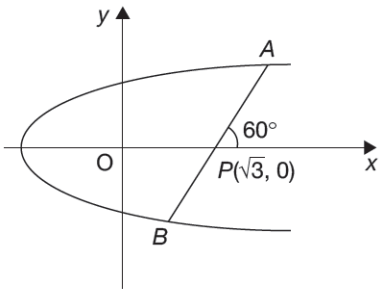
hence equation of the line is $y + 3 = 1(x + 2) \Rightarrow x - y = 1$

and $$y + 3 = 3(x + 2) \Rightarrow 3x - y + 3 = 0.$$

Example 16

If the line $y-\sqrt{3}x+3=0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of $PA \cdot PB$ {where $P \equiv (\sqrt{3}, 0)$}

Solution



Slope of line $y-\sqrt{3}x+3=0$ is $\sqrt{3}$.

If line makes an angle θ with x-axis, then $\tan\theta = \sqrt{3}$

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$\therefore \quad \theta = 60°$

$$\frac{x-\sqrt{3}}{\cos 60°} = \frac{y-0}{\sin 60°} = r$$

$$\Rightarrow \quad \left(\sqrt{3}+\frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$$

be a point on the parabola $y^2 = x + 2$ then

$$\frac{3}{4}r^2 = \sqrt{3} + \frac{r}{2} + 2 \Rightarrow 3r^2 - 2r - 4(2+\sqrt{3}) = 0$$

$$\therefore \quad PA \cdot PB = r_1 r_2 = \left|\frac{-4(2+\sqrt{3})}{3}\right| = \frac{4(2+\sqrt{3})}{3}$$

Example 17

If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines, then k is:

(A) 3 **(B)** 4 **(C)** –1 **(D)** –4

Solution

$$m_1 = -\frac{1}{4} \qquad m_2 = -\frac{4}{k}$$

Two lines are perpendicular if $m_1 m_2 = -1$

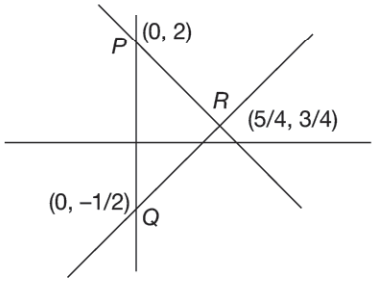
$$\Rightarrow \quad \left(-\frac{1}{4}\right)\times\left(-\frac{4}{k}\right) = -1 \Rightarrow k = -1$$ **Ans. (C)**

Example 18

A line L passes through the points (1, 1) and (0, 2) and another line M that is perpendicular to L passes through the point (0, –1/2).The area of the triangle formed by these lines with y-axis is:

(A) 25/8 **(B)** 25/16
(C) 25/4 **(D)** 25/32

Solution



Equation of the line L is $y - 1 = \frac{-1}{1}(x-1) \Rightarrow y = -x + 2$

Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at P and Q, then $PQ = 5/2$.

Also x-coordinate of their point of intersection $R = 5/4$

$\therefore$ Area of the $\Delta PQR = \frac{1}{2}\left(\frac{5}{2}\times\frac{5}{4}\right) = \frac{25}{16}$ **Ans. (B)**

Example 19

If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is:

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

Solution

A straight line is parallel to y-axis, if its y-coefficient is zero, i.e., $4 - k = 0$ i.e., $k = 4$ **Ans. (D)**

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Know Your Progress - 3

1. **(i)** Reduce the line $2x - 3y + 5 = 0$,
   - **(a)** In slope- intercept form and hence find slope and Y-intercept
   - **(b)** In intercept form and hence find intercepts on the axes.
   - **(c)** In normal form and hence find perpendicular distance from the origin and angle made by the perpendicular with the positive x-axis.

   (ii) Find distance of point A (2, 3) measured parallel to the line $x - y = 5$ from the line $2x + y + 6 = 0$.

2. **(i)** Find the angle between the lines $3x + y - 7 = 0$ and $x + 2y - 9 = 0$.

   (ii) Find the line passing through the point (2, 3) and perpendicular to the straight line $4x - 3y = 10$.

   (iii) Find the equation of the line which has positive y-intercept 4 units and is parallel to the line $2x - 3y - 7 = 0$. Also find the point where it cuts the x-axis.

   (iv) Classify the following pairs of lines as coincident, parallel or intersecting:
   - **(a)** $x + 2y - 3 = 0$ and $-3x - 6y + 9 = 0$
   - **(b)** $x + 2y + 1 = 0$ and $2x + 4y + 3 = 0$
   - **(c)** $3x - 2y + 5 = 0$ and $2x + y - 5 = 0$

Locus

Locus

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

(a) Equation of Locus: The equation to a locus is the relation that exists between the coordinates of any point on the path, and that holds for no other point except those lying on the path.

(b) Procedure for finding the equation of the locus of a point:

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminant contains only h, k, and known quantities.

(iv) Replace h by x, and k by y, in the eliminant. The resulting equation would be the equation of the locus of P.

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Equation of Circle

A circle is the locus of a point that moves in a plane in such a way that its distance from a fixed point (in the same given plane) remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

Standard Equations of the Circle

Central Form

If (h, k) is the centre and r is the radius of the circle, then its equation is $\mathbf{(x-h)^2 + (y-k)^2 = r^2}$

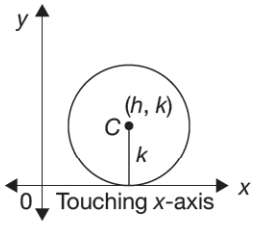
Special Cases:

(i) If centre is origin $(0, 0)$ and radius is 'r', then equation of circle is $\mathbf{x^2 + y^2 = r^2}$ and this is called the standard form.

(ii) If radius of circle is zero then equation of circle is $(x-h)^2 + (y-k)^2 = 0$. Such circle is called **zero circle** or **point circle**.

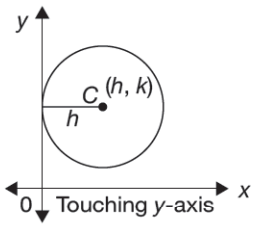
(iii) When circle touches x-axis, then equation of the circle is

$$(x-h)^2 + (y-k)^2 = k^2.$$



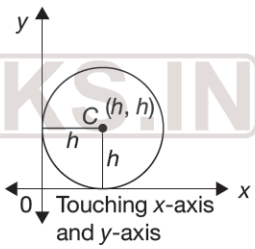
(iv) When circle touches y-axis, then equation of circle is

$$\mathbf{(x-h)^2 + (y-k)^2 = h^2}.$$



(v) When circle touches both the axes (x-axis and y-axis), then equation of circle is

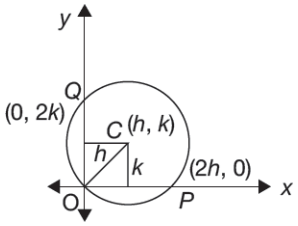
$$\mathbf{(x-h)^2 + (y-h)^2 = h^2}.$$



(vi) When circle passes through the origin and centre of the circle is (h, k), then radius $\sqrt{h^2 + k^2} = r$ and intercept cut on x-axis $OP = 2h$,and intercept cut on y-axis is $OQ = 2k$ and equation of circle is

$$\mathbf{(x-h)^2 + (y-k)^2 = h^2 + k^2}$$

or $$\mathbf{x^2 + y^2 - 2hx - 2ky = 0}$$



Note

Centre of the circle may exist in any quadrant hence for general cases use ± sign before h and k.

General Equation of Circle

$x^2 + y^2 + 2gx + 2fy + c = 0$. Where g, f, c are constants and centre is $(-g, -f)$

i.e., $\left(-\frac{\text{coefficient of } x}{2}, \frac{\text{coefficient of } y}{2}\right)$

and radius $r = \sqrt{g^2 + f^2 - c}$

Note

(i) If $(g^2 + f^2 - c) > 0$, then r is real and positive and the circle is a real circle.

(ii) If $(g^2 + f^2 - c) = 0$, then radius $r = 0$ and circle is a point circle.

(iii) If $(g^2 + f^2 - c) < 0$, then r is imaginary and then circle is also an imaginary circle with real centre.

(iv) $x^2 + y^2 + 2gx + 2fy + c = 0$ has three constants and to get the equation of the circle at least three conditions should be known $\Rightarrow$ A unique circle passes through three non-collinear points.

(v) The general second degree in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if:

- coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- coefficient of $xy = 0$ or $h = 0$
- $(g^2 + f^2 - c) \geq 0$ (for a real circle)

Intercepts Cut by the Circle on Axes

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on:

(i) x-axis $= 2\sqrt{g^2 - c}$

(ii) y-axis $= 2\sqrt{f^2 - c}$

Note

(i) If the circle cuts the x-axis two distinct points, then $g^2 - c > 0$

(ii) If the circle cuts the y-axis at two distinct points, then $f^2 - c > 0$

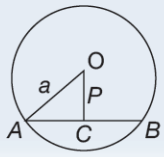
(iii) If circle touches x-axis then $g^2 = c$.

(iv) If circle touches y-axis then $f^2 = c$.

(v) Circle lies completely above or below the x-axis, then $g^2 < c$

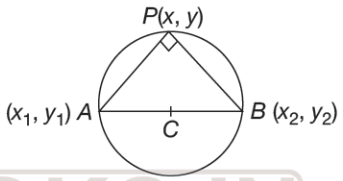
(vi) Circle lies completely to the right or left to the y-axis, then $f^2 < c$.

(vii) Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - P^2}$, where a is the radius and P is the length of perpendicular from the centre of the chord.



Equation of Circle in Diameter Form

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of the diameter of the circle and $P(x, y)$ is the point other than A and B on the circle then from geometry, we know the $\angle APB = 90°$.



$\Rightarrow$ (Slope of PA) $\times$ (Slope of PB) $= -1$

$\Rightarrow \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Note

This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2).

Equation of Circle in Parametric Forms

(i) The parametric equation of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$; $\theta \in [0, 2\pi)$ and $(r\cos\theta, r\sin\theta)$ are called the parametric co-ordinates.

(ii) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $x = h + r\cos\theta$, $y = k + r\sin\theta$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are

$x = -g + \sqrt{g^2 + f^2 - c}\cos\theta$,

$y = -f + \sqrt{g^2 + f^2 - c}\sin\theta$ where θ is parameter.

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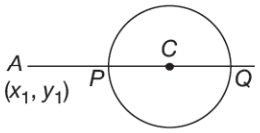
Note

Equation of a straight line joining two point α and β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

Position of a Point w.r.t Circle

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1, y_1) then:
Point (x_1, y_1) lies outside the circle or on the circle or inside the circle according as



$$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$$

or $$S_1 >, =, < 0$$

(b) The greatest and the least distance of a point A from a circle with centre C and radius r is $AC + r$ and $|AC - r|$, respectively.

Power of a Point w.r.t. Circle

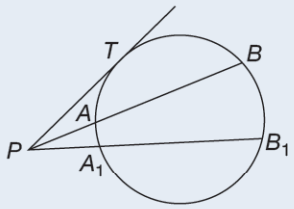
Theorem: The power of point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1 where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note

If P outside, inside, or on the circle then power of point is positive, negative, or zero, respectively.

If from a point $P(x_1, y_1)$, inside or outside the circle, a secant be drawn intersecting then circle in two points A and B, then $PA \cdot PB$ = constant. The product $PA \cdot PB$ is called power of point $P(x_1, y_1)$ w.r.t. the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, i.e., for number of secants

$$PA \cdot PB = PA_1 \cdot PB_1 = PA_2 \cdot PB_2 = \ldots = PT^2 = S_1$$

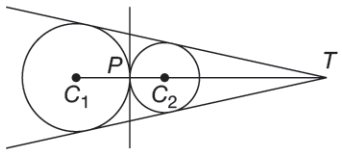


Direct and Transverse Common Tangents

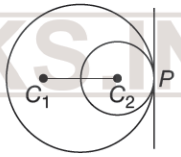
Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres, then:

Both Circles will Touch

(i) Externally if $C_1C_2 = r_1 + r_2$ i.e., the distance between their centres is equal to sum of their radii and point P and T divides C_1C_2 in the ratio $r_1 : r_2$ (internally and externally, respectively). In this case, there are **three common tangents**.



(ii) **Internally** if $C_1C_2 = |r_1 - r_2|$ i.e., the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ externally and in this case there will be only **one common tangent**.



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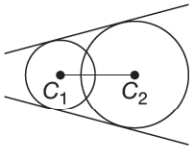
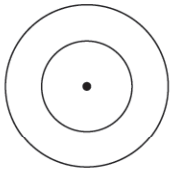
The Circles will Intersect

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents**.



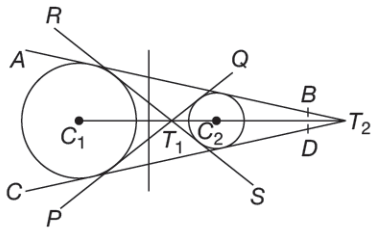
The Circles will not Intersect

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$. In this case, there will be no common tangent.



(ii) When circles are apart from each other, then $C_1C_2 > r_1 + r_2$ and in this case there will be **four common**

tangents or indirect or internal common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB and CD are called direct or external common tangents. These lines meet C_1C_2 produced on T_2. Thus, T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.



Note

Length of direct common tangent $= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent $= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

Example 22

Find the centre and the radius of the circles

(a) $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

(b) $x^2 + y^2 + 2x \sin\theta + 2y \cos\theta - 8 = 0$

(c) $2x^2 + lxy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$, for some λ.

Solution

(a) We rewrite the given equation is

$$x^2 + y^2 - \frac{8}{3}x - \frac{10}{3}y + 1 = 0$$

$$\Rightarrow g = -\frac{4}{3},\ f = -\frac{5}{3},\ c = 1$$

Hence, the centre is $\left(\frac{4}{3}, \frac{5}{3}\right)$ and the radius is

$$\sqrt{\frac{16}{9} + \frac{25}{9} - 1} = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3} \text{ units}$$

(b) $x^2 + y^2 + 2x \sin\theta + 2y\cos\theta - 8 = 0$. Centre of this circle is $(-\sin\theta, -\cos\theta)$

Radius $= \sqrt{\sin^2\theta + \cos^2\theta + 8} = \sqrt{1 + 8} = 3$ units

(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$

We rewrite the equation as

$$x^2 + \frac{\lambda}{2}xy + y^2 + \left(\frac{\lambda - 4}{2}\right)x + 3y - \frac{5}{2} = 0 \qquad \text{(i)}$$

Since, there is no term of xy in the equation of circle $\Rightarrow \frac{\lambda}{2} = 0 \Rightarrow \lambda = 0$

So, equation (i) reduces to $x^2 + y^2 - 2x + 3y - \frac{5}{2} = 0$

$\therefore$ centre is $\left(1, -\frac{3}{2}\right)$

Radius $= \sqrt{1 + \frac{9}{4} + \frac{5}{2}} = \frac{\sqrt{23}}{2}$ units.

Example 23

If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is:

(A) 3/2 **(B)** 3/4 **(C)** 1/10 **(D)** 1/20

Solution

The diameter of the circle is perpendicular distance between the parallel lines (tangents)

$3x - 4y + 4 = 0$ and $3x - 4y - \frac{7}{2} = 0$ and so it is equal to $\frac{4 + 7/2}{\sqrt{9 + 16}} = \frac{3}{2}$.

Hence, radius is $\frac{3}{4}$. **Ans. (B)**

Example 24

If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

Solution

Centre of circle = (–3/2, –2). This lies on diameter $y = 2x + m$

$\Rightarrow$ $-2 = (-3/2) \times 2 + m \Rightarrow m = 1$

Example 25

The equation of a circle that passes through the point (1, –2) and (4, –3) and whose centre lies on the line $3x + 4y = 7$ is

(A) $15(x^2 + y^2) - 94x + 18y - 55 = 0$

(B) $15(x^2 + y^2) - 94x + 18y + 55 = 0$

(C) $15(x^2 + y^2) + 94x - 18y + 55 = 0$

(D) None of these

Solution

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

Hence, substituting the points, (1, –2) and (4, –3) in equation (i)

$$5 + 2g - 4f + c = 0 \qquad \text{(ii)}$$

$$25 + 8g - 6f + c = 0 \qquad \text{(iii)}$$

centre $(-g, -f)$ lies on line $3x + 4y = 7$

Hence $-3g - 4f = 7$

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Solving for g, f, c, we get

Here $g = \frac{-47}{15}, f = \frac{9}{15}, c = \frac{55}{15}$

Hence, the equation is $15(x^2 + y^2) - 94x + 18y + 55 = 0$

Ans. (B)

Example 26

A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Solution

Let the centre of the circle be (α, β). It lies on the line $y = x - 1$

$$\Rightarrow \quad \beta = \alpha - 1.$$

Hence, the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow$ the equation of the circle is $(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$

It passes through (7, 3)

$$\Rightarrow \quad (7 - \alpha)^2 + (4 - \alpha)^2 = 9$$

$$\Rightarrow \quad 2\alpha^2 - 22\alpha + 56 = 0$$

$$\Rightarrow \quad \alpha^2 - 11\alpha + 28 = 0$$

$$\Rightarrow \quad (\alpha - 4)(\alpha - 7) = 0$$

$$\Rightarrow \quad \alpha = 4, 7$$

Hence, the required equations are

$x^2 + y^2 - 8x - 6y + 16 = 0$ and $x^2 + y^2 - 14x - 12y + 76 = 0$.

Ans.

Example 27

Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on $L_1$and L_2 are equal, then which of the following equations can represent L_1?

(A) $x + y = 0$ **(B)** $x - y = 0$

(C) $x + 7y = 0$ **(D)** $x - 7y = 0$

Solution

Let L_1 be $y = mx$

Lines $L_1$and L_2 will be at equal distances from centre of the circle centre of the circle is $\left(\frac{1}{2}, -\frac{3}{2}\right)$

$$\Rightarrow \quad \frac{\left|\frac{1}{2}m + \frac{3}{2}\right|}{\sqrt{1+m^2}} = \frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}} \Rightarrow \frac{(m+3)^2}{(1+m^2)} = 8$$

$$\Rightarrow \quad 7m^2 - 6m - 1 = 0$$

$$\Rightarrow \quad (m - 1)(7m + 1) = 0$$

$$\Rightarrow \quad m = 1, m = -\frac{1}{7}$$

$$\Rightarrow \quad y = x, 7y + x = 0$$ **Ans. (B, C)**

Example 28

If $P(2, 8)$ is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ that neither touches nor intersects the axes, then set for p is:

(A) $p < -1$ **(B)** $p < -4$ **(C)** $p > 96$ **(D)** ϕ

Solution

For internal point $p(2, 8)$, $4 + 64 - 4 + 32 - p < 0 \Rightarrow p > 96$

And x intercept $= 2\sqrt{1+p}$, therefore $1 + p < 0$

$\Rightarrow p < -1$ and y intercept $= 2\sqrt{4+p} \Rightarrow p < -4$ **Ans. (D)**

Example 29

Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$,

Solution

Given circles are

$$x^2 + y^2 + 2ax + c^2 = 0 \quad (1)$$

and $$x^2 + y^2 + 2by + c^2 = 0 \quad (2)$$

Let C_1 and C_2 be the centre of circle (i) and (ii), respectively, r_1 and r_2 be their radii, then

$C_1 = (-a, 0), C_2 = (0, -b), r = \sqrt{a^2 - c^2}$ $r_2 = \sqrt{b^2 - c^2}$

Here we find the two circle touch other internally or externally.

for touch, $|C_1C_2| = |r_1 \pm r_2|$

or $$\sqrt{(a^2 + b^2)} = |\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}$$

On squaring

$$a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$$

or $$c^2 = \pm\sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$$

Again squaring,

$$c^2 = a^2b^2 - c^2(a^2 + b^2) + c^4$$

or $$c^2(a^2 + b^2) = a^2b^2$$

or $$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$$

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Know Your Progress - 4

1. Centres of the three circles $x^2 + y^2 - 4x - 6y - 14 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$, and $x^2 + y^2 - 10x - 16y + 7 = 0$

(A) are the vertices of a right triangle
(B) the vertices of an isosceles triangle that are not regular
(C) vertices of a regular triangle
(D) are collinear

2. $y - 1 = m_1(x - 3)$ and $y - 3 = m_2(x - 1)$ are two family of straight lines, at right angled to each other. The locus of their point of intersection is

(A) $x^2 + y^2 - 2x - 6y + 10 = 0$
(B) $x^2 + y^2 - 4x - 4y + 6 = 0$
(C) $x^2 + y^2 - 2x - 6y + 6 = 0$
(D) $x^2 + y^2 - 4x - 4y - 6 = 0$

3. Suppose that the equation of the circle having $(-3, 5)$ and $(5, -1)$ as end points of a diameter is $(x - a)^2 + (y - b)^2 = r^2$. Then $a + b + r$, $(r > 0)$ is

(A) 8 **(B)** 9
(C) 10 **(D)** 11

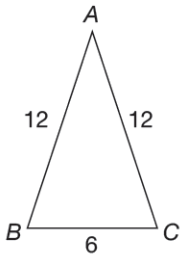
4. B and C are fixed points having co-ordinates $(3, 0)$ and $(-3, 0)$, respectively. If the vertical angle BAC is $90°$, then the locus of the centroid of the ΔABC has the equation:

(A) $x^2 + y^2 = 1$ **(B)** $x^2 + y^2 = 2$
(C) $9(x^2 + y^2) = 1$ **(D)** $9(x^2 + y^2) = 4$

5. The area of an equilateral triangle inscribed in the circle $x^2 + y^2 - 2x = 0$ is

(A) $\frac{3\sqrt{3}}{4}$ **(B)** $\frac{3\sqrt{3}}{2}$
(C) $\frac{3\sqrt{3}}{8}$ **(D)** none

6. The radius of the circle passing through the vertices of the triangle ABC, is



(A) $\frac{8\sqrt{15}}{5}$ **(B)** $\frac{3\sqrt{15}}{5}$
(C) $3\sqrt{5}$ **(D)** $3\sqrt{2}$

7. In the xy plane, the segment with end points $(3, 8)$ and $(-5, 2)$ is the diameter of the circle. The point $(k, 10)$ lies on the circle for

(A) no value of k
(B) exactly one integral k
(C) exactly one non integral k
(D) two real values of k

8. $(6, 0)$, $(0, 6)$, and $(7, 7)$ are the vertices of a triangle. The circle inscribed in the triangle has the equation

(A) $x^2 + y^2 - 9x + 9y + 36 = 0$
(B) $x^2 + y^2 - 9x - 9y + 36 = 0$
(C) $x^2 + y^2 + 9x - 9y + 36 = 0$
(D) $x^2 + y^2 - 9x - 9y - 36 = 0$

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Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. If two opposite vertices of a square are (5, 4) and (1, .6), then the coordinates of its remaining two vertices is :
(A) (–2, 2) and (5, 3) **(B)** (8, –3) and (–2, 1)
(C) (8, 6) and (3, 5) **(D)** (1, –3) and (2, 5)

2. If the middle points of the sides of a triangle be (–2, 3), (4, –3), and (4, 5), then centroid of triangle is:
(A) $\left(\frac{5}{3}, 2\right)$ **(B)** $\left(\frac{5}{6}, 1\right)$
(C) $\left(1, \frac{5}{6}\right)$ **(D)** $\left(2, \frac{5}{3}\right)$

3. If the vertices of a triangle be (2, 1), (5, 2), and (3, 4) then its circumcenter is:
(A) $\left(\frac{13}{2}, \frac{9}{2}\right)$ **(B)** $\left(\frac{13}{4}, \frac{9}{4}\right)$
(C) $\left(\frac{9}{4}, \frac{13}{4}\right)$ **(D)** None

4. The orthocentre of the triangle ABC is 'B' and the circumcentre is 'S' (a, b). If A is the origin then the co-ordinates of C are:
(A) $(2a, 2b)$ **(B)** $\left(\frac{a}{2}, \frac{b}{2}\right)$
(C) $\left(\sqrt{a^2+b^2}, 0\right)$ **(D)** None

5. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$, and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then:
(A) $a = 2, b = 4$ **(B)** $a = 3, b = 4$
(C) $a = 2, b = 3$ **(D)** $a = 3, b = 5$

6. If $A(2, 2)$, $B(-4, -4)$, $C(5, -8)$ are the vertices of any triangle, the length of median passes through C will be:
(A) $\sqrt{65}$ **(B)** $\sqrt{117}$
(C) $\sqrt{85}$ **(D)** $\sqrt{113}$

7. If the mid-point of the line-segment joining the points (–7, 14) and $(K, 4)$ is (a, b), where $2a + 3b = 5$, then the value of K is:
(A) $K = 15$ **(B)** $K = -7$
(C) $K = -15$ **(D)** $K = 10$

8. Value of m for which the point $P(m, 6)$ divides the join of $A(-4, 3)$ and $B(2, 8)$ is:
(A) 5 **(B)** $\frac{3}{2}$
(C) $-\frac{2}{5}$ **(D)** None

9. The incentre of the triangle formed by (0, 0), (5, 12), (16, 12) is:
(A) (7, 9) **(B)** (9, 7)
(C) (–9, 7) **(D)** (–7, 9)

10. The incentre of the triangle with vertices $(1, \sqrt{3})$, (0, 0), and (2, 0) is:
(A) $\left(1, \frac{\sqrt{3}}{2}\right)$ **(B)** $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
(C) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ **(D)** $\left(1, \frac{1}{\sqrt{3}}\right)$

11. Orthocentre of triangle with vertices (0, 0), (3, 4), and (4, 0) is:
(A) $\left(-3, \frac{3}{4}\right)$ **(B)** (3, 12)
(C) $\left(3, \frac{3}{4}\right)$ **(D)** (3, 9)

12. If four points $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$, and $D(x, 3x)$ are given in such a way that $\frac{\text{Area}(\Delta DBC)}{\text{Area}(\Delta ABC)} = \frac{1}{2}$, then the value of x is:
(A) $\frac{3}{8}$ or $\frac{-14}{8}$ **(B)** 2 or –3
(C) $\frac{11}{8}$ or $\frac{-3}{8}$ **(D)** None of these

13. If the co-ordinates of two points A and B are (3, 4) and (5, –2) respectively, then the co-ordinates of any point P if $PA = PB$ and Area of $\Delta PAB = 10$ is:
(A) (7, 2) or (1, 0) **(B)** (–7, 2) or (3, 0)
(C) (7, –2) or (5, 0) **(D)** (7, –2) or (–1, 0)

14. A straight line through $P(1, 2)$ is such that its intercept between the axis is bisected at P. Its equation is:
(A) $x + 2y = 5$ **(B)** $x - y + 1 = 0$
(C) $x + y - 3 = 0$ **(D)** $2x + y - 4 = 0$

15. A line L is perpendicular to the line $5x - y = 1$ and the area of the triangle formed by the line L and coordinate axes is 5. The equation of the line L is:
(A) $x + 5y = 5$ **(B)** $x + 5y = \pm 5\sqrt{2}$
(C) $x - 5y = 5$ **(D)** $x - 5y = 5\sqrt{2}$

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16. If PS is the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$, and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is:

(A) $2x - 9y - 7 = 0$ **(B)** $2x - 9y - 11 = 0$

(C) $2x + 9y - 11 = 0$ **(D)** $2x + 9y + 7 = 0$

17. Locus of a point such that the sum of its distance from the points $(0, 2)$ and $(0, -2)$ is 6:

(A) $4x^2 + 6y^2 = 5$ **(B)** $5x^2 + 9y^2 = 50$

(C) $9x^2 + 5y^2 = 45$ **(D)** None

18. Two opposite vertices of a rectangle are $(1, 3)$ and $(5, 1)$. If the other two vertices of the rectangle lie on the line $y - x + l = 0$, then $l =$

(A) -1 **(B)** 2

(C) 1 **(D)** None of these

19. The points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y = 10$, are:

(A) $(3, 1)$ **(B)** $(7, 11)$

(C) $(-7, 11)$ **(D)** $(1, 3)$

20. Which of the following pair(s) of straight lines are mutually perpendicular?

(A) $3x - 4y + 12 = 0$ and $4x + 3y - 6 = 0$

(B) $4x + 5y - 8 = 0$ and $3x - 2y + 4 = 0$

(C) $\frac{x - x_1}{2} = \frac{y - y_1}{3}$ and $\frac{x - x_2}{3} = \frac{y - y_2}{-2}$

(D) $\frac{x - x_1}{5} = \frac{y - y_1}{-4}$ and $\frac{x - x_2}{4} = \frac{y - y_2}{5}$

21. Given the points $A(0, 4)$ and $B(0, -4)$, the equation of the locus of the point $P(x, y)$ such that $|AP - BP| = 6$ is:

(A) $9x^2 - 7y^2 + 63 = 0$

(B) $9x^2 - 7y^2 - 63 = 0$

(C) $7x^2 - 9y^2 + 63 = 0$

(D) $7x^2 - 9y^2 - 63 = 0$

22. The line segment joining the points $(3, -4)$ and $(1, 2)$ is trisected at the points P and Q. If the coordinates of P and Q are $(p, -2)$ and $(5/3, q)$ respectively, then find the value of p and q.

(A) 0 **(B)** 1

(C) 2 **(D)** 3

23. If the three vertices of a parallelogram are $(a + b, a - b)$, $(2a + b, 2a - b)$ and $(a - b, a + b)$, then find the coordinates of fourth vertex.

(A) $(-a, a)$ **(B)** $(-b, b)$

(C) $(-a, b)$ **(D)** $(-b, a)$

PART – II KVPY Level Problems

1. Coordinates of the vertices of a triangle ABC are $(12, 8)$, $(-2, 6)$, and $(6, 0)$ then the **correct** statement is:

(A) triangle is right but not isosceles

(B) triangle is isosceles but not right

(C) triangle is obtuse

(D) the product of the abscissa of the centroid, orthocenter and circumcenter is 160.

2. Find the value of x_1 if the distance between the points $(x_1, 2)$ and $(3, 4)$ be 8.

(A) $3 \pm 2\sqrt{15}$ **(B)** $3 \pm \sqrt{15}$

(C) $2 \pm 3\sqrt{15}$ **(D)** $2 \pm \sqrt{15}$

3. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$, and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then:

(A) $a = 2, b = 4$ **(B)** $a = 3, b = 4$

(C) $a = 2, b = 3$ **(D)** $a = 3, b = 5$

4. The four points whose co-ordinates are $(2, 1)$, $(1, 4)$, $(4, 5)$, $(5, 2)$ form:

(A) a rectangle that is not a square

(B) a trapezium that is not a parallelogram

(C) a square

(D) a rhombus that is not a square

5. The length of a line segment AB is 10 units. If the co-ordinates of one extremity are $(2, -3)$ and the abscissa of the other extremity is 10, then the sum of all possible values of the ordinate of the other extremity is:

(A) 3 **(B)** -4

(C) 12 **(D)** -6

6. If A and B are the points $(-3, 4)$ and $(2, 1)$, then the co-ordinates of the point C on AB produced such that $AC = 2BC$ are:

(A) $(2, 4)$ **(B)** $(3, 7)$

(C) $(7, -2)$ **(D)** $\left(-\frac{1}{2}, \frac{5}{2}\right)$

7. The orthocenter of the triangle ABC is 'B' and the circumcenter is 'S' (a, b). If A is the origin then the co-ordinates of C are:

(A) $(2a, 2b)$ **(B)** $\left(\frac{a}{2}, \frac{b}{2}\right)$

(C) $\left(\sqrt{a^2 + b^2}, 0\right)$ **(D)** none

8. A particle begins at the origin and moves successively in the following manner as shown,
1 unit to the right, 1/2 unit up, 1/4 unit to the right, 1/8 unit down, 1/16 unit to the right etc.

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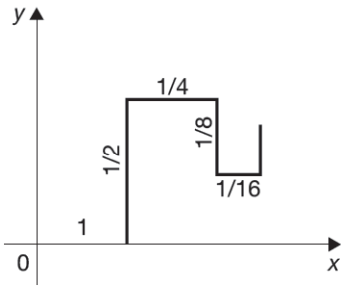

The length of each move is half the length of the previous move and movement continues in the 'zigzag' manner infinitely. The co-ordinates of the point to which the 'zigzag' converges is -

(A) (4/3, 2/3) **(B)** (4/3, 2/5)
(C) (3/2, 2/3) **(D)** (2, 2/5)

9. The medians of a triangle meet at (0, –3) and its two vertices are at (–1, 4) and (5, 2). Then the third vertex is at:

(A) (4, 15) **(B)** (–4, –15)
(C) (–4, 15) **(D)** (4, –15)

10. If the two vertices of a triangle are (7, 2) and (1, 6) and its centroid is (4, 6) then the coordinate of the third vertex are (a, b). The value of $(a + b)$, is:

(A) 13 **(B)** 14
(C) 15 **(D)** 16

11. If in triangle ABC, $A \equiv (1,10)$, circumcentre $\equiv \left(-\frac{1}{3}, \frac{2}{3}\right)$ and orthocentre $\equiv \left(\frac{11}{3}, \frac{4}{3}\right)$ then the co-ordinates of mid-point of side opposite to A is:

(A) (1, –11/3) **(B)** (1, 5)
(C) (1, –3) **(D)** (1, 6)

12. Suppose ABC is a triangle with 3 acute angles A, B, and C. The point whose coordinates are $(\cos B - \sin A, \sin B - \cos A)$ can be in the:

(A) first and 2nd quadrant
(B) second the 3rd quadrant
(C) third and 4th quadrant
(D) second quadrant only

13. Consider the points $P(2, -4)$; $Q(4, -2)$; and $R(7, 1)$. The points P, Q, R:

(A) form an equilateral triangle
(B) form a right-angled triangle
(C) form an isosceles triangle that is not equilateral
(D) are collinear.

14. A triangle has two of its vertices at (0, 1) and (2, 2) in the cartesian plane. Its third vertex lies on the x-axis. If the area of the triangle is 2 square units, then the sum of the possible abscissae of the third vertex is:

(A) –4 **(B)** 0
(C) 5 **(D)** 6

15. A point $P(x, y)$ moves so that the sum of the distance from P to the coordinate axes is equal to the distance from P to the point $A(1, 1)$. The equation of the locus of P in the first quadrant is:

(A) $(x+1)(y+1) = 1$ **(B)** $(x+1)(y+1) = 2$
(C) $(x-1)(y-1) = 1$ **(D)** $(x-1)(y-1) = 2$

16. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a ΔABC. If the centroid of ΔABC moves on the line $2x + 3y = 1$, then the locus of the vertex C is:

(A) $2x + 3y = 9$ **(B)** $2x - 3y = 7$
(C) $3x + 2y = 5$ **(D)** $3x - 2y = 3$

17. A stick of length 10 units rests against the floor and a wall of a room. If the stick begins to slide on the floor, then the locus of its middle point is:

(A) $x^2 + y^2 = 2.5$ **(B)** $x^2 + y^2 = 25$
(C) $x^2 + y^2 = 100$ **(D)** none

18. Given the points $A(0, 4)$ and $B(0, -4)$, the equation of the locus of the point P such that $|AP - BP| = 6$ is:

(A) $9x^2 - 7y^2 + 63 = 0$
(B) $9x^2 - 7y^2 - 63 = 0$
(C) $7x^2 - 9y^2 + 63 = 0$
(D) $7x^2 - 9y^2 - 63 = 0$

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19. A line passes through (2, 2) and cuts a triangle of area 9 square units from the first quadrant. The sum of all possible values for the slope of such a line, is:

(A) –2.5 **(B)** –2
(C) –1.5 **(D)** –1

20. The diagonals of a parallelogram $PQRS$ are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then $PQRS$ must be a:

(A) rectangle **(B)** square
(C) cyclic quadrilateral **(D)** rhombus

21. A and B are any two points on the positive x and y axis, respectively, satisfying $2(OA) + 3(OB) = 10$. If P is the middle point of AB then the locus of P is:

(A) $2x + 3y = 5$ **(B)** $2x + 3y = 10$
(C) $3x + 2y = 5$ **(D)** $3x + 2y = 10$

22. The co-ordinates of the orthocentre of the triangle bounded by the lines, $4x - 7y + 10 = 0$; $x + y = 5$, and $7x + 4y = 15$ is:

(A) (2, 1) **(B)** (–1, 2)
(C) (1, 2) **(D)** (1, –2)

23. If the x intercept of the line $y = mx + 2$ is greater than $1/2$, then the gradient of the line lies in the interval:
(A) $(-1, 0)$ **(B)** $(-1/4, 0)$
(C) $(-\infty, -4)$ **(D)** $(-4, 0)$

24. The greatest slope along the graph represented by the equation $4x^2 - y^2 + 2y - 1 = 0$ is:
(A) -3 **(B)** -2
(C) 2 **(D)** 3

25. The extremities of the base of an isosceles triangle ABC are the points $A(2, 0)$ and $B(0, 1)$. If the equation of the side AC is $x = 2$ then the slope of the side BC is:
(A) $\frac{3}{4}$ **(B)** $\frac{4}{3}$
(C) $\frac{3}{2}$ **(D)** $\sqrt{3}$

26. Number of lines that can be drawn through the point $(4, -5)$ so that its distance from $(-2, 3)$ will be equal to 12 is equal to:
(A) 0 **(B)** 1
(C) 2 **(D)** 3

27. Two mutually perpendicular straight lines through the origin from an isosceles triangle with the line $2x + y = 5$. Then the area of the triangle is:
(A) 5 **(B)** 3
(C) 5/2 **(D)** 1

28. Point 'P' lies on the line $\lambda\{(x, y) \mid 3x + 5y = 15\}$. If '$P$' is also equidistant from the coordinate axes, then P can be located in which of the four quadrants?
(A) I only **(B)** II only
(C) I or II only **(D)** IV only

29. The area of the parallelogram formed by the lines $3x + 4y = 7a$; $3x + 4y = 7b$; $4x + 3y = 7c$ and $4x + 3y = 7d$ is:
(A) $\frac{|(a-b)(c-d)|}{7}$ **(B)** $|(a-b)(c-d)|$
(C) $\frac{|(a-b)(c-d)|}{49}$ **(D)** $7|(a-b)(c-d)|$

30. If x_1, y_1 are the roots of $x^2 + 8x - 20 = 0$, x_2, y_2 are the roots of $4x^2 + 32x - 57 = 0$ and x_3, y_3 are the roots of $9x^2 + 72x - 112 = 0$, then the points, $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3):
(A) are collinear
(B) form an equilateral triangle
(C) form a right-angled isosceles triangle
(D) are concyclic

31. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π square units, the equation of the circle is:
(A) $x^2 + y^2 + 2x - 2y - 62 = 0$
(B) $x^2 + y^2 - 2x + 2y - 62 = 0$
(C) $x^2 + y^2 - 2x + 2y - 47 = 0$
(D) $x^2 + y^2 + 2x - 2y - 47 = 0$

32. Let C be the circle with centre $(0, 0)$ and radius 3 units. The equation of the locus of the mid-points of the chords of the circle C that subtend an angle of $\frac{2\pi}{3}$ at its centre is:
(A) $x^2 + y^2 = 1$ **(B)** $x^2 + y^2 = \frac{27}{4}$
(C) $x^2 + y^2 = \frac{9}{4}$ **(D)** $x^2 + y^2 = \frac{3}{2}$

33. The point diametrically opposite to the point $(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is:
(A) $(3, -4)$ **(B)** $(-3, 4)$
(C) $(-3, -4)$ **(D)** $(3, 4)$

34. Three distinct points A, B, and C are given in the 2-dimensional coordinate plane such that the ratio of the distance of any one of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $\frac{1}{3}$. Then, the circumcentre of the triangle ABC is at the point:
(A) $\left(\frac{5}{2}, 0\right)$ **(B)** $\left(\frac{5}{3}, 0\right)$
(C) $(0, 0)$ **(D)** $\left(\frac{5}{4}, 0\right)$

35. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is:
(A) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$
(B) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
(C) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
(D) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

36. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if:
(A) $-85 < m < -35$ **(B)** $-35 < m < 15$
(C) $15 < m < 65$ **(D)** $35 < m < 85$

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37. The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is:

(A) $x^2 + y^2 + x + y - 2 = 0$

(B) $x^2 + y^2 - 2x - 2y + 1 = 0$

(C) $x^2 + y^2 - x - y = 0$

(D) $x^2 + y^2 + 2x + 2y - 7 = 0$

38. The length of the diameter of the circle that touches the x-axis at the point (1, 0) and passes through the point (2, 3) is:

(A) 5/3 **(B)** 10/3

(C) 3/5 **(D)** 6/5

39. If the circle $x^2 + y^2 - 6x - 8y + \left(25 - a^2\right) = 0$ touches the axis of x, then a equals:

(A) ±4 **(B)** ±3

(C) 0 **(D)** ±2

40. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is:

(A) 10 **(B)** $5\sqrt{2}$

(C) $5\sqrt{3}$ **(D)** 5

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EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. The centre of a clock is taken as origin. At 4.30 pm, the equation of line along minute hand is $x = 0$. Therefore, at this instant, the equation of line along the hour hand will be **[NSEJS-2011]**

(A) $x - y = 0$ **(B)** $x + y = 0$

(C) $y = \sqrt{2x}$ **(D)** $y = \dfrac{x}{\sqrt{2}}$

2. Consider the point $A(a, b + c)$, $(b, c + a)$ and $C(c, a + b)$. The area of ΔABC is **[NSEJS-2011]**

(A) $2(a^2 + b^2 + c^2)$ **(B)** $\dfrac{a^2 + b^2 + c^2}{6}$

(C) $2(ab + bc + ca)$ **(D)** none of these

3. Find the equation of the line parallel to $4x + 3y = 5$ and having x-intercept (–3) **[NSEJS-2012]**

(A) $3x + 4y + 12 = 0$ **(B)** $3x + 4y = 12$

(C) $4x + 3y - 12 = 0$ **(D)** $4x + 3y + 12 = 0$

4. In the $x - y$ plane let A be the point (5.0) and L be the line $y = \dfrac{x}{3}$. The number of points p on the line L such that triangle OAP is isosceles is (O being the origin) **[NSEJS-2013]**

(A) 2 **(B)** 3

(C) 4 **(D)** 5

5. If $A(p, q + r)$, $B(q, r + p)$ and $C(r, p + q)$ are points, then area of triangle ABC **[NSEJS-2015]**

(A) $p^2 + q^2 + r^2$ **(B)** $(p + q + r)^2$

(C) $\dfrac{1}{2}(pq + qr + rp)$ **(D)** zero

PART – II Previous Years' KVPY Problems

1. Let S be the circle in XY-plane that touches the X-axis at point A, the Y-axis at point B and the unit circle $x^2 + y^2 = 1$ at point C externally. If O denotes the origin, then the angle OCA equals **[KVPY-2017]**

(A) $\dfrac{5\pi}{8}$ **(B)** $\dfrac{\pi}{2}$

(C) $\dfrac{3\pi}{4}$ **(D)** $\dfrac{3\pi}{4}$

2. Let A_1, A_2, A_3 be regions in the XY-plane defined by **[KVPY-2017]**

$A_1 = \{(x, y): x^2 + 2y^2 \le 1\}$

$A_2 = \{(x, y): |x^3| + 2\sqrt{2}\,|y|^3 \le 1\}$

$A_a = \{(x, y): \max(|x|, \sqrt{2}\,|y| \le 1\}$ Then,

(A) $A_1 \supset A_2 \supset A_3$ **(B)** $A_3 \supset A_1 \supset A_2$

(C) $A_2 \supset A_3 \supset A_1$ **(D)** $A_3 \supset A_2 \supset A_1$

3. Let $x_0\ y_0$ be fixed real numbers such that $x_0^2 + y_0^2 > 1$. If x, y are arbitrary real numbers such that $x^2 + y^2 \le 1$, then the minimum value of $(x - x_0)^2 + (y - y_0)^2$ is **[KVPY-2018]**

(A) $\left(\sqrt{x_0^2 + y_0^2} - 1\right)^2$ **(B)** $x_0^2 + y_0^2 - 1$

(C) $(|x_0| + |y_0| - 1)^2$ **(D)** $(|x_0| + |y_0|)^2 - 1$

PART – III Previous Years' PRMO Problems

1. Let R denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ divide R into four regions R_i, $i = 1, 2, 3,$. If R_i denotes the area of the region R_i and if $|R_1||R_2||R_3||R_4|$, determine $|R_1| - |R_2| - |R_3| + |R_4|$. **[PRE-RMO 2019]**

ANSWER KEYS

Know Your Progress - 1

1. **(i)** $PQ = \sqrt{34}$ units; **(ii)** $x = 6$ or $x = 0$ **(iii)** 11, –7
2. **(i) (a)** (2, 1) **(b)** (7, 16)
   (ii) (a) 2 : 3 (internally) **(b)** 9 : 4 (externally); **(c)** 8 : 7 (internally)
3. **(i) (a)** $\left(\frac{5}{3}, 3\right)$; **(b)** $\left(-\frac{9}{2}, \frac{15}{2}\right)$, $\frac{5\sqrt{10}}{2}$ **(c)** (14, –6)

Know Your Progress - 2

1. 25 square units;
2. 132 square units;

Know Your Progress - 3

1. **(i) (a)** $y = \frac{2x}{3} + \frac{5}{3}, \frac{2}{3}, \frac{5}{3}$; **(b)** $\frac{x}{(-5/2)} + \frac{y}{(5/3)} = 1, -\frac{5}{2}, \frac{5}{3}$; **(c)** $-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}, \frac{5}{\sqrt{13}}, \alpha = \pi - \tan^{-1}\left(\frac{3}{2}\right)$;
   (ii) $13\sqrt{2}/3$ units
2. **(i)** $\theta = 135°$ or $45°$ **(ii)** $3x + 4y = 18$ **(iii)** $2x - 3y + 12 = 0$, (–6, 0)
   (iv) (a) Coincident, **(b)** Parallel, **(c)** Intersecting

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Know Your Progress - 4

1. D 2. B 3. A 4. A 5. A 6. A 7. B 8. B

Practice Exercise

EXERCISE – I

PART – I NSEJS Level Problems

1. B	2. D	3. B	4. A	5. C	6. C	7. C	8. C	9. A	10. D
11. C	12. C	13. A	14. D	15. B	16. D	17. C	18. C	19. A, C	
20. A, C, D		21. A	22. A	23. B					

PART – II KVPY Level Problems

1. D	2. A	3. C	4. C	5. D	6. C	7. A	8. B	9. B	10. B
11. A	12. D	13. D	14. A	15. B	16. A	17. B	18. A	19. A	20. D
21. A	22. C	23. D	24. C	25. A	26. A	27. A	28. C	29. D	30. A
31. C	32. C	33. C	34. D	35. C	36. B	37. C	38. B	39. A	40. C

EXERCISE – II

PART – I Previous Years' NSEJS Problems

1. B **2.** D **3.** D **4.** C **5.** D

PART – II Previous Years' KVPY Problems

1. A **2.** D **3.** A

PART – III Previous Years' PRMO Problems

1. 48

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Chapter

10 Inequalities

Learning Outcomes

After reading this chapter, you will be able to understand

- Algebraic Inequalities
- Modulus Inequalities
- AM, GM and HM Inequalities
- Log and Exponential Inequalities
- Advanced Inequalities Based on Positive Numbers

Inequalities Involving Modulus Function

Properties of Modulus Function

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(i) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$, where a is positive.

(ii) $|x| \leq a \Rightarrow x \in [-a, a]$, where a is positive

(iii) $|x| > |y| \Rightarrow x^2 > y^2$

(iv) $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$

(*Triangle Inequality of Modulus*)

(v) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$

(vi) $|x - y| = |x| + |y| \Rightarrow xy \leq 0$

Logarithmic Inequalities

Points to Remember

(i) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if} \quad a > 1 \\ x > y & \text{if} \quad 0 < a < 1 \end{cases}$

(ii) If $a > 1$, then

(a) $\log_a x < p \Rightarrow 0 < x < a^p$

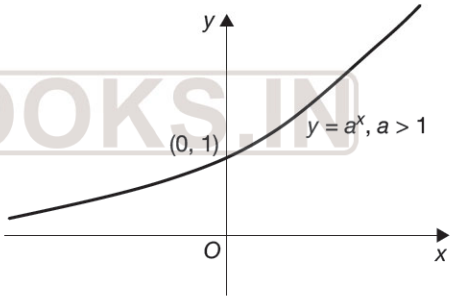
(b) $\log_a x > p \Rightarrow x > a^p$

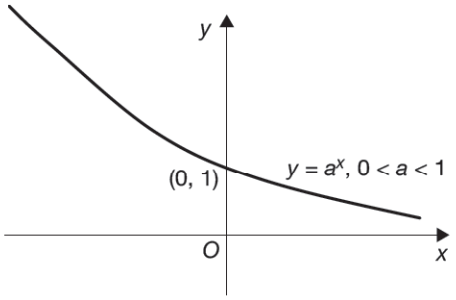
(iii) If $0 < a < 1$, then

(a) $\log_a x < p \Rightarrow x > a^p$

(b) $\log_a x > p \Rightarrow 0 < x < a^p$

Exponential Inequations





If $a^{f(x)} > b \Rightarrow \begin{cases} f(x) > \log_a b \text{ when } a > 1 \\ f(x) < \log_a b \text{ when } 0 < a < 1 \end{cases}$

Inequalities Based on RMS, AM, GM and HM

Note

Given $a_1, a_2, a_3, \ldots, a_n$ are positive numbers.

$$\text{Root Mean Square, RMS} = \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}$$

$$\text{Arithmetic Mean, AM} = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

$$\text{Geometric Mean, GM} = (a_1\, a_2\, a_3 \ldots a_n)^{1/n}$$

$$\text{Harmonic Mean, HM} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} \ldots \frac{1}{a_n}}$$

We have a relation,

$$\text{RMS} \geq \text{AM} \geq \text{GM} \geq \text{HM}$$

Equality holds when $a_1 = a_2 = a_3 \ldots = a_n$

Cauchy Schwartz Inequality

If $a_1, a_2, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n$ are real numbers, then

$$\left(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2\right)$$
$$\geq \left(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n\right)^2$$

Proof: It comes from a result of quadratic expression.

We know that

$$(a_1x - b_1)^2 + (a_2x - b_2)^2 + (a_3x - b_3)^2 + \ldots + (a_nx - b_n)^2 \geq 0. \qquad \forall\, x \in R$$

$$\Rightarrow \quad \left(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2\right)x^2 - 2\left(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n\right)x + \left(b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2\right) \geq 0 \qquad \forall\, x \in R$$

Since coefficient of $x^2 > 0$ then $D \leq 0$

$$\Rightarrow \quad 4(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 - 4\left(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2\right)\left(b_1^2 + b_1^2 + \ldots + b_n^2\right) \leq 0.$$

$$\Rightarrow \quad \left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

[Hence Proved]

Equality holds when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots = \frac{a_n}{b_n} = x$

Titu's Lemma on Cauchy – Schwartz Inequality

Let $a_1, a_2, \ldots, a_n \in R$ and $x_1, x_2, x_3, \ldots, x_n \in R^+$

Then $\left(\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \frac{a_3^2}{x_3} + \ldots + \frac{a_n^2}{x_n}\right) \geq \frac{\left(a_1 + a_2 + \ldots + a_n\right)^2}{\left(x_1 + x_2 + x_3 + \ldots + x_n\right)}$

Proof:

$$\left(\left(\frac{a_1}{\sqrt{x_1}}\right)^2 + \left(\frac{a_2}{\sqrt{x^2}}\right)^2 + \left(\frac{a_3}{\sqrt{x_3}}\right)^2 + \ldots + \left(\frac{a_n}{\sqrt{x_n}}\right)^2\right) \cdot \left[\left(\sqrt{x_1}\right)^2 + \left(\sqrt{x_2}\right)^2 + \ldots + \left(\sqrt{x_n}\right)^2\right] \geq (a_1 + a_2 + \ldots + a_n)^2$$

$$\Rightarrow \left(\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \ldots + \frac{a_n^2}{x_n}\right) \geq \frac{\left(a_1 + a_2 + a_3 + \ldots + a_n\right)^2}{\left(x_1 + x_2 + x_3 + \ldots + x_n\right)}$$

Equality holds when $\frac{\frac{a_1}{\sqrt{x_1}}}{\sqrt{x_1}} = \frac{a_1}{x_1} = \frac{a_2}{x_2} = \frac{a_3}{x_3} = \ldots = \frac{a_n}{x_n}$

Example 1

If x satisfies $|x-1| + |x-2| + |x-3| \geq 6$, then

(A) $0 \leq x \leq 4$ **(B)** $x \leq -2$ or $x \geq 4$

(C) $x \leq 0$ or $x \geq 4$ **(D)** none of these

Solution

Case I: $x \leq 1$, then

$$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow x \leq 0$$

Hence $x \leq 0$ (i)

Case II: $1 < x \leq 2$, then

$$x - 1 + 2 - x + 3 - x \geq 6 \Rightarrow x \leq -2$$

But $1 < x < 2 \Rightarrow$ No solution (ii)

Case III: $2 < x \leq 3$, then

$$x - 1 + x - 2 + 3 - x \geq 6 \Rightarrow x \geq 6$$

But $2 < x < 3 \Rightarrow$ No solution (iii)

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Case IV: $x > 3$, then

$$x-1+x-2+x-3 \geq 6 \Rightarrow x \geq 4$$

Hence $x \geq 4$ (iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \leq 0$ or $x \geq 4$. **Ans. (C)**

Example 2

Solve for x:

(a) $||x-1|+2| \leq 4$.

(b) $\dfrac{x-4}{x+2} \leq \left|\dfrac{x-2}{x-1}\right|$

Solution

(a)
$$||x-1|+2| \leq 4$$
$$\Rightarrow \quad -4 \leq |x-1|+2 \leq 4$$
$$\Rightarrow \quad -6 \leq |x-1| \leq 2$$
$$\Rightarrow \quad |x-1| \leq 2$$
$$\Rightarrow \quad -2 \leq x-1 \leq 2$$
$$\Rightarrow \quad -1 \leq x \leq 3$$
$$\Rightarrow \quad x \in [-1, 3] \quad \textbf{Ans.}$$

(b) **Case 1:** Given inequation will be satisfied for all x such that

$$\frac{x-4}{x+2} \leq 0$$
$$\Rightarrow \quad x \in (-2, 4] - \{1\} \quad \text{(i)}$$

(**Note:** $\{1\}$ is not in domain of RHS)

Case 2: $\dfrac{x-4}{x+2} > 0$

$$\Rightarrow \quad x \in (-\infty, -2) \cup (4, \infty) \quad \text{(ii)}$$

Given inequation becomes

$\dfrac{x-2}{x-1} \geq \dfrac{x-4}{x+2}$ or $\dfrac{x-2}{x-1} \leq -\dfrac{x-4}{x+2}$

on solving we get $x \in (-2, 4/5) \cup (1, \infty)$

on solving we get $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii) we get

$$x \in (4, \infty) \quad \text{(iii)}$$
$$x \in \phi$$

Hence, solution of the original inequation: $x \in (-2, \infty) - \{1\}$ (taking union of (i) and (iii)) **Ans.**

Example 3

The equation $|x| + \left|\dfrac{x}{x-1}\right| = \dfrac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ **(B)** $(1, \infty)$

(C) $(-1, 1)$ **(D)** $(-\infty, \infty)$

Solution

$$\frac{x^2}{|x-1|} = \left|x + \frac{x}{x-1}\right|$$

$\because |x| + \left|\dfrac{x}{x-1}\right| = \left|x + \dfrac{x}{x-1}\right|$ is true only if $\left(x \cdot \dfrac{x}{x-1}\right) \geq 0$

$$\Rightarrow \quad x \in \{0\} \cup (1, \infty) \quad \textbf{Ans. (A,B)}$$

Example 4

Solve for x, if $\sqrt{x^2-3x+2} > x-2$.

Solution

$$\begin{bmatrix} \begin{cases} x^2-3x+2 \geq 0 \\ x-2 \geq 0 \\ (x^2-3x+2) > (x-2)^2 \end{cases} \\ \text{or} \\ \begin{cases} x^2-3x+2 \geq 0 \\ x-2 < 0 \end{cases} \end{bmatrix} \Rightarrow \begin{bmatrix} \begin{cases} (x-1)(x-2) \geq 0 \\ (x-2) \geq 0 \Rightarrow x > 2 \\ x-2 > 0 \end{cases} \\ \text{or} \\ \begin{cases} (x-1)(x-2) \geq 0 \\ \Rightarrow x \leq 1 \\ x-2 < 0 \end{cases} \end{bmatrix}$$

Hence, solution set of the original inequation is $x \in R - (1, 2]$ **Ans.**

Example 5

Solve for x:

(a) $\log_{0.5}(x^2-5x+6) \geq -1$

(b) $\log_{1/3}(\log_4(x^2-5)) > 0$

Solution

(a) $\log_{0.5}(x^2-5x+6) > -1$

$$\Rightarrow \quad 0 < x^2-5x+6 \leq (0.5)^{-1}$$
$$\Rightarrow \quad 0 < x^2-5x+6 \leq 2$$
$$\begin{cases} x^2-5x+6 > 0 \\ x^2-5x+6 \leq 2 \end{cases} \Rightarrow x \in [1, 2) \cup (3, 4]$$

Hence, solution set of original inequation: $x \in [1,2) \cup (3,4]$ **Ans.**

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(b) $\log_{1/3}(\log_4(x^2-5)) > 0$

$$\Rightarrow \quad 0 < \log_4(x^2-5) < 1$$

$$\begin{cases} 0 < \log_4(x^2-5) \Rightarrow x^2-5 > 1 \\ \log_4(x^2-5) < 1 \Rightarrow 0 < x^2-5 < 4 \end{cases}$$

$$\Rightarrow \quad 1 < (x^2-5) < 4$$

$$\Rightarrow \quad 6 < x^2 < 9$$

$$\Rightarrow \quad x \in (-3, -\sqrt{6}) \cup (\sqrt{6}, 3)$$

Hence, solution set of original inequation:

$x \in (-3, -\sqrt{6}) \cup (\sqrt{6}, 3)$ **Ans.**

Example 6

Solve for x: $\log_2 x \le \dfrac{2}{\log_2 x - 1}$

Solution

Let $\log_2 x = t$

$$t \le \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \le 0$$

$$\Rightarrow \quad \frac{t^2-t-2}{t-1} \le 0$$

$$\Rightarrow \quad \frac{(t-2)(t+1)}{(t-1)} \le 0$$

$$\Rightarrow \quad t \in (-\infty, -1] \cup (1, 2]$$

or $\log_2 x \in (-\infty, -1] \cup (1, 2]$

or $x \in \left(0, \frac{1}{2}\right] \cup (2, 4]$ **Ans.**

Example 7

Solve the inequation: $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

Solution

This inequation is equivalent to the collection of the systems

$$\begin{bmatrix} \begin{cases} 2x+3 > 1 \\ 0 < x^2 < 2x+3 \end{cases} \\ \text{or} \\ \begin{cases} 0 < 2x+3 < 1 \\ x^2 > 2x+3 > 0 \end{cases} \end{bmatrix} \Rightarrow \begin{bmatrix} \begin{cases} x > -1 \\ (x-3)(x+1) < 0 \text{ and } x \ne 0 \end{cases} \\ \text{or} \\ \begin{cases} -\frac{3}{2} < x < -1 \\ (x-3)(x+1) > 0 \end{cases} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \begin{cases} x > -1 \\ -1 < x < 3 \end{cases} \Rightarrow -1 < x < 3 \text{ and } x \ne 0 \\ \text{or} \\ \begin{cases} -\frac{3}{2} < x < -1 \\ x < -1 \quad \text{or} \quad x > 3 \end{cases} \Rightarrow -\frac{3}{2} < x < -1 \end{bmatrix}$$

Hence, solution of the original inequation is

$$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$$ **Ans.**

Example 8

Solve for x: $2^{x+2} > \left(\frac{1}{4}\right)^{\frac{1}{x}}$

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Solution

We have $2^{x+2} > 2^{-2/x}$. Since the base $2 > 1$, we have $x + 2 > -\dfrac{2}{x}$

(The sign of the inequality is retained).

Now

$$x + 2 + \frac{2}{x} > 0 \Rightarrow \frac{x^2+2x+2}{x} > 0$$

$$\Rightarrow \quad \frac{(x+1)^2+1}{x} > 0 \Rightarrow x \in (0, \infty)$$ **Ans.**

Example 9

Solve for x: $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution

We have $\left(\frac{5}{4}\right)^{1-x} < \left(\frac{16}{25}\right)^{2\left(1-\sqrt{x}\right)}$ or $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4\left(1-\sqrt{x}\right)}$

Since, the base $0 < \dfrac{4}{5} < 1$, the inequality is equivalent to the inequality

$$x - 1 > 4(1 + \sqrt{x}) \Rightarrow \frac{x-5}{4} > \sqrt{x}$$

Now, RHS is positive

$$\Rightarrow \quad \frac{x-5}{4} > 0 \Rightarrow x > 5 \qquad \text{(i)}$$

We have $\frac{x-5}{4} > \sqrt{x}$ both sides are positive, so squaring both sides

$$\Rightarrow \quad \frac{(x-5)^2}{16} > x$$

or $$\frac{(x-5)^2}{16} - x > 0$$

or $$x^2 - 26x + 25 > 0$$

or $$(x-25)(x-1) > 0$$

$$\Rightarrow \quad x \in (-\infty, 1) \cup (25, \infty) \qquad \text{(ii)}$$

Intersection of (i) and (ii) gives $x \in (25, \infty)$ **Ans.**

Example 10

If a, b are the positive numbers and $3a + b = 5$ then find maximum value of

(i) a^3b **(ii)** a^2b **(iii)** ab^3

Also explain holding of equality.

Solution

(i) $$\frac{a+a+a+b}{4} \geq (a \cdot a \cdot a \cdot b)^{1/4}$$

$$\Rightarrow \quad \frac{3a+b}{4} \geq (a^3b)^{1/4}$$

$$\Rightarrow \quad \frac{5^4}{4^4} \geq a^3b$$

Equality holds when $a = b$

and $3a + b = 5 \Rightarrow a = \frac{5}{4} = b$ **Ans.**

(ii) $$\frac{\frac{3a}{2}+\frac{3a}{2}+b}{3} \geq \left(\frac{3a}{2} \cdot \frac{3a}{2} \cdot b\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \frac{3a+b}{3} \geq \left(\frac{9}{4}a^2b\right)^{\frac{1}{3}} \Rightarrow \left(\frac{5}{3}\right)^3 \geq \frac{9}{4}a^2b$$

$$\Rightarrow \quad \frac{500}{243} \geq a^2b.$$

Equality holds when $\frac{3a}{2} = b$

$3a + b = 5 \Rightarrow 3b = 5 \Rightarrow b = \frac{5}{3}$ and $a = \frac{10}{9}$ **Ans.**

(iii) $$\frac{3a+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}}{4} \geq \left(3a \cdot \frac{b}{3} \cdot \frac{b}{3} \cdot \frac{b}{3}\right)^{1/4}$$

$$\Rightarrow \quad \frac{3a+b}{4} \geq \left(\frac{ab^3}{2^2}\right)^{1/4}$$

$$\Rightarrow \quad \left(\frac{5}{4}\right)^4 \times 3^2 \geq ab^3$$

Equality holds when $3a = \frac{b}{3}$ **Ans.**

Example 11

If $a, b, c \in R^+$. Prove that $\frac{a^3}{4b^2} + \frac{b}{2c} + \frac{81c^2}{4a^3} \geq 3$

Solution

$$\frac{\frac{a^3}{4b^2}+\frac{b}{4c}+\frac{b}{4c}+\frac{81c^2}{4a^3}}{4} \geq \left(\frac{a^3}{4b^2} \times \frac{b}{4c} \times \frac{b}{4c} \times \frac{81c^2}{4a^3}\right)^{1/4}$$

$$\Rightarrow \quad \frac{a^3}{4b^2} + \frac{b}{2c} + \frac{81c^2}{4a^3} \geq 4 \times \left(\frac{81}{4^4}\right)^{1/4}$$

$$\Rightarrow \quad \frac{a^3}{4b^2} + \frac{b}{2c} + \frac{81c^2}{4a^3} \geq (81)^{1/4} \geq 3$$

Example 12

$a > 0, b > 0$ and $a + b = 1$, show that $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq 12.5$

Solution

Let $a = \sin^2\theta$ and $b = \cos^2\theta$

$$\therefore \quad \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 = \sin^4\theta + \cos^4\theta + \sec^4\theta + \operatorname{cosec}^4\theta + 4$$

$$= (\sin^2\theta + \cos^2\theta) - 2\sin^2\theta\cos^2\theta + (1 + \tan^2\theta)^2 + (1 + \cot^2\theta)^2 + 4$$

$$= 1 - 2\sin^2\theta \cdot \cos^2\theta + \tan^4\theta + \cot^4\theta + 2(\tan^2\theta + \cot^2\theta) + 6.$$

$$= 7 + (\tan^4\theta + \cot^4\theta) + 2(\tan^2\theta + \cot^2\theta) - \frac{1}{2}\sin^2 2\theta$$

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$$0 \le \frac{1}{2}\sin^2 2\theta \le \frac{1}{2} \Rightarrow -\frac{1}{2}\sin^2\theta \ge -\frac{1}{2}$$

[Equality at $\sin 2\theta = 1 \Rightarrow \theta = 45°$]

$$\tan^4\theta + \cos^4\theta \ge 2\sqrt{\tan^4\theta \cdot \cot^4\theta}$$

$\Rightarrow$ $\tan^4\theta + \cot^4\theta \ge 2$ [Equality at $\theta = 45°$]

and $\tan^2\theta + \cot^2\theta \ge 2$ [Equality at $\theta = 45°$]

So, $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \ge 7 + 2 + 2 \times 2 - \frac{1}{2}$

$$\Rightarrow \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \ge 12.5$$

Example 13

If a, b, x, y are positive rational numbers such that $\frac{1}{x} + \frac{1}{y} = 1$.

Then, prove that $\frac{a^x}{x} + \frac{b^y}{y} \ge ab$

Solution

Let LCM of denominators of x and y be k then kx and ky will be positive integers

Let $a^x, a^x, a^x \ldots ky$ times and $b^y, b^y, b^y, \ldots, kx$ times

So

$$\text{AM} = \frac{(a^x + a^x + \ldots (ky \text{ times})) + (b^y + b^y + \ldots (kx \text{ times}))}{kx + ky}$$

$$= \frac{a^x \cdot ky + b^y \cdot kx}{kx + ky} = \frac{a^x \cdot y + b^y \cdot x}{x + y}$$

$$\text{GM} = \left((a^x)^{ky} \cdot (b^y)^{kx}\right)^{\frac{1}{kx+ky}}$$

$$\text{GM} = (ab)^{\frac{xy}{x+y}} = ab \left\{\frac{1}{x} + \frac{1}{y} = 1 \Rightarrow \frac{x+y}{xy} = 1\right\}$$

Now, as AM $\ge$ GM

$\therefore$ $\frac{a^x y + b^y x}{x + y} \ge ab$ [But $x + y = xy$]

So, $\frac{a^x y + b^y x}{xy} \ge ab$

$\Rightarrow$ $\frac{a^x}{x} + \frac{b^y}{y} \ge ab$ [Hence Proved]

Aliter:

$x = \frac{p}{q_1}$ and $y = \frac{p}{q_2}$; $\frac{1}{x} + \frac{1}{y} = 1$

$[p = q_1 + q_2]$; $p, q_1, q_2 \in I$

$$\text{LHS} = \frac{a^x}{x} + \frac{b^y}{y} = \frac{(a^{\frac{p}{q_1}})q_1}{p} + \frac{(b^{\frac{p}{q_2}})q_2}{p} = \frac{(a^{\frac{p}{q_1}})q_1 + (b^{\frac{p}{q_2}})q_2}{q_1 + q_2}$$

$$\text{LHS} \ge \left((a^{\frac{p}{q_1}})^{q_1} (b^{\frac{p}{q_2}})^{q_2}\right)^{\frac{1}{q_1+q_2}} \quad \text{(AM} \ge \text{GM)}$$

Example 14

If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that

$$a + b \ge 6(2^{1/3} + 4^{1/3}).$$

Solution

Let α, β, γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots must be negative.

Let $\alpha_1 = -\alpha, \alpha_2 = -\beta, \alpha_3 = -\gamma$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = \frac{a}{2}$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \frac{b}{2}$$

$$\alpha_1\alpha_2\alpha_3 = 2$$

Applying AM $\ge$ GM, we have

$$\frac{\alpha_1\alpha_2\alpha_3}{3} \ge (\alpha_1\alpha_2\alpha_3)^{1/3}$$

$$\Rightarrow a \ge 6 \times 2^{1/3}$$

Also, $\frac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3}$

$$\Rightarrow b \ge 6 \times 4^{1/3}$$

Therefore, $a + b \ge 6(2^{1/3} + 4^{1/3})$.

Example 15

If $a_i > 0 \ \forall \ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$, then prove that

$$(1 + a_1)(1 + a_2)(1 + a_3) \ldots (1 + a_n) \ge 2^n$$

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Solution

Using A.M. ≥ G.M.

$$1 + a_1 \geq 2\sqrt{a_1}$$

$$1 + a_2 \geq 2\sqrt{a_2}$$

$$1 + a_n \geq 2\sqrt{a_n}$$

$$\Rightarrow \quad (1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq 2^n (a_1 a_2 a_3 \ldots a_n)^{1/2}$$

As $a_1 a_2 a_3 \ldots a_n = 1$

Hence, $(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq 2^n$.

Example 16

Let $a, b, c \in R^+$ prove that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.

Solution

$$\Rightarrow \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a^2}{ab+ac} + \frac{b^2}{bc+ab} + \frac{c^2}{ac+bc}$$

$$\geq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

[Using Titu's Lemma]

$$\Rightarrow \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

We know that,

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

$$\Rightarrow \quad \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \geq 0.$$

$$\Rightarrow \quad \frac{(a+b+c)^2}{ab+bc+ca} \geq 3$$

Hence, $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.

Example 17

Let $a, b \in R^+$. Prove that $8(a^4 + b^4) \geq (a + b)^4$.

Solution

$$8\left(a^4 + b^4\right) = 8\left[\frac{\left(a^2\right)^2}{1} + \frac{\left(b^2\right)^2}{1}\right] \geq 8\frac{\left(a^2 + b^2\right)^2}{(1+1)} = 4\left(a^2 + b^2\right)^2$$

Now, $4\left(a^2 + b^2\right) = 4\left(\frac{a^2}{1} + \frac{b^2}{1}\right)^2 \geq 4\left(\frac{(a+b)^2}{(1+1)}\right)^2 = (a+b)^4$.

[Using Titu's Lemma]

Hence, proved. [This can also be proved by RMS ≥ AM.]

Example 19

Prove that: $-4 \leq 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \leq 10$, for all values of θ.

Solution

We have,

$$5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3}$$

$$- 3\sin\theta\sin\frac{\pi}{3} = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$$

Since,

$$-\sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \leq \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$$

$$\leq \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad -7 \leq \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \leq 7 \ \forall \ \theta.$$

$$\Rightarrow \quad -7 \leq 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) \leq 7 \ \forall \ \theta.$$

$$\Rightarrow \quad -7 + 3 \leq 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \leq 7 + 3 \ \forall \ \theta.$$

$$\Rightarrow \quad -4 \leq 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \leq 10 \ \forall \ \theta.$$

Example 20

Find the maximum value of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$:

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

Solution

We have

$$1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$$

$$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta) = 1$$

$$+ \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)(\cos\theta + \sin\theta)$$

$$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) \cdot \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

$\therefore$ maximum value $= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) \cdot \sqrt{2} = 4$ **Ans. (D)**

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Practice Exercise

EXERCISE – I

PART – I KVPY Level Problems

1. The number of solution of the equation $e^{2x} + e^{x} + e^{-2x} + e^{-x} = 3\,(e^{-2x} + e^{x})$ is

(A) 0 **(B)** 2
(C) 1 **(D)** more than 2

2. The solution of the inequation $|x^2 - 2x - 3| < |x^2 - x + 5|$ is:

(A) $(-\infty, 5)$
(B) $(-\infty, 2) \cup (3, 8) \cup (8, \infty)$
(C) $(-8, \infty)$
(D) $(3, 8)$

3. The minimum value of $f(x) = |x - 1| + |x - 2| + |x - 3|$ is equal to:

(A) 1 **(B)** 2
(C) 3 **(D)** 0

4. The complete solution set of the inequation $\sqrt{x+18} < 2 - x$ is:

(A) $[-18, -2]$ **(B)** $(-\infty, -2) \cup (7, \infty)$
(C) $(-18, 2) \cup (7, \infty)$ **(D)** $[-18, -2)$

5. Solution of the inequality, $x - 3 < \sqrt{x^2 + 4x - 5}$ is:

(A) $(-\infty, -5] \cup [1, \infty)$
(B) $(-5, 3]$
(C) $(-\infty, -5] \cup \left(\frac{7}{5}, \infty\right)$
(D) $\left(\frac{7}{5}, \infty\right)$

6. If $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$, then x lies in the interval

(A) $(2, \infty)$ **(B)** $(1, 2)$
(C) $(1, \infty)$ **(D)** none of these

7. The solution set of the inequality

$\log_{\frac{5}{8}}\left(2x^2 - x - \frac{3}{8}\right) \geq 1$ is:

(A) $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$
(B) $\left[-\frac{1}{2}, 1\right]$
(C) $\left[-\frac{1}{2}, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$
(D) $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, \infty\right)$

8. Solution set of the inequality, $2 - \log_2(x^2 + 3x) \geq 0$ is:

(A) $[-4, 1]$
(B) $[-4, -3) \cup (0, 1]$
(C) $(-\infty, -3) \cup (1, \infty)$
(D) $(-\infty, -4) \cup [1, \infty)$

9. If $\log_{1/3}\left(\frac{3x-1}{x+2}\right)$ is less than unity then x must lie in the interval:

(A) $(-\infty, -2) \cup (5/8, \infty)$
(B) $(-2, 5/8)$
(C) $(-\infty, -2) \cup (1/3, 5/8)$
(D) $(-2, 1/3)$

10. If a, b, c are positive real numbers such that $ab^2c^3 = 64$ then minimum value of $\left(\frac{1}{a} + \frac{2}{b} + \frac{3}{c}\right)$ is equal to:

(A) 6 **(B)** 2
(C) 3 **(D)** None of these

11. If $abcd = 1$ where a, b, c, d are positive reals then the minimum value of $a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd$ is

(A) 6 **(B)** 10
(C) 12 **(D)** 20

12. **Statement-1:** If $27\,abc \geq (a + b + c)^3$ and $3a + 4b + 5c = 12$ then $\frac{1}{a^2} + \frac{1}{b^3} + \frac{1}{c^5} = 10$; where a, b, c are positive real numbers.

Statement-2: For positive real numbers A.M. $\geq$ G.M.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

13. If a, b, and c are positive real numbers then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is greater than or equal to:

(A) 3 **(B)** 6
(C) 27 **(D)** 5

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14. If $x \in R$, the numbers $(5^{1+x} + 5^{1-x})$, $a/2$, $(25^x + 25^{-x})$ form an A.P., then 'a' must lie in the interval:
(A) [1, 5] **(B)** [2, 5]
(C) [5, 12] **(D)** [12, ∞)

15. Let x, y, z be positive real numbers satisfying $x+y+z=1$ prove that $xy(x+y)^2 + yz(y+z)^2 + zx(z+x)^2 \geq 4xyz$

16. If a, b, c are positive fractions and $a + b + c = 1$, prove that $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq 64$

17. Show that: $\tan A + \tan B + \tan C \geq 3\sqrt{3}$, $A, B, C \in \left[0, \frac{\pi}{2}\right]$

18. For all $x \in R$, find the range of the function, $f(x) = \cos x\,(\sin x + \sqrt{\sin^2 x + \sin^2 \alpha}\,)$; $\alpha \in [0, \pi]$

19. Solve the inequality: $\log_{1/2}(x+1) > \log_2(2-x)$.

20. Solve the inequality: $\log_x 2 \cdot \log_{2x} 2 \cdot \log_2 4x > 1$.

21. Prove that the triangle ABC is equilateral iff $\cot A + \cot B + \cot C = \sqrt{3}$.

22. If $A + B + C = \pi$; prove that $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1$.

23. If x, y, z are positive integer and $x + y + z = 1$, prove that $\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right) \geq 8$

24. If $2a + b + 3c = 1$ and $a > 0$, $b > 0$, $c > 0$, then find the greatest value of $a^4b^2c^2$ and obtain the corresponding values of a, b, c.

25. If a, b, c, d are positive numbers in H.P. then prove that $a + d \geq b + c$

26. Prove that for positive quantities a, b, c we have $\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leq \frac{a+b+c}{2}$

27. If a, b, c are natural numbers, prove that $\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c} \geq a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$

28. If $a \neq b$ and a, b, x, y are positive rationals such that $x + y = 1$ then prove that $a^x b^y < ax + by$

29. Find all positive real numbers x, y, z such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2018}$$

$$2y - 2z + \frac{1}{x} = \frac{1}{2018}$$

$$2z - 2x + \frac{1}{y} = \frac{1}{2018}$$

Solve the following inequalities from 30 to 33:

30. $(\log_2 x)^4 - \left(\log_{\frac{1}{2}} \frac{x^5}{4}\right)^2 - 20\log_2 x + 148 < 0.$

31. $x^{1/\log x} \cdot \log x < 1$

32. $(\log 100x)^2 + (\log 10x)^2 + \log x \leq 14$

33. $\log_{1/5}(2x^2 + 5x + 1) < 0$

34. The set of all real numbers x for which $x^2 - |x+2| + x > 0$, is
(A) $(-\infty, -2) \cup (2, \infty)$ **(B)** $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$
(C) $(-\infty, -1) \cup (1, \infty)$ **(D)** $(\sqrt{2}, \infty)$

35. If $A = \sin^2 x + \cos^4 x$, then for all real x:
(A) $1 \leq A \leq 2$ **(B)** $\frac{3}{4} \leq A \leq \frac{13}{16}$
(C) $\frac{3}{4} \leq A \leq 1$ **(D)** $\frac{13}{16} \leq A \leq 1$

36. If $a_1, a_2, a_3 \ldots a_n \in R^+$ and $a_1 \cdot a_2 \cdot a_3 \ldots a_n = 1$, then minimum value of $(1 + a_1 + a_1^2)(1 + a_2 + a_2^2)(1 + a_3 + a_3^2) \ldots (1 + a_n + a_n^2)$ is equal to:
(A) 3^{n+1} **(B)** 3^n
(C) 3^{n-1} **(D)** none of these

37. Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is:
(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$
(C) 1 **(D)** $\frac{m+n}{6mn}$

PART – II PRMO Level Problems

1. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

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(A) $\frac{1}{PS}+\frac{1}{ST}<\frac{2}{\sqrt{QS\times SR}}$

(B) $\frac{1}{PS}+\frac{1}{ST}>\frac{2}{\sqrt{QS\times SR}}$

(C) $\frac{1}{PS}+\frac{1}{ST}<\frac{4}{QR}$

(D) $\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$

2. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$ then

(A) $\tan^2 x=\frac{2}{3}$ **(B)** $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$

(C) $\tan^2 x=\frac{1}{3}$ **(D)** $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

3. Let $b_i>1$ for $i=1, 2, \ldots, 101$. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in Arithmetic progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in A.P. such that $a_1=b_1$ and $a_{51}=b_{51}$. If $t=b_1+b_2+\ldots+b_{51}$ and $S=a_1+a_2+\ldots+a_{51}$ then

(A) $s>t$ and $a_{101}>b_{101}$
(B) $s>t$ and $a_{101}<b_{101}$
(C) $s<t$ and $a_{101}>b_{101}$
(D) $s<t$ and $a_{101}<b_{101}$

4. Which of the following statements is(are) correct?

(A) $7^{1/7}>(42)^{1/14}>1$

(B) $\log_3(5)\log_7(9)\log_{11}(13)>-2$

(C) $\sqrt{99+70\sqrt{2}}+\sqrt{99-70\sqrt{2}}$ is rational

(D) $\frac{1}{\log_4 3}+\frac{1}{\log_7 3}>3$

5. If $f(x)=\frac{\sin 3x}{\sin x}$, $x\neq n\pi$, then the range of values of $f(x)$ for real values of x is:

(A) $[-1, 3]$ **(B)** $(-\infty, -1]$
(C) $(3, +\infty)$ **(D)** $[-1, 3)$

6. Maximum and minimum value of $2\sin^2\theta-3\sin\theta+2$ is:

(A) $\frac{1}{4}, -\frac{7}{4}$ **(B)** $\frac{1}{4}, \frac{21}{4}$

(C) $\frac{21}{4}, -\frac{3}{4}$ **(D)** $7, \frac{7}{8}$

7. For $\theta\in(0, \pi/2)$, the maximum value of $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{6}\right)$ is attained at $\theta=$

(A) $\frac{\pi}{12}$ **(B)** $\frac{\pi}{6}$

(C) $\frac{\pi}{3}$ **(D)** $\frac{\pi}{4}$

8. Minimum value of the expression $\cos^2\theta-(6\sin\theta\cos\theta)+3\sin^2\theta+2$, is:

(A) $4+\sqrt{10}$ **(B)** $4-\sqrt{10}$
(C) 0 **(D)** 4

9. $\sec^2\theta=\frac{4xy}{(x+y)^2}$ is true if and only if:

(A) $x+y\neq 0$ **(B)** $x=y, x\neq 0$
(C) $x=y$ **(D)** $x\neq 0, y\neq 0$

10. If $a>0$, then minimum value of $a+2a^2+a^3+15+a^{-1}+a^{-3}+a^{-4}$ is

11. If $a, b, c, d>0$ such that $a+2b+3c+4d=50$, then find the maximum value of $\left(\frac{a^2b^4c^3d}{16}\right)^{1/10}$.

12. Given that the cubic $ax^3-ax^2+9bx-b=0$ $(a\neq 0)$ has all three positive roots. Find the harmonic mean of the roots independent of a and b, hence deduce that the root are all equal. Find also the minimum value of $(a+b)$ if a and $b\in N$.

13. **(a)** If $y=10\cos^2 x-6\sin x\cos x+2\sin^2 x$, then find the greatest and least value of y.

(b) If $y=1+2\sin x+3\cos^2 x$, find the maximum and minimum values of $y\ \forall\ x\in R$.

(c) If $y=9\sec^2 x+16\operatorname{cosec}^2 x$, find the minimum value of y for all permissible value of x.

(d) If $a\leq 3\cos\left(\theta+\frac{\pi}{3}\right)+5\cos\theta+3\leq b$, find a and b, where a is the minimum value and b is the maximum value.

14. If M and m denotes maximum and minimum value of $\sqrt{49\cos^2\theta+\sin^2\theta}+\sqrt{49\sin^2\theta+\cos^2\theta}$, then find the value of $(M+m)$.

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15. If x and y are real number such that $x^2 + 2xy - y^2 = 6$, find the minimum value of $(x^2 + y^2)^2$.

16. Let α, β, γ be distinct real numbers such that

$$a\alpha^2 + b\alpha + c = (\sin\theta)\alpha^2 + (\cos\theta)\alpha$$

$$a\beta^2 + b\beta + c = (\sin\theta)\beta^2 + (\cos\theta)\beta$$

$$a\gamma^2 + b\gamma + c = (\sin\theta)\gamma^2 + (\cos\theta)\gamma$$

(where $a, b, c \in R$)

Find the maximum value of the expression $\dfrac{a^2+b^2}{a^2+3ab+5b^2}$

17. Given $x, y \in R$, $x^2 + y^2 > 0$. If the maximum and minimum value of the expression $E = \dfrac{x^2+y^2}{x^2+xy+4y^2}$ are M and m, and A denotes the average value of M and m, compute $(2007)A$.

Solve the following inequalities from 18 to 23:

18. $\log_{1/2} x + \log_3 x > 1$

19. $\log_{x^2}(2+x) < 1$

20. $\log_x \dfrac{4x+5}{6-5x} < -1$

21. $(\log_{|x+6|} 2) \cdot \log_2(x^2 - x - 2) \geq 1$

22. $\log_3 \dfrac{|x^2-4x|+3}{x^2+|x-5|} \geq 0$

23. $\log_{\frac{x+6}{3}}\left(\log_2\left\{\dfrac{x-1}{x+2}\right\}\right) > 0$

24. Prove that from the equality $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a+b}$ follows the relation; $\dfrac{\sin^8\alpha}{a^3} + \dfrac{\cos^8\alpha}{b^3} = \dfrac{1}{(a+b)^3}$.

25. 3 natural numbers x, y, z are in G.P. and x, y^2, z are in A.P., then the minimum value of $x + 2y + z$ is equal to

26. If the roots of the equation, $x^3 + px^2 + qx - 1 = 0$ form an increasing G.P. where p and q are real, then

(A) $p + q = 0$

(B) $p \in (-3, \infty)$

(C) one of the roots is unity

(D) one root is smaller than 1 and one root is greater than 1.

27. Find the minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}}$ for $x > 0$.

28. Find out the values of 'a' for which any solution of the inequality, $\dfrac{\log_3(x^2-3x+7)}{\log_3(3x+2)} < 1$ is also a solution of the inequality, $x^2 + (5-2a)x \leq 10a$.

29. Solve the inequality $\log_{\log_2\left(\frac{x}{2}\right)}(x^2 - 10x + 22) > 0$.

30. Find the set of values of 'y' for which the inequality, $2\log_{0.5} y^2 - 3 + 2x\log_{0.5} y^2 - x^2 > 0$ is valid for at least one real value of 'x'.

31. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then the difference between the maximum and minimum value of u^2 is given by:

(A) $2(a^2+b^2)$ **(B)** $2\sqrt{a^2+b^2}$

(C) $(a+b)^2$ **(D)** $(a-b)^2$

32. If a, b, c are positive real numbers, then $\left[(1+a)(1+b)(1+c)\right]^7 > 7^k a^4 b^4 c^4$. Find k.

33. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$ such that $\left[\alpha^6\right] = 3k$. Find k. Here $[x]$ denotes the G.I.F.

34. x, y, z are positive reals such that

$$\frac{x^2+yz}{y+z} + \frac{x^2+yz}{y+z} + \frac{x^2+yz}{y+z} \geq m(x+y+z).$$

Find $m+12$.

35. For $a, b, c, d \in \mathbb{R}^+$, $a^3 + b^3 + c^3 + d^3 = 10$. If $a^4 + b^4 + c^4 + d^4 \geq \sqrt[3]{n}$. Find $\dfrac{n}{100}$.

36. $\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq m\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^n$. Find $19m + 23n$.

37. Given a, b are real numbers such that $a + b = 17$. If the minimum value of $2^a + 4^b$ is $m\sqrt[3]{4}$, then find the sum of digits of m.

38. Find the sum of all values of x satisfying the equation

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$$\left(\sqrt{x^2-5x+6}+\sqrt{x^2-5x+4}\right)^{\frac{x}{2}}$$

$$+\left(\sqrt{x^2-5x+6}-\sqrt{x^2-5x+4}\right)^{\frac{x}{2}}=2^{\frac{x+4}{4}}$$

39. For real numbers $a, b, c, d, e > 1$, $\frac{a^2}{c-1}+\frac{b^2}{d-1}+\frac{c^2}{e-1}+\frac{d^2}{a-1}+\frac{e^2}{b-1}\geq u$. Find $\frac{u^2}{100}$.

40. Let a, b, c, x, y, z be positive reals satisfying $\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=1$ such that $abc\leq(n)^3$ and $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\leq 1$ such that $\left(x^2+1\right)\left(y^2+1\right)\left(z^2+1\right)\geq(m)^3$. Find $2n+m$.

EXERCISE – II

PART – I Previous Years' PRMO Problems

1. What is the greatest integer not exceeding the sum $\sum_{n=1}^{1599}\frac{1}{\sqrt{n}}$? **[PRMO-2019]**

2. Positive integers x, y, z satisfy $xy+z=160$. Compute the smallest possible value of $x+yz$. **[PRMO-2019]**

3. What is the largest positive integer n such that $\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}}\geq n(a+b+c)$. **[PRMO-2018]**

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ANSWER KEYS

Practice Exercise

EXERCISE – I

PART – I KVPY Level Problems

1. C **2.** C **3.** B **4.** D **5.** A **6.** A **7.** A **8.** B **9.** A **10.** C
11. B **12.** D **13.** A **14.** D

18. $-\sqrt{1+\sin^2\alpha} \le f(x) \le \sqrt{1+\sin^2\alpha}$ **19.** $-1 < x < \frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2} < x < 2$

20. $2^{-\sqrt{2}} < x < 2^{-1}$; $1 < x < 2^{\sqrt{2}}$ **24.** $\frac{1}{9.2^{16}}$; $a = \frac{1}{4}, b = \frac{1}{4}, c = \frac{1}{12}$

29. $x = y = z = 2018$ **30.** $x \in \left(\frac{1}{16}, \frac{1}{8}\right) \cup (8, 16)$ **31.** $(0, 1) \cup (1, 10^{1/10})$

32. $\frac{1}{\sqrt{10}^9} \le x \le 10$ **33.** $(-\infty, -2.5) \cup (0, \infty)$

34. B **35.** (3) **36.** (2) **37.** (1)

PART – II PRMO Level Problems

1. B, D **2.** A, B **3.** B **4.** A, B, D **5.** D **6.** D **7.** A **8.** B **9.** B
10. 22 **11.** 5 **12.** 28

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13. (a) $y_{max} = 11$, $y_{min} = 1$; (b) $y_{max} = \frac{13}{3}$, $y_{min} = -1$; (c) 49; (d) $a = -4$ and $b = 10$

14. 18 **15.** 18 **16.** 2 **17.** 1338

18. $0 < x < 3^{1/1-\log 3}$ (where base of log is 2) **19.** $-2 < x < -1, -1 < x < 0, 0 < x < 1, x > 2$

20. $\frac{1}{2} < x < 1$ **21.** $x < -7, -5 < x \le -2, x \ge 4$ **22.** $x \le -\frac{2}{3}$; $\frac{1}{2} \le x \le 2$

23. $(-6, -5) \cup (-3, -2)$ **25.** 4 **26.** A, C, D **27.** $Y_{min} = 6$ **28.** $a \ge \frac{5}{2}$

29. $x \in (3, 5-\sqrt{3}) \cup (7, \infty)$ **30.** $(-\infty, -2\sqrt{2}) \cup \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(0, \frac{1}{\sqrt{2}}\right) \cup (2\sqrt{2}, \infty)$ **31.** D

32. 7 **33.** 1 **34.** 13 **35.** 25 **36.** 42 **37.** 12 **38.** 5 **39.** 4 **40.** 6

EXERCISE – II

PART – I Previous Years' PRMO Problems

1. 78 **2.** 50 **3.** 14

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